Some isoperimetric inequalities on $\mathbb{R}^N$ with respect to weights $|x|^\alpha$

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joint work with:

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Motivation:

L. Caffarelli, R. Kohn, L. Nirenberg (1984):

$\exists C > 0$, such that $\forall u \in C_0^\infty(\mathbb{R}^N)$,

$$C \left\| |x|^b u \right\|_{L^q} \leq \left\| |x|^a |\nabla u| \right\|_{L^p}^{\lambda} \cdot \left\| |x|^c u \right\|_{L^r}^{1-\lambda},$$

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where:

$$p, q, r \geq 1, \quad 0 < \lambda \leq 1,$$

$$\frac{1}{p} + \frac{a}{N}, \ \frac{1}{q} + \frac{b}{N}, \ \frac{1}{r} + \frac{c}{N} > 0,$$

$$\frac{1}{q} + \frac{b}{N} = \lambda \left(\frac{1}{p} + \frac{a-1}{N} \right) + (1-\lambda) \left(\frac{1}{r} + \frac{c}{N} \right),$$

$+$ some further conditions.

What is the best constant C in the inequality?

We study the case $\lambda = 1$ here:

$$C \left\| |x|^b u \right\|_{L^q} \leq \left\| |x|^a |\nabla u| \right\|_{L^p}$$

where: $p, q \geq 1,$

$$\frac{1}{p} + \frac{a-1}{N} = \frac{1}{q} + \frac{b}{N} > 0$$

and $0 \leq a - b \leq 1.$

For $p = 1$:

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This is equivalent to the following

Weighted isoperimetric inequality:

$$C\left(\int_{\Omega} |x|^{bq} \, dx\right)^{1/q} \leq \int_{\partial\Omega} |x|^a \, \mathscr{H}_{N-1}(dx)$$

for smooth sets $\Omega \subset \mathbb{R}^N$.

Isoperimetric problem:

Let $N \in \mathbb{N}$, $k + N - 1 > 0$ and $l + N > 0$.
Define

$$\mathcal{R}(\Omega) := \frac{\displaystyle\int_{\partial\Omega} |x|^k \, \mathscr{H}_{N-1}(dx)}{\left(\displaystyle\int_{\Omega} |x|^l \, dx\right)^{\frac{k+N-1}{l+N}}}, \quad (\Omega \text{ smooth}).$$

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Note: $\mathcal{R}(\Omega) = \mathcal{R}(t\Omega) \quad \forall t > 0$.

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Note: $\mathcal{R}(\Omega) = \mathcal{R}(t\Omega) \quad \forall t > 0$.

Find $C := \inf \left\{ \mathcal{R}(\Omega) : \ \Omega \subset \mathbb{R}^N, \ \Omega \neq \emptyset, \ \text{smooth} \right\}$.

A necessary condition for $C > 0$:

If $C > 0$ then $l\,\frac{N-1}{N} \leq k$.

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A necessary condition for radiality:

Define: $C^{rad} := \mathcal{R}(B_1)$.

$$\text{If} \quad l > k - 1 + \frac{N-1}{k+N-1} =: l^* \quad \text{then} \quad C^{rad} > C.$$

Weighted isoperimetric inequality:

We have $C = C^{rad}$ in each one of the following cases:

(i) $N \geq 1$ and $l + 1 \leq k$. S. Howe (2015).
See also
M.F. Betta, B., A. Mercaldo, M.R. Posteraro (1999),
for the case $l = 0$, $k \geq 1$.

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(ii) $N \geq 2, \quad k \leq l + 1$ and $l \dfrac{N-1}{N} \leq k \leq 0$;
N. Chiba, T. Horiuchi (2015).
See also
A. Diaz, N. Harman, S. Howe, D. Thompson (2012),
for the case $N = 2$.

(iii) $N \geq 2, \quad 0 \leq k \leq l+1$ and

$$l \leq k - 1 + \frac{N-1}{k+N-1} \quad (= l^*);$$

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$N \geq 3, \quad 0 \leq k \leq l+1$ and

$$l \leq \frac{(k+N-1)^3}{(k+N-1)^2 - (N-1)^2/N} - N =: l_1 \ (< l^*).$$

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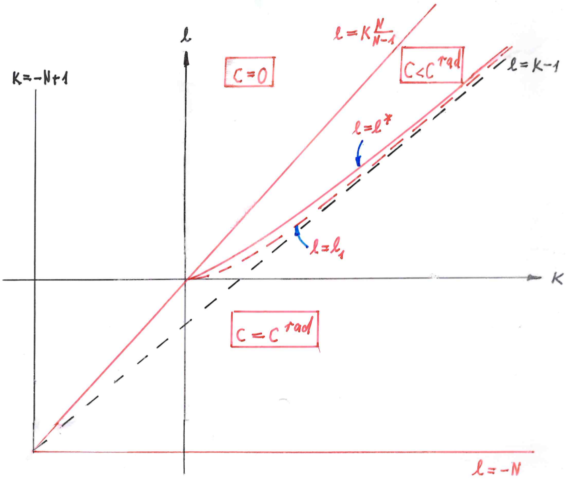
Note also: $l^*(0,N) = l_1(0,N) = 0$ and
$l^*(k,N) - l_1(k,N) = O\left(\frac{1}{k}\right)$ as $k \to +\infty$.

Weighted isoperimetric inequality: values of K and ℓ



CKN inequalities:

Let $p \geq 1, \quad q \geq 1, \quad a, b \in \mathbb{R},$ and such that $a > 1 - \frac{N}{p}$ and $0 \leq a - b \leq 1,$ where

$$b = b(a, p, q, N) := N\left(\frac{1}{p} - \frac{1}{q}\right) + a - 1.$$

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Define

$$E \quad := \quad \frac{\displaystyle\int_{\mathbb{R}^N} |x|^{ap} |\nabla v|^p \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |x|^{bq} |v|^q \, dx\right)^{p/q}}, \qquad v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\},$$

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$$S := \inf\{E(v) : \, v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}\}, \quad \text{and}$$
$$S^{rad} := \inf\{E(v) : \, v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}, \, v \text{ radial}\}.$$

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If $p = 1$, $a = k$ and $q = \dfrac{l+N}{k+N-1}$ then

$$S = C \qquad \text{and} \qquad S^{rad} = C^{rad}.$$

The case $p = q$, Hardy–Sobolev inequality:

Let $1 < p = q$ ($\Leftrightarrow a - b = 1$). Then

$$S = S^{rad} = \left(\frac{N}{p} - 1 + a\right)^p; \qquad \text{G. Hardy (1919).}$$

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The case $q = p^*$, Sobolev inequality:

Let $N > p$, $q = p^* := \dfrac{Np}{N - p}$ and $a = b = 0$. Then

$S = S^{rad}$; G. Talenti (1976).

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Define $a^* \in \left(1 - \dfrac{N}{p}, +\infty\right)$ by

$$\left(\frac{N}{p} - 1 + a^*\right)^2 = (N-1)\left(\frac{1}{q-p} - \frac{1}{q+p'}\right),$$

where $p' = \dfrac{p}{p-1}$.

Symmetry breaking:

If $a > a^*$ then $S < S^{rad}$;

P. Caldiroli, R. Musina (2013).

Symmetry:

If $1 - \dfrac{N}{p} < a \leq a^*$ then $S = S^{rad}$; ABCMP.

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See also:

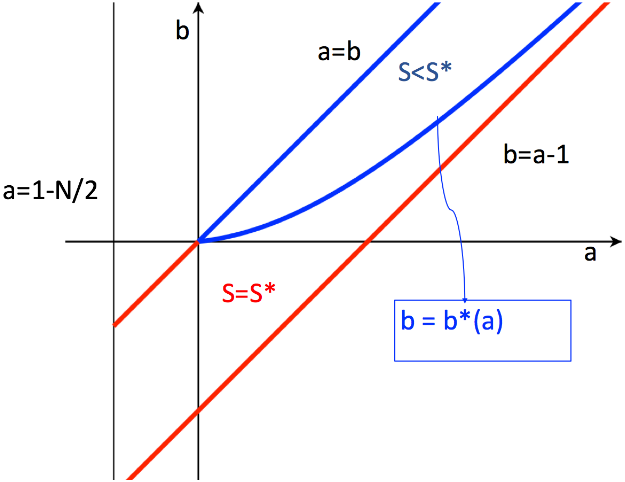
F. Catrina, Z. Wang (2001),

V. Felli, M. Schneider (2003) and

J. Dolbeault, M. Esteban, M. Loss (2015), arXiv, for $p = 2$.

T. Horiuchi, P. Kumlin (2012) for $a \leq 0$.

CKN inequalities, $N > 2 = p$: values of a and b



Lorentz spaces

If $u : \mathbb{R}^N \to \mathbb{R}$ is measurable and $u^\star$ is its Schwarz symmetrization, then the decreasing rearrangement of u is given by

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For every $r \in (0, \infty)$ define

$$\begin{aligned} \|u\|_{r,q} &:= \left(\int_0^{+\infty} \left[u^*(s)\, s^{1/r} \right]^q \frac{ds}{s} \right)^{1/q} \quad \text{if } q \in (0,\infty), \text{ and} \\ \|u\|_{r,\infty} &:= \sup_{s>0} u^\star(s)\, s^{1/r} \quad \text{if } q = +\infty. \end{aligned}$$

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The Lorentz space $L^{r,q}(\mathbb{R}^N)$ is the collection of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $\|u\|_{r,q}$ is finite.

Best constant in an imbedding inequality:

Let $0 \le a \le 1 + \dfrac{N}{q} - \dfrac{N}{p}$ and $r := \dfrac{Np}{N - p + ap}$. Then

$$\left(\int_{\mathbb{R}^N} |x|^{ap} |\nabla u|^p \, dx\right)^{1/p} \geq (\omega_N)^{-b/N} \left(S^{rad}\right)^{1/p} \|u\|_{r,q}$$

$$\forall u \in C_0^\infty(\mathbb{R}^N);$$ ABCMP.

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$$\forall u \in C_0^\infty(\mathbb{R}^N); \quad \text{ABCMP.}$$

For the case $a = 0$ see: A. Alvino (1977).

Some tools in the proofs:

1. μ_l– symmetrization:

(a) Define a measure μ_l by $d\mu_l(x) = |x|^l \, dx$.
If $M \subset \mathbb{R}^N$ is measurable with $\mu_l(M) < \infty$,
then let $M^\star$ be the ball B_R such that $\mu_l(B_R) = \mu_l(M)$.

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(b) If $u : \mathbb{R}^N \to \mathbb{R}$ is a measurable function such that

$$\mu_l\, \{|u| > t\} < \infty \quad \forall t > 0,$$

then let $u^\star$ be the nonnegative, radially symmetric and radially decreasing function satisfying

$$\mu_l\{u^\star > t\} = \mu_l\{|u| > t\} \quad \forall t > 0.$$

Then:

$$\int_{\mathbb{R}^N} |u|^m |x|^l \, dx = \int_{\mathbb{R}^N} |u^\star|^m |x|^l \, dx \quad \forall u \in C_0^\infty(\mathbb{R}^N), \ \forall m > 0,$$

and

$C = C^{rad} \iff \mathcal{R}(\Omega) \geq \mathcal{R}(\Omega^\star)$

for all nonempty smooth sets Ω.

2. Another description of C:

Define

$$\mathcal{Q}(v) := \frac{\displaystyle\int_{\mathbb{R}^N} |x|^k |\nabla v| \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |x|^l |v|^{\frac{l+N}{k+N-1}} \, dx \right)^{\frac{k+N-1}{l+N}}}, \quad \forall v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}.$$

Then

$$C = \inf \left\{ \mathcal{Q}(v) : \, v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\} \right\}.$$

3. The case $k \leq 0$; Horiuchi's trick:

Let $u \in C_0^\infty(\mathbb{R}^N)$,

$$J := \int_{\mathbb{R}^N} |x|^k |\nabla u| \, dx,$$

and define y, z and v by:

$y := x|x|^{\frac{k}{N-1}}$, $z := |y|$ and $v(y) := u(x), \quad (x \in \mathbb{R}^N)$.

Using N–dimensional spherical coordinates we evaluate

$$\begin{aligned} J &= \int_{\mathscr{S}^{N-1}} \int_0^{+\infty} r^{k+N-1} \sqrt{u_r^2 + \frac{|\nabla_\theta u|^2}{r^2}}\, dr\, d\theta \\ &= \int_{\mathscr{S}^{N-1}} \int_0^{+\infty} z^{N-1} \sqrt{v_z^2 + \frac{|\nabla_\theta v|^2}{z^2} \frac{(N-1)^2}{(k+N-1)^2}}\, dz\, d\theta \\ &\geq \int_{\mathscr{S}^{N-1}} \int_0^{+\infty} z^{N-1} \sqrt{v_z^2 + \frac{|\nabla_\theta v|^2}{z^2}}\, dz\, d\theta \\ &= \int_{\mathbb{R}^N} |\nabla_y v|\, dy. \end{aligned}$$

4. The case $k > 0$; a crucial inequality:

Let x, y, u, v, J be as in no. **3.**
The mapping

$$t \longmapsto \log \left(\int_{\mathscr{S}^{N-1}} \int_0^{+\infty} z^{N-1} \sqrt{v_z^2 + t\frac{|\nabla_\theta v|^2}{z^2}}\, dz\, d\theta \right)$$

is concave.

Hence we obtain for every $A \in \left[0, \frac{(N-1)^2}{(k+N-1)^2}\right]$,

$$\begin{aligned} J &= \int_{\mathscr{S}^{N-1}} \int_0^{+\infty} z^{N-1} \sqrt{v_z^2 + \frac{|\nabla_\theta v|^2}{z^2} \frac{(N-1)^2}{(k+N-1)^2}} \, dz \, d\theta \\ &\geq \left(\int_{\mathscr{S}^{N-1}} \int_0^{+\infty} z^{N-1} \sqrt{v_z^2 + \frac{|\nabla_\theta v|^2}{z^2}} \, dz \, d\theta \right)^A \cdot \\ &\quad \cdot \left(\int_{\mathscr{S}^{N-1}} \int_0^{+\infty} z^{N-1} |v_z| \, dz \, d\theta \right)^{1-A} \\ &= \left(\int_{\mathbb{R}^N} |\nabla_y v| \, dy \right)^A \cdot \left(\int_{\mathbb{R}^N} |v_z| \, dy \right)^{1-A}. \end{aligned}$$

5. Pólya-Szegö principle:

Assume that $C = C^{rad}$, and let $p \in [1, +\infty)$ and $u \in C_0^\infty(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N} |\nabla u|^p |x|^{pk+(1-p)l}\, dx \geq \int_{\mathbb{R}^N} |\nabla u^*|^p |x|^{pk+(1-p)l}\, dx,$$

(u^* = μ_l–symmetrization of u);
ABCMP.

Sketch of the proof in the special case

$p = 2, \quad l = 0, \quad 0 \le k \le 1$:

$$
\begin{aligned}
I &:= \int_{\mathbb{R}^N} |\nabla u|^2 |x|^{2k}\, dx, \quad u \in C_0^\infty(\mathbb{R}^N),\ u \ge 0, \\
I^* &:= \text{same for } u^\star.
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By the co-area formula,

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By the Cauchy-Schwarz inequality,

$$\left(\int_{u=t} |x|^k \right)^2 \leq \left(\int_{u=t} |\nabla u| |x|^{2k} \right) \cdot \left(\int_{u=t} |\nabla u|^{-1} \right),$$

where equality holds if u is replaced by $u^\star$.

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where equality holds if u is replaced by $u^\star$.

Hence,

$$\begin{aligned} I &\geq \int_0^\infty \left(\int_{u=t} |x|^k\right)^2 \cdot \left(\int_{u=t} |\nabla u|^{-1}\right)^{-1} dt, \\ I^* &= \int_0^\infty \left(\int_{u^\star=t} |x|^k\right)^2 \cdot \left(\int_{u^\star=t} |\nabla u^\star|^{-1}\right)^{-1} dt. \end{aligned}$$

Since $\displaystyle\int_{u>t} dx = \int_{u^\star>t} dx$,

Fleming-Rishel's formula gives

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Finally, the weighted isoperimetric inequality tells us that

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$\Longrightarrow I \geq I^*$. □

Thank you !