Bounds for the spectrum of the magnetic Laplacian

Bruno Colbois (joint work with Alessandro Savo)

Luminy, November 23, 2016

Let (M, g) be a complete Riemannian manifold and let Ω be a compact domain of M with smooth boundary $\partial\Omega$, if non empty (if M is closed, we can choose $\Omega = M$ and $\partial\Omega = \emptyset$).

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$$\nabla^A_X u = \nabla_X u - iA(X)u \tag{1}$$

for all vector fields X on Ω and for all $u \in C^\infty(\Omega, \mathbb{C})$.

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If ϕ is a function on Ω, Δ_A and $\Delta_{A+d\phi}$ are unitarily equivalent and will have the same spectrum. If $A = 0$, we recover the usual Laplacian.

Example

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We have $B = bdx^1 \wedge dx^2$. *The magnetic field is constant.*

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Let $\Omega \subset \mathbb{R}^2$ *be a domain with a hole, say around a point* $a = (a_1, a_2)$*. As potential, consider*

$$A_{a,\gamma}(x_1, x_2) = \gamma\left(\frac{-(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2} dx^1 + \frac{(x_1 - a_1)}{(x_1 - a_1)^2 + (x_2 - a_2)^2} dx^2\right.$$

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Then, $A_{a,\gamma}$ *is a closed 1-form, and* $dA_{a,\gamma} = B = 0$*.*

The circulation of $A_{a,\gamma}$ *is* γ*. For a simple closed curve* c *around* a*, the circulation of* A *is given by*

$$\Phi_c^A = \frac{1}{2\pi}\int_c A.$$

If the boundary of Ω is non empty, we will consider Neumann magnetic conditions, that is:

$$\nabla_N^A u = 0 \quad \text{on} \quad \partial\Omega, \tag{3}$$

where N denotes the inner unit normal. Then Δ_A is self-adjoint, and admits a discrete spectrum

$$0 \leq \lambda_1(\Delta_A) \leq \lambda_2(\Delta_A) \leq \ldots \to \infty.$$

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For a domain $\Omega \subset \mathbb{R}^2$ with a hole around a point $a = (a_1, a_2)$ and circulation $\frac{1}{2}$, what does occur if a approach the boundary? (Noris, Terracini, Bonnaillie, Felli, ...)

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Let $\Omega \subset \mathbb{R}^2$ be a region with smooth boundary, which is homeomorphic to a disk with k holes. They look at a potential A with $dA = 0$.

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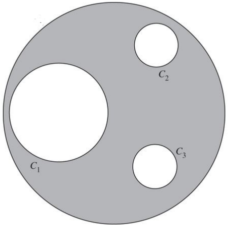
Let $\Omega \subset \mathbb{R}^2$ be a region with smooth boundary, which is homeomorphic to a disk with k holes. They look at a potential A with $dA = 0$. Let c_i a closed path which parametrise the boundary of the i^{th} hole, and $\Phi^A_{c_i} = \frac{1}{2\pi} \int_{c_i} A$.



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$\lambda_1(\Delta_A) = 0$ if and only if $\Phi^A_{c_i} \in \mathbb{Z}$ for all i. (See also Shikagawa for compact manifolds).

Moreover, when $k = 1$, $\lambda_1(\Delta_A, \Omega)$ is maximal when $\Phi^A_c = \frac{1}{2}$.

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A Cheeger type inequality (by Carsten Lange, Shiping Liu, Norbert Peyerimhoff, Olaf Post);

Lower bound thanks to Bochner methods (by Michela Egidi, Shiping Liu, Florentin Münch, Norbert Peyerimhoff).

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- highlighting the role of the circulation $\Phi_{c_i}^A = \frac{1}{2\pi}\int_{c_i} A$.

Remark

On the sphere S^2 *(with its canonical metric), there exists a family* A_k *of potentials, such that* $\lambda_1(\Delta_{A_k}) \to \infty$ *as* $k \to \infty$. *If* $B_k = dA_k$, *we have* $\|B_k\|_2 \to \infty$ *as* $k \to \infty$ *(Besson-C-Courtois).*

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The 1- forms $\delta\psi$ and h are L^2-orthogonal on Ω:

$$\int_\Omega \langle \delta\psi, h\rangle dvol_g = 0.$$

Let Ω be a domain in (M, g). We choose a family of closed curves $(c_1, ..., c_m)$, basis of the homology of degree 1 of Ω and we consider the dual basis of harmonic 1-forms $A_1, ..., A_m \in Har_1(\Omega)$: we have

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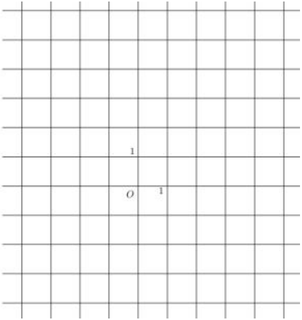
and we denote by $d(\Phi^A, \mathbb{Z}^m)$ the Euclidean distance between Φ^A and the Euclidean lattice:

$$d(\Phi^A, \mathbb{Z}^m)^2 = min\{\sum_{j=1}^{m} (\Phi_{c_j}^A - k_j)^2 : (k_1, ..., k_m) \in \mathbb{Z}^m\}.$$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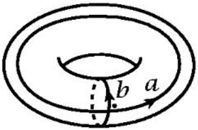
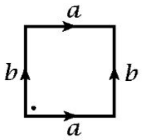
Example of a torus:





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If $A = \beta_1 A_1 + \beta_2 A_2$,

$$d(\Phi^A, \mathbb{Z}^2)^2 = min\{(\beta_1 - k_1)^2 + (\beta_2 - k_2)^2 : (k_1, k_2) \in \mathbb{Z}^2\}$$

We also introduce the lattice generated by the dual basis $(A_1, ..., A_m)$:

$$\mathcal{L}_{\mathbf{Z}} = \{k_1 A_1 + \cdots + k_m A_m : k_j \in \mathbf{Z}\}$$

which is an abelian subgroup of $Har_1(\Omega)$. Given $A \in Har_1(\Omega)$, we define its minimum distance to the lattice $\mathcal{L}_{\mathbf{Z}}$ by the formula:

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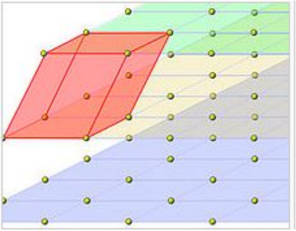

For the example of a torus:

If A is an harmonic form, $A = \beta_1 dx^1 + \beta_2 dx^2$,

$$d(A, \mathcal{L}_{\mathbf{Z}})^2 = \min \left\{ \|(k_1 \frac{2\pi}{\alpha_1} - \beta_1)^2 + (k_2 \frac{2\pi}{\alpha_2} - \beta_2)^2\|^2,\ (k_1, k_2) \in \mathbb{Z}^2 \right\}$$

Our results: Ω domain of (M, g).

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Recall that we write

$$A = \delta\psi + h$$

with $dh = \delta h = 0$ and $d(\delta\psi) = B$.

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We have

$$\lambda_1(\Delta_A) \leq \frac{d(A, \mathcal{L}_{\mathbf{Z}})^2}{|\Omega|} + \frac{\|B\|_2^2}{\lambda_{1,1}(\Omega)|\Omega|},$$

where $\lambda_{1,1}$ denotes the first nonzero eigenvalue of the Laplacian on co-exact 1-forms and $|\Omega|$ denotes the volume of Ω.

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In particular, if $B = 0$, we get

$$\lambda_1(\Delta_A) \leq \frac{d(A, \mathcal{L}_{\mathbf{Z}})^2}{|\Omega|},$$

and this inequality is sharp (equality in the case of a flat rectangular torus).

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The term $\lambda_{1,1}$ reflects the presence of the geometry. In his estimates for surfaces, L. Erdös has a term depending on the curvature and injectivity radius of the surface. However, in dimension 2, $\lambda_{1,1}$ is equal to the first eigenvalue of the Laplacian on functions, and this is no longer the case in higher dimensions.

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with

$$C_1(\Omega, A) \leq \frac{c_1(n)}{|\Omega|}\left(d(A, \mathcal{L}_{\mathbf{Z}})^2 + \frac{\|B\|^2}{\lambda_{1,1}(\Omega)}\right).$$

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3. The estimate is compatible with the Weyl law.

Lower bounds for λ_1.

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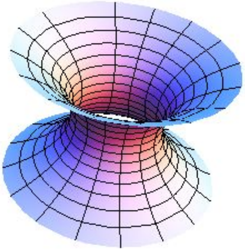
A *Riemannian cylinder* is a domain (Ω, g) diffeomorphic to $[0,1] \times \mathbb{S}^1$, endowed with a Riemannian metric g. We denote by Σ_1 and Σ_2 the boundaries of the cylinder.



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It is clear that, in the definition of the constant K, we can assume that the range of ψ is the interval $[0, 1]$, and that $\psi = 0$ on Σ_1 and $\psi = 1$ on Σ_2.

Theorem

Let (Ω, g) be a Riemannian cylinder, and let A be a closed 1-form on Ω. Assume that Ω is K-foliated by the level curves of the smooth function $\psi \in \mathcal{F}_\Omega$. Then:

$$\lambda_1(\Omega, A) \geq \frac{4\pi^2}{KL^2} \cdot d(\Phi^A, \mathbf{Z})^2,$$

where L is the maximum length of a level curve of ψ and Φ^A is the flux of A across any of the boundary components of Ω.

Theorem

Let (Ω, g) be a Riemannian cylinder, and let A be a closed 1-form on Ω. Assume that Ω is K-foliated by the level curves of the smooth function $\psi \in \mathcal{F}_\Omega$. Then:

$$\lambda_1(\Omega, A) \geq \frac{4\pi^2}{KL^2} \cdot d(\Phi^A, \mathbf{Z})^2,$$

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Note that $K \geq 1$; we will see that in many interesting situations (for example, for revolution cylinders) one has in fact $K = 1$. However, in full generallity, it is difficult to estimate K.

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$$\lambda_1(\Omega, A) \geq \frac{4\pi^2\beta^2}{B^2L^2} d(\Phi^A, \mathbb{Z})^2$$

where β denotes the minimum of the distance between Σ_1 and Σ_2, B the maximum of the distance between Σ_1 and Σ_2 and L the length of the outer boundary.

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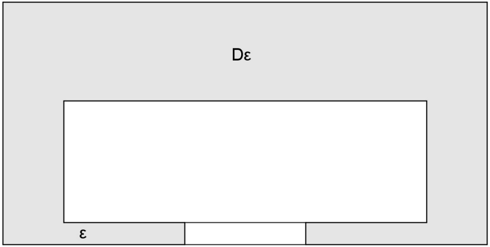

Figure : $\lambda_1 \to 0$ as $\epsilon \to 0$

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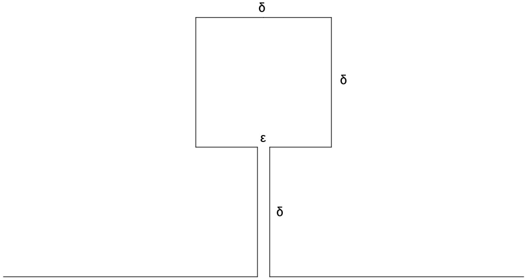

Figure : A local deformation implying $\lambda_1 \to 0$

Sketch of the proofs.

Upper bounds for λ_1:

Case 1: if $A = h$ with $dh = \delta h = 0$ and, moreover, $\frac{1}{2\pi}\int_c h \in \mathbb{Z}$ for all closed curved c, we have

$$A = h = n_1 A_1 + ... + n_k A_k$$

where $(A_1, ..., A_k)$ is the dual basis of harmonic forms and $n_1, ..., n_k \in \mathbb{Z}$.

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Then, $\phi(x)$ does depend on the path between x_0 and x only up to a factor 2π, and

$$u(x) = e^{i\phi(x)}$$

is an eigenfunction.

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Recall that

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In general, $h \notin \mathcal{L}_{\mathbf{Z}}$. We choose $\omega \in \mathcal{L}_{\mathbf{Z}}$ minimizing $d(A, \mathcal{L}_{\mathbf{Z}})$, and consider the same function as in case 1:

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$$\lambda_1(\Delta_A) \leq \frac{\|w - h\|^2 + \|\delta\psi\|^2}{|\Omega|}.$$

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This implies

$$\lambda_1(\Delta_A) \leq \frac{d(A, \mathcal{L}_{\mathbf{Z}})^2}{|\Omega|} + \frac{\|B\|^2}{\lambda_{1,1}(\Omega)|\Omega|}.$$

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We use the same strategy as in the second step. We choose $\omega \in \mathcal{L}_{\mathbf{Z}}$ minimizing $d(A, \mathcal{L}_{\mathbf{Z}})$, and consider the function

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The test functions will be of the type fu where f is a real smooth function on Ω.

We have

$$
\begin{aligned}
(d - iA)(fu) &= udf + fdu - ihuf - iuf\,\delta\psi \\
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We have to control the Rayleigh quotient

$$R(fu) \leq 2\left(\frac{\int_\Omega |df|^2}{\int_\Omega f^2} + \frac{\int_\Omega f^2|\omega - h - \delta\psi|^2}{\int_\Omega f^2}\right).$$

So, we are lead to control the Rayleigh quotient

$$R(f) = 2\frac{\int_\Omega |df|^2 + Vf^2}{\int_\Omega f^2}, \quad \text{where } V = |\omega - h - \delta\psi|^2.$$

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Thus, the problem is now to find an upper bound for the spectrum of the operator $\Delta + V$, where Δ is the usual Laplacian acting on functions and $V = |\omega - h - \delta\psi|^2$ is a nonnegative potential.

The proof follows word for word what is done in the case where the potential is equal to 0.

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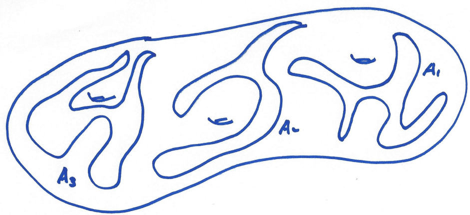
The idea is to construct disjointly supported domains $\Omega_1, ..., \Omega_k$ on Ω and to associate a test function for the Rayleigh quotient to each of these domains.



The fact that the potential V is positif implies that we can choose the domains Ω_i with

$$\int_{\Omega_i} V dvol_g \leq c \frac{\int_\Omega V}{k}.$$

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We use the foliation as follow: we restrict the potential A on the cylinder to each circle of the foliation, and this allows us to estimate the spectrum of the Riemannian cylinder in comparison with the spectrum of circles.

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The presence of K reflects the complexity of the geometry of the cylinder.

Thank you!