On the selection of solutions to a nonlinear PDE system

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(joint work with G. Pisante)

Shape Optimization and Isoperimetric and Functional Inequalities

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Implicit PDE

$$\begin{cases} F_i(Du) = 0 & \text{in } \Omega, i = 1, \ldots, m \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

- Ω is an open bounded subset of $\mathbb{R}^n$
- $u : \Omega \to \mathbb{R}^N, N \geq 1$
- $F_i : \mathbb{R}^{N\times n} \to \mathbb{R}, i = 1, ..., m$ and $\varphi : \overline{\Omega} \to \mathbb{R}^N$ given
- $\{\xi : F_i(\xi) = 0\}$ compact

The quasilinear problems are excluded!

1. scalar case $(N = 1)$: viscosity solutions (Crandall-Lions), pyramidal construction (Cellina)
2. vectorial case $(N > 1)$: Baire category method (Dacorogna-Marcellini), Gromov integration (Müller-Sverak)

Examples: eikonal equation, potential wells, singular value problems...

Scalar case ($N = 1$)

VISCOSITY APPROACH:

1. $$\begin{cases} |Du| = 1 & \text{a.e. in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \qquad (EE)$$

If $|D\varphi| \le 1$ there exists a unique viscosity solution, given by

$$u(x) = \inf_{y \in \partial\Omega} \{\varphi(y) + |x - y|\}$$

2. $$\begin{cases} \left|\dfrac{\partial u}{\partial x_1}\right| = \left|\dfrac{\partial u}{\partial x_2}\right| = 1 & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (SEE)$$

If Ω is a rectangle whose sides are parallel to $x_2 = \pm x_1$, then $d_{l^1}(\cdot, \partial\Omega)$ is a viscosity solution.

PYRAMIDAL CONSTRUCTION:

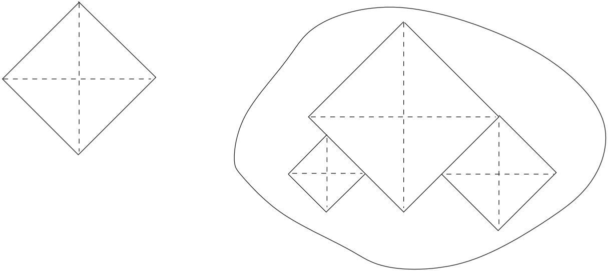
Theorem (Cellina)

Let $E = \{\xi : F_i(\xi) = 0\}$. If $D\varphi \in E \cup \operatorname{int} \operatorname{co} E$, then there exists a solution.

... but there exist infinitely many solutions! For example, for

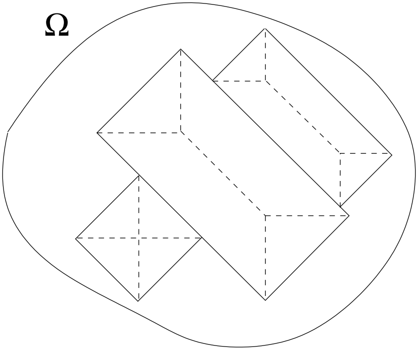
$$(SEE) \qquad \begin{cases} \left|\dfrac{\partial u}{\partial x_1}\right| = \left|\dfrac{\partial u}{\partial x_2}\right| = 1 & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

here is a solution:



Litterature

B. Dacorogna and P. Marcellini, 2004: they give an explicit covering of Ω, made up of rectangles R_i



putting on each one the viscosity solution.
What are the global properties of such a solution?

$$\mathcal{S} := \left\{ v \in W_0^{1,\infty}(\Omega) : \left| \frac{\partial v}{\partial x_1} \right| = \left| \frac{\partial v}{\partial x_2} \right| = 1 \ a.e. \right\}$$

Define a good functional $\mathcal{F}$ over $\mathcal{S}$

$$\Downarrow$$

The functions $v \in \mathcal{S}$ minimizing $\mathcal{F}$ are *selected solutions* of (SEE).

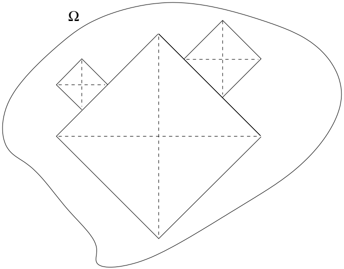
Warning: $\mathcal{S}$ is not convex!

EXAMPLE: $\mathcal{F}(v) = \int_\Omega |v|^p$, has neither a minimizer nor a maximizer over $\mathcal{S}$.

Let $v \in \mathcal{S}$. Then, for $i = 1, 2$

- $\dfrac{\partial v}{\partial x_i}$ is an $L^\infty(\Omega)$ function
- which takes just two values (a.e. in Ω): ± 1

Minimize the measure of the discontinuity set of $\dfrac{\partial v}{\partial x_i}, i = 1, 2$



$\mathcal{E} = \{v \in \mathcal{S} : Dv \in SBV_{loc}(\Omega)\}$

Can we minimize $\mathcal{H}^t\left(J_{\frac{\partial v}{\partial x_1}} \cup J_{\frac{\partial v}{\partial x_2}}\right)$ over $\mathcal{E}$, for $t \geq 1$?

1. $t = 1$

   Let $\Omega = (0,1)^2$ and $v \in \mathcal{E}$; define, for $t \in (0, 1/2)$

   $$\begin{array}{rccc} w_t: & [0,1] & \mapsto & \mathbb{R} \\ & s & \mapsto & v(s,t) \end{array}$$

   Since $|v(\cdot)| \leq dist_{l^1}(\cdot, \partial\Omega)$ then w_t' has at least $\left[\frac{1}{2t}\right]$ jumps.

   $$\mathcal{H}^1\left(J_{\frac{\partial v}{\partial x_1}}\right) \geq \int_0^{1/2} \mathcal{H}^0(J_{w_t'})\, dt \geq \int_0^{1/2} \left[\frac{1}{2t}\right] dt = +\infty$$

2. $t > 1$

   By the properties of the Hausdorff measures, since $v \in \mathcal{E}$, one has $\mathcal{H}^t\left(J_{\frac{\partial v}{\partial x_1}} \cup J_{\frac{\partial v}{\partial x_2}}\right) = 0$, for every $t > 1$.

IDEA: we measure $J_{\frac{\partial v}{\partial x_i}}$ using a weight $h \in C_0(\Omega)$. Assume that Ω is *admissible*, i.e.:

- Ω is a bounded C^1_{pw} domain
- there exists a finite number of points $\partial\Omega$ such that the normal ν satisfies $|\nu_1| = |\nu_2|$ and a finite number of segments parallel to $x_2 = \pm x_1$.

THEOREM 2 (G. Pisante and G.C.)

Let $\Omega \subset \mathbb{R}^2$ be a admissible domain. Then

$$\inf \left\{ \sum_{i=1}^{2} \int_{\Omega} d_1(x, \partial\Omega) \; d \left| D \frac{\partial v}{\partial x_i} \right| (x) \, , \quad v \in \mathcal{E}(\Omega) \right\}$$

has a solution.

N.B.: If $\partial\Omega$ is composed of a finite number of segments parallel to $x_2 = \pm x_1$, then one can consider $h \equiv 1$.

A vectorial problem:

$$\begin{cases} Du \in \mathcal{O}(2) & \Omega \\ u = 0 & \partial\Omega \end{cases} \qquad (OP)$$

N.B.: $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $|Du|^2 = 2; |det(Du)| = 1$

- Existence of solutions: Baire category approach, Gromov integration approach...infinitely many solutions!!! Pyramidal construction???
- In $\Omega = (-2,2)^2$, Dacorogna, Marcellini and Paolini constructed an explicit solution with a fractal behaviour at $\partial\Omega$ of

$$\begin{cases} Du \in E & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

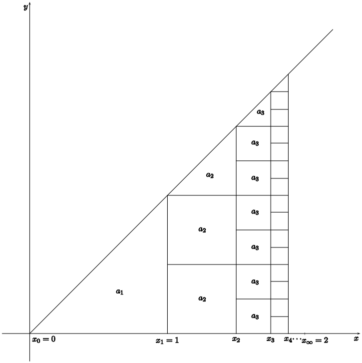
where $E \subset \mathcal{O}(2)$ is the set of matrices $e_i, i = 1\ldots, 8$

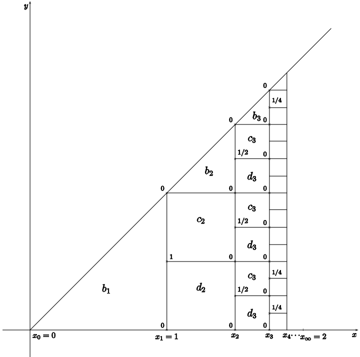
$$\left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

An explicit solution $u = (u^1, u^2)$ in $T = \{(x, y) \in \mathbb{R}^2 : x \geq y \geq 0\}$:

$$a(x,y) = \min\{1 \pm x, 1 \pm y\}, \qquad b(x,y) = \max\{1 - |x|, 1 - |y|\},$$

$$c(x,y) = \begin{cases} 1 - |x|, & \text{if } |x| \leq y \\ 1 - y, & \text{if } |y| \leq -x \\ 1 - x, & \text{if } |x| \leq -y \\ 1 - |y|, & \text{if } |y| \leq x \end{cases} \qquad d(x,y) = \begin{cases} 1 - x, & \text{if } |x| \leq y \\ 1 + y, & \text{if } |y| \leq -x \\ 1 - |x|, & \text{if } |x| \leq -y \\ 1 - |y|, & \text{if } |y| \leq x \end{cases}$$





We have to deal with possible fractalisations in Ω!!

- Γ_u: "good" points in the sense that $x_0 \in \Gamma_u$ if there exists a ball $B(x_0) \subset \Omega$ centered at x_0 such that $\{x : Du = e_i\} \cap B(x_0)$ is a Caccioppoli partition of $B(x_0)$, that is, the sum of the perimeters of $\{Du = e_i\}$ in $B(x_0)$ is finite.

- Σ^u_∞: "bad" points in the sense that $\Sigma^u_\infty = \partial\Omega \cup (\Omega \setminus \Gamma_u)$.

We will consider the set $\mathcal{S}$ of solutions u to (OP) such that

H1) $\Sigma^u_\infty \cap \Omega_\delta \cup \partial\Omega_\delta$ is connected.

H2) Σ^u_∞ is locally of finite $\mathcal{H}^1$ measure in Ω.

H3) For $c > 0$, there exists a positive function $\phi : (0, \infty) \to (0, \infty)$ such that for any $\delta > 0$ and for any u, $\mathcal{H}^1 - a.e.\ x \in \Sigma^u_\infty \cap \Omega_\delta$ we can find at least three indices $i_1,\ i_2,\ i_3 \in \{1, \dots, 8\}$ with the property that for every $r < \phi(\delta)$ we have

$$\mathcal{L}^2(B(x, r) \cap \{Du = e_{i_s}\}) > cr^2\,, \quad s \in \{1, 2, 3\}.$$

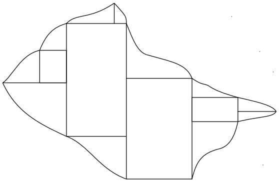
Example: construct a Vitali covering of Ω made up of squares in which you define the solution of Dacorogna, Marcellini and Paolini.

$$\mathcal{F}(u) = \int_\Omega dist(x,\partial\Omega)\chi_{\Sigma^u_\infty}\, d\mathcal{H}^1 + \sum_{i,j=1}^{2}\int_\Omega \left[dist(x,\Sigma^u_\infty)\right]^\alpha d|Du^j_{x_i}|$$

- 1st term: $\lim_{\delta\to 0}\int_{\Omega_\delta} dist(x,\partial\Omega)\chi_{\Sigma^u_\infty}\, d\mathcal{H}^1$
  Similar to the functional for (SEE)!!
- 2nd term: $\lim_{\delta\to 0}\lim_{h\to 0}\sum_{i,j=1}^{2}\int_{\Omega_\delta\setminus(\Sigma^u_\infty)_h}\left[dist(x,\Sigma^u_\infty)\right]^\alpha d|Du^j_{x_i}|$

N.B.: α depends on the geometry of Ω.

Ω : compatible domain



α-compatible triangular domains:

$$T_h := \left\{(s,t) \in \mathbb{R}^2 : a \le s \le b, h(b) \le t \le h(s)\right\}$$

where $h : [a,b] \to \mathbb{R}$ is a $C^1([a,b])$ function with $h'(t) < 0$ and $\alpha > 0$ satisfies

$$2\left[\max\left\{\frac{1}{1+\frac{1}{c_1}}, \frac{1}{1+c_2}\right\}\right]^{\alpha+1} < 1,$$

with $-c_1 = \min\limits_{x_1 \in [a,b]} h'$, $\quad -c_2 = -\max\limits_{x_1 \in [a,b]} h'$.

Theorem (G.C. and G. Pisante)

There exists $u \in \mathcal{S}$ which minimizes $\mathcal{F}$.

Thank you for your attention!