# Existence and uniqueness of dynamic evolutions for a peeling test in dimension one

Gianni Dal Maso, Giuliano Lazzaroni, and Lorenzo Nardini
SISSA, Trieste, Italy

Shape optimization, Isoperimetric and Functional Inequalities
Luminy, November 21, 2016

- I will present some results on a simple one dimensional model of a dynamic peeling test for a thin film, initially attached to a rigid substrate.
- Our main motivation for the study of this problem is to develop the mathematical tools for a model of dynamic crack growth, which combines the equations of elasto-dynamics for the displacement (out of the crack) with an evolution law which connects the crack growth with the displacement.
- This simplified model was considered in the book by Freund: Dynamic fracture mechanics (1990). It exhibits some of the relevant mathematical difficulties due to the time dependence of the domain of the wave equation.
- Examples of solutions of this model were recently studied by Dumouchel, Marigo, Charlotte (2008) and Bargellini, Dumouchel, Lazzaroni, Marigo (2012). These examples show that the quasistatic limit of this model is highly nontrivial.

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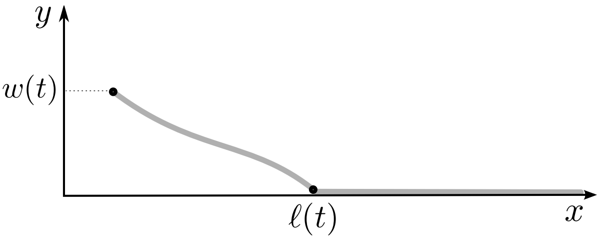
# Geometry of the peeling process

- A thin film is initially attached to a planar rigid substrate. The peeling process is assumed to depend only on one variable. This hypothesis is crucial for our analysis, since we frequently use d'Alembert's formula for the wave equation.
- The film is described by a curve (in grey in the figure), which represents its intersection with a vertical plane with horizontal coordinate $x$ and vertical coordinate $y$. The positive $x$-axis represents the substrate as well as the reference configuration of the film.



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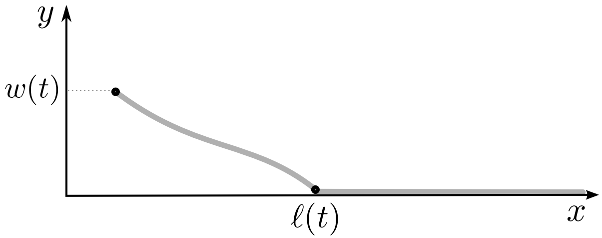
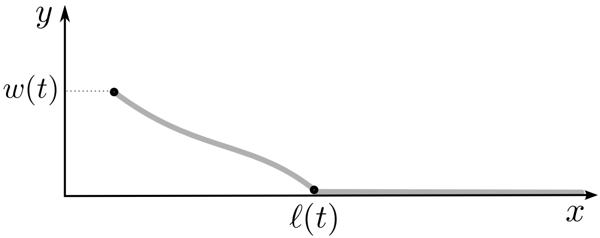
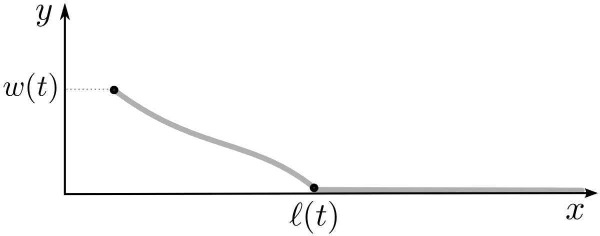


# Assumptions on the peeling process




- In its deformed configuration the film at time $t \geq 0$ is parametrised by $(h(t,x), u(t,x))$, horizontal and vertical displacement of the point at $x$.
- The film is assumed to be perfectly flexible, inextensible, and glued to the rigid substrate on the half line $\{x \geq \ell(t)\,,\ y = 0\}$, where $\ell(t)$ is a nondecreasing function which represents the debonding front, with $\ell_0 := \ell(0) > 0$.
- At $x = 0$ we prescribe a time-dependent vertical displacement $u(t, 0) = w(t)$ and a fixed tension so that the speed of sound in the film is constant.

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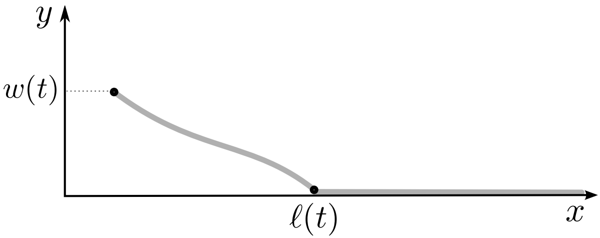


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# Equation satisfied by the vertical displacement

- Using the linear approximation and the inextensibility it turns out that $h$ can be expressed in terms of $u$ as

$$h(t,x) = \frac{1}{2}\int_x^{+\infty} u_x(t,z)^2 dz,$$

and $u$ solves the problem

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) = 0, & t > 0\,,\ 0 < x < \ell(t), \\ u(t,0) = w(t), & t > 0, \\ u(t,\ell(t)) = 0, & t > 0, \end{cases}$$

where we normalised the speed of sound to one.

- The system is supplemented by the initial conditions

$$\begin{cases} u(0,x) = u_0(x), & 0 < x < \ell_0, \\ u_t(0,x) = u_1(x), & 0 < x < \ell_0, \end{cases}$$

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# Solution with a prescribed debonding front

- On the debonding front $\ell\colon [0,+\infty) \to [\ell_0,+\infty)$, with $\ell(0)=\ell_0$, we assume that for every $T>0$ there exists $0<L_T<1$ such that

$$0 \le \ell(t_2)-\ell(t_1) \le L_T(t_2-t_1) \text{ for every } 0 \le t_1 < t_2 \le T.$$

- On the prescribed displacement $w(t)$ at $x=0$ we assume that $w \in H^1(0,T)$ for every $T>0$.
- As for the initial conditions, on the displacement we assume that $u_0 \in H^1(0,\ell_0)$, and on the velocity we assume that $u_1 \in L^2(0,\ell_0)$.
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**Theorem (DM-Lazzaroni-Nardini 2016)**

*The boundary value problem for the displacement $u$ with prescribed debonding front $\ell$ has a one and only one solution.*

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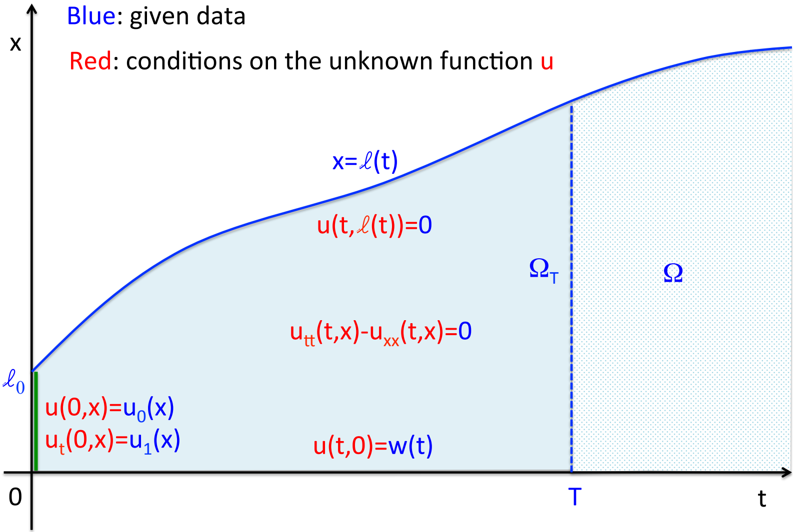

Blue: given data
Red: conditions on the unknown function u
x
$x=\ell(t)$
$u(t,\ell(t))=0$
$\Omega_T$
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$u_{tt}(t,x)-u_{xx}(t,x)=0$
$\ell_0$
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$u(t,0)=w(t)$
0
T
t

# Equivalent formulation

- By d'Alembert's formula, $u$ is a solution in $\Omega_T$ if and only if

$$u(t,x) = f(t-x) + g(t+x) \quad \text{for a.e. } (t,x) \in \Omega_T$$

for some functions $f \in H^1_{loc}(-\ell_0, T)$ and $g \in H^1_{loc}(0, T+\ell_0)$.

- The boundary condition $u(t,0) = w(t)$, together with the continuity of $f$, $g$, and $w$, gives $w(t) = f(t) + g(t)$ for every $t \in (0,T)$.
- Therefore, $u$ is a solution in $\Omega_T$ satisfying the boundary condition $u(t,0) = w(t)$ if and only if

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- The boundary condition $u(t,\ell(t)) = 0$ is satisfied if and only if

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# Determining $f$ on $[-\ell_0, \ell_0]$

- Formula $u(t,x) = w(t+x) - f(t+x) + f(t-x)$ and the corresponding formula for $u_t(t,x)$ allow us to obtain the values of $f(s)$, for $-\ell_0 \leq s \leq \ell_0$, from the initial conditions $u_0$ and $u_1$. More precisely,

$$f(s) = \begin{cases} w(s) - \dfrac{u_0(s)}{2} - \dfrac{1}{2}\displaystyle\int_0^s u_1(x)\,dx - w(0) + \dfrac{u_0(0)}{2} & \text{for } s \in [0, \ell_0], \\ \dfrac{u_0(-s)}{2} - \dfrac{1}{2}\displaystyle\int_0^{-s} u_1(x)\,dx - \dfrac{u_0(0)}{2} & \text{for } s \in [-\ell_0, 0]. \end{cases}$$

- To conclude the proof of existence and uniqueness of the solution $u$, we have to show that $f$ can be extended in a unique way to a function $f$ defined on $[-\ell_0, +\infty)$ and satisfying

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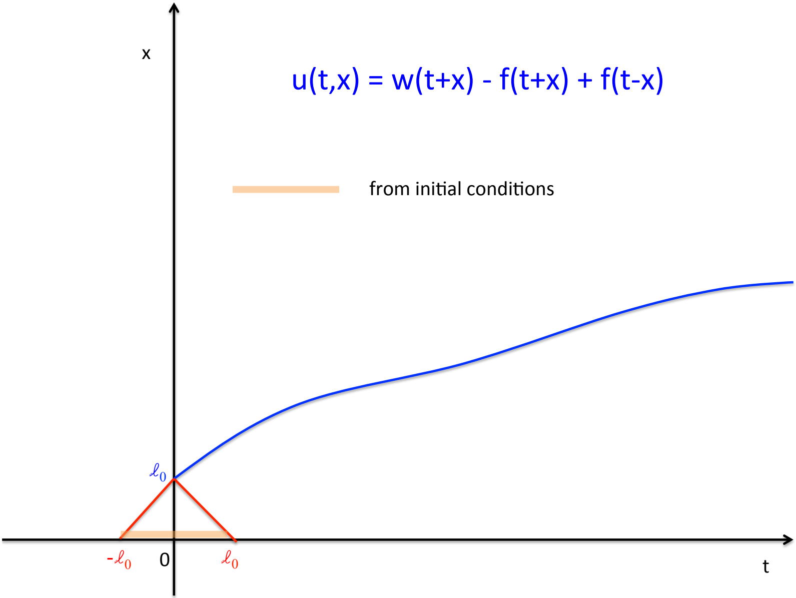
# Extending f to larger intervals



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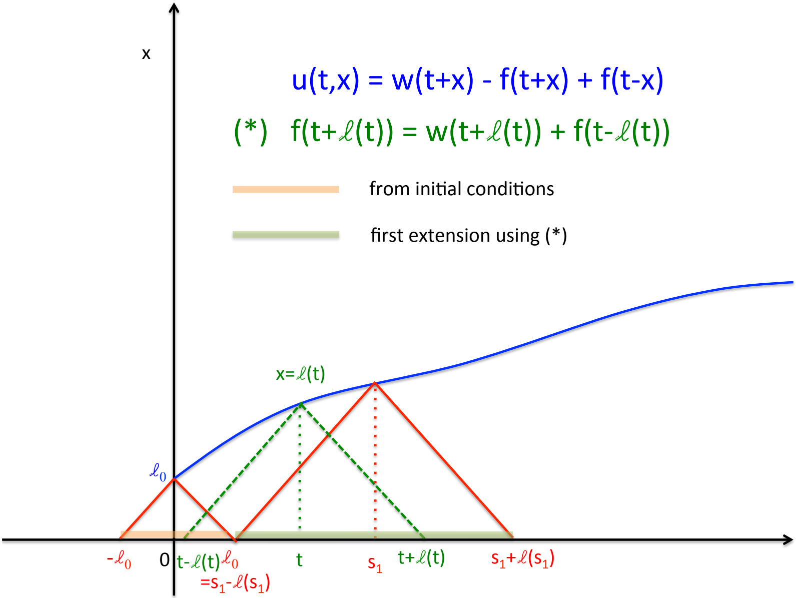
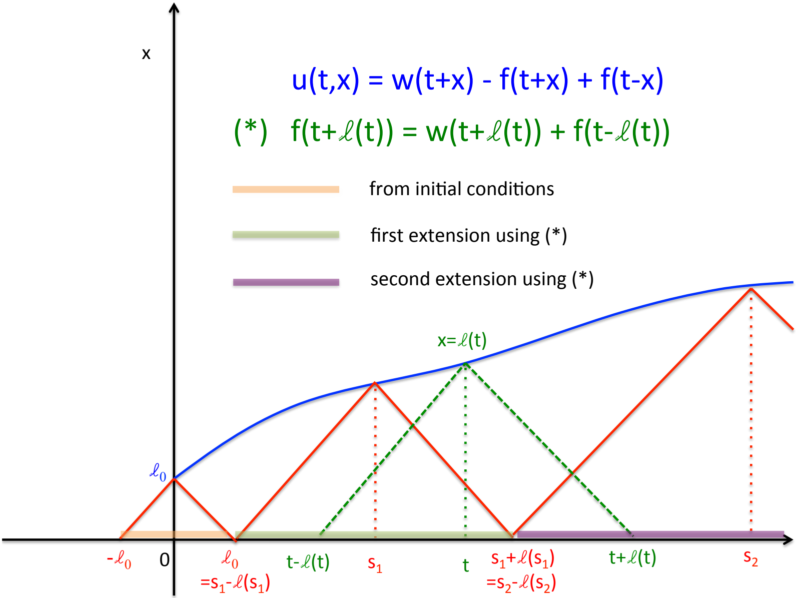


# Extending f to larger intervals

$$u(t,x) = w(t+x) - f(t+x) + f(t-x)$$

$$(*) \quad f(t+\ell(t)) = w(t+\ell(t)) + f(t-\ell(t))$$



# Determining the debonding front

- So far the debonding front $\ell$ was prescribed. We assume now that only $u_0$, $u_1$, and $w$ are given. The evolution of the debonding front $\ell$ has to be determined on the basis of an additional energy criterion.
- To formulate this criterion we fix once and for all the initial conditions $u_0$ and $u_1$ and we consider the energy of $u$ as a functional depending on $\ell$ and $w$. More precisely,

$$\mathcal{E}(t;\ell,w) := \frac{1}{2}\int_0^{\ell(t)} u_x(t,x)^2\,dx + \frac{1}{2}\int_0^{\ell(t)} u_t(t,x)^2\,dx,$$

  where $u$ is the unique solution corresponding to $u_0$, $u_1$, $\ell$ and $w$. The first term is the potential energy and the second one is the kinetic energy at time $t$.
- A crucial role is played by the dynamic energy release rate, which is defined as a (sort of) partial derivative of $\mathcal{E}$ with respect to the elongation of the debonded region.

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- A crucial role is played by the dynamic energy release rate, which is defined as a (sort of) partial derivative of $\mathcal{E}$ with respect to the elongation of the debonded region.

# Determining the debonding front

- So far the debonding front $\ell$ was prescribed. We assume now that only $u_0$, $u_1$, and $w$ are given. The evolution of the debonding front $\ell$ has to be determined on the basis of an additional energy criterion.
- To formulate this criterion we fix once and for all the initial conditions $u_0$ and $u_1$ and we consider the energy of $u$ as a functional depending on $\ell$ and $w$. More precisely,

$$\mathcal{E}(t;\ell,w) := \frac{1}{2}\int_0^{\ell(t)} u_x(t,x)^2\,dx + \frac{1}{2}\int_0^{\ell(t)} u_t(t,x)^2 dx,$$

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# Dynamic energy release rate

- To define the dynamic energy release rate $G_\alpha(t_0)$ at time $t_0$ for a given speed $0 < \alpha < 1$ of the debonding front, we modify the debonding front $\ell$ and the prescribed displacement $w$ using two functions $\lambda$ and $z$ such that $\lambda(t) = \ell(t)$ for $t \le t_0$, $\dot\lambda(t_0+) = \alpha$, $z(t) = w(t)$ for $t \le t_o$, and $z(t) = w(t_0)$ for $t > t_0$.
- We define

$$G_\alpha(t_0) := \lim_{t \to t_0^+} \frac{\mathcal{E}(t_0;\lambda,z) - \mathcal{E}(t;\lambda,z)}{\lambda(t) - \lambda(t_0)},$$

the decrease of (potential + kinetic) energy per unit length of debonding.
- We prove that, given $\ell$, $w$, $\lambda$, and $z$ as above, the limit exists for a.e. $t_0 > 0$ and depends on $\lambda$ only through $\alpha$. Moreover, we prove that

$$G_\alpha(t_0) = 2\frac{1-\alpha}{1+\alpha}\dot f(t_0 - \ell(t_0))^2,$$

where $f$ is the function in the representation formula for the solution: $u(t,x) = w(t+x) - f(t+x) + f(t-x)$.

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- The energy dissipated to debond $[x_1, x_2]$ is given by $\int_{x_1}^{x_2} \kappa(x)\,dx$, where $\kappa\colon [0,+\infty) \to (0,+\infty)$ is a prescribed function, representing the local toughness of the glue between the film and the substrate.
- The energy criterion for the debonding front, called Griffith's criterion, is

$$\begin{cases} \text{a) } \dot\ell(t) \geq 0, \\ \text{b) } G_{\dot\ell(t)}(t) \leq \kappa(\ell(t)), \\ \text{c) } \big(G_{\dot\ell(t)}(t) - \kappa(\ell(t))\big)\dot\ell(t) = 0. \end{cases}$$

- a) asserts that the debonding can only grow (unidirectionality).
  b) states that the dynamic energy release rate is always bounded by the local toughness (otherwise more debonding would have occurred).
  c) says that the debonding front can incrsease with a positive speed only if the energy released by the vibrations of the film is totally dissipated by the debonding process.

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# Coupled problem

- Given $u_0$, $u_1$, and $w$, we want to find a solution $(u, \ell)$ to the coupled problem: $u$ solves the wave equation in the time dependent domain determined by $\ell$ and $\ell$ satisfies the Griffith's criterion with the dynamic energy release rate corresponding to $u$.
- The strategy for the proof of these results is to write Griffith's criterion as an ordinary differential equation for $\ell$ depending on the unknown function $f$. More precisely, Griffith's criterion is equivalent to

$$\begin{cases} \dot\ell(t) = \dfrac{2\dot f(t-\ell(t))^2 - \kappa(\ell(t))}{2\dot f(t-\ell(t))^2 + \kappa(\ell(t))} \vee 0, & \text{for a.e. } t > 0, \\ \ell(0) = \ell_0. \end{cases} \qquad (\star\star)$$

- Therefore we have to find $f$ and $\ell$ so that

$$f(t+\ell(t)) = w(t+\ell(t)) + f(t-\ell(t)) \qquad (\star)$$

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The next theorem gives existence and uniqueness for the coupled system when the local toughness $\kappa$ is constant.

**Theorem (DM-Lazzaroni-Nardini 2016)**

*Let* $u_0 \in C^{0,1}([0, \ell_0])$, *let* $u_1 \in L^\infty(0, \ell_0)$, *let* $w \in C^{0,1}([0, T])$ *for every* $T > 0$, *and let* $\kappa > 0$ *be a constant. Assume the following compatibility conditions:* $u_0(0) = w(0)$ *and* $u_0(\ell_0) = 0$. *Then, there exists a unique solution* $(u, \ell)$ *of the coupled problem with* $(u, \ell) \in H^1(\Omega_T) \times C^{0,1}([0, T])$ *for every* $T > 0$. *Moreover, for every* $T > 0$ *one has* $u \in C^{0,1}(\overline{\Omega}_T)$ *and there exists* $L_T < 1$ *such that* $0 \le \dot{\ell}(t) \le L_T$ *for a.e.* $t \in (0, T)$.

# Existence and uniqueness for constant toughness

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# Differential equation for $\ell$

- On $[-\ell_0, \ell_0]$ we have

$$f(s) = \begin{cases} w(s) - \dfrac{u_0(s)}{2} - \dfrac{1}{2}\displaystyle\int_0^s u_1(x)\,dx - w(0) + \dfrac{u_0(0)}{2} & \text{for } s \in [0, \ell_0], \\ \dfrac{u_0(-s)}{2} - \dfrac{1}{2}\displaystyle\int_0^{-s} u_1(x)\,dx - \dfrac{u_0(0)}{2} & \text{for } s \in [-\ell_0, 0]. \end{cases}$$

- Our assumptions guarantee only that $f \in C^{0,1}([-\ell_0, \ell_0])$. Therefore we have to justify existence and uniqueness of a local solution to the differential equation

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# The autonomous differential equation

- Set $z(t) := t - \ell(t)$. Then the following problems are equivalent

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where $F(z) := 1 - \dfrac{(2\dot f(z)^2 - \kappa) \vee 0}{2\dot f(z)^2 + \kappa}$.

- Since $\dot f$ is bounded, $1/F$ is bounded on $[-\ell_0, \ell_0]$. The standard formula for autonomous problems implies that the Cauchy problem for $z$ has a unique solution $z \in C^{0,1}([0, s_1])$ and that this solution satisfies

$$\int_{-\ell_0}^{z(t)} \frac{dz}{F(z)} = t \quad \text{for every } t \in [0, s_1], \text{ where } s_1 = \int_{-\ell_0}^{\ell_0} \frac{dz}{F(z)}.$$

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# Construction of $f$ and $\ell$

(*) $f(t+\ell(t)) = w(t+\ell(t)) + f(t-\ell(t))$

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f from initial conditions

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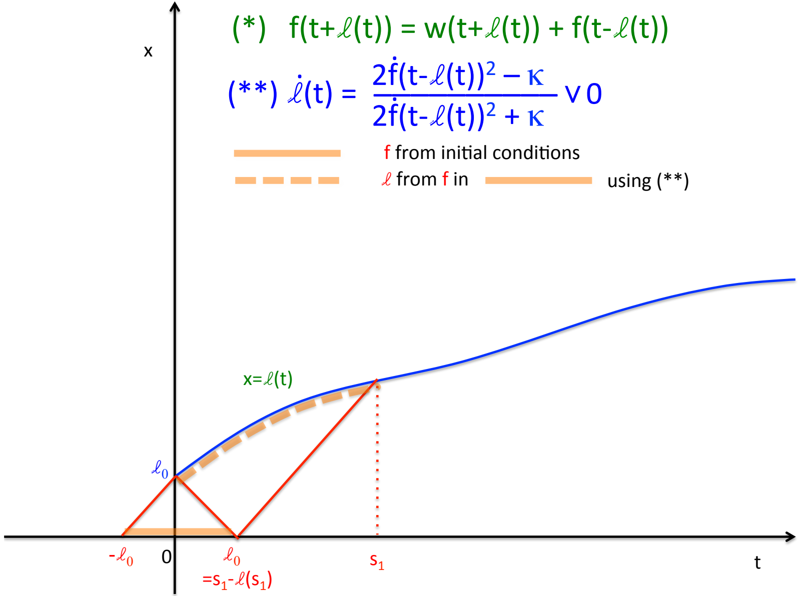

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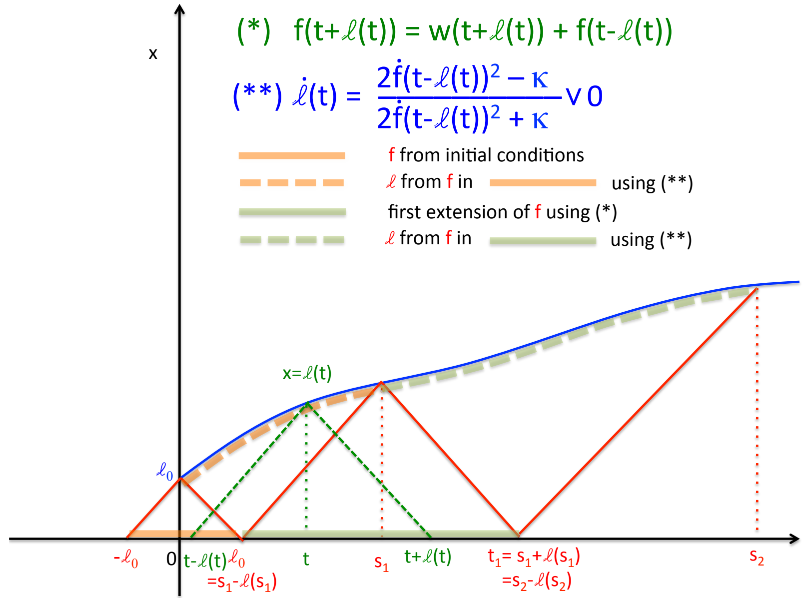

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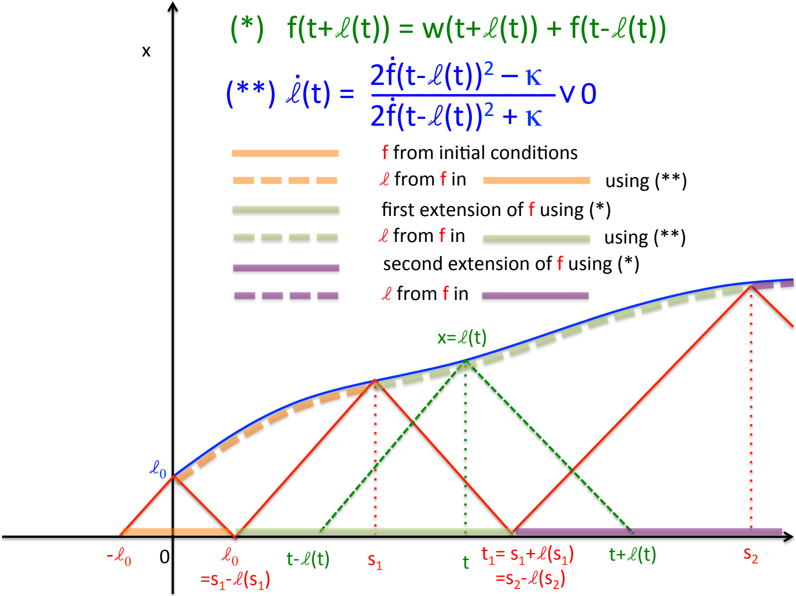

The previous result can be extended to a local toughness $\kappa$ depending on $x$.

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*Let* $u_0 \in C^{0,1}([0,\ell_0])$, *let* $u_1 \in L^\infty(0,\ell_0)$, *let* $w \in C^{0,1}([0,T])$ *for every* $T>0$, *and let* $\kappa \in C^{0,1}([0,a];[c_1,c_2])$ *for every* $a>0$, *where* $0<c_1<c_2<+\infty$ *are independent of* $a$. *Assume the following compatibility conditions:* $u_0(0)=w(0)$ *and* $u_0(\ell_0)=0$.
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*Then, there exists a unique solution* $(u,\ell)$ *of the coupled problem such that* $(u,\ell) \in H^1(\Omega_T) \times C^{0,1}([0,T])$ *for every* $T > 0$. *Moreover, for every* $T > 0$ *one has* $u \in C^{0,1}(\overline{\Omega}_T)$ *and there exists* $L_T < 1$ *such that* $0 \le \dot{\ell}(t) \le L_T$ *for a.e.* $t \in (0,T)$.

# The case of a nonconstant $\kappa$

The previous result can be extended to a local toughness $\kappa$ depending on $x$.

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# Changes in the proof

- Set $z(t) := t - \ell(t)$. Then the following problems are equivalent

$$\begin{cases} \dot\ell(t) = \dfrac{2\dot f(t-\ell(t))^2 - \kappa(\ell(t))}{2\dot f(t-\ell(t))^2 + \kappa(\ell(t))} \vee 0, \\ \ell(0) = \ell_0. \end{cases} \qquad \begin{cases} \dot z(t) = \dfrac{2\kappa(t-z)}{2\dot f(z)^2 + \kappa(t-z)} \wedge 1, \\ z(0) = -\ell_0. \end{cases}$$

- Any solution of the second problem must satisfy $\dot z > 0$ a.e. and therefore $t \mapsto z(t)$ is invertible. The equation solved by the inverse function $t(z)$ is

$$\frac{dt}{dz} = \left(\frac{1}{2} + \frac{\dot f(z)^2}{\kappa(t-z)}\right) \vee 1 =: \Phi(z,t),$$

with initial condition $t(-\ell_0) = 0$.

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# Conclusion of the proof

- We can apply classical results of ordinary differential equations and obtain a unique solution $z \mapsto t(z)$.
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THANK YOU FOR YOUR ATTENTION!