Wulff shape characterizations in overdetermined anisotropic elliptic problems

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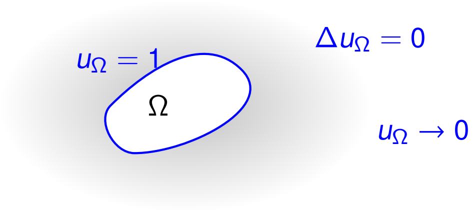
joint work with Giulio Ciraolo and Paolo Salani

Capacitary problem

$\Omega \rightsquigarrow$ conductor
$u_\Omega \rightsquigarrow$ electrostatic potential
$|Du_\Omega| \rightsquigarrow$ intensity of the field;



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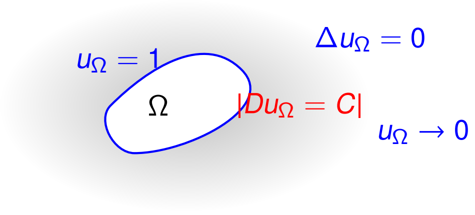

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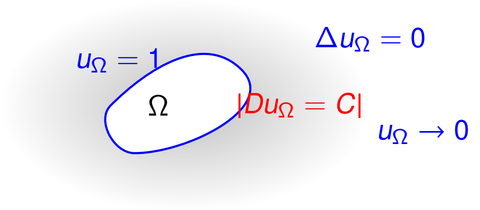

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where $\mathcal{A} = \{v \in C_0^\infty(\mathbb{R}^N),\ v \geq 1 \text{ in } \Omega\}$.

$Cap(\Omega)$ measures the capacitance of Ω: the total charge Ω can hold while maintaining a given potential energy,with respect to an idealized ground at infinity.

$\rightsquigarrow$ the Laplace operator reflects the the linearity of the electrical conductivity law, determined by the isotropy of the dielectric.

$\rightsquigarrow$ what happens in an anisotropic dielectic or for potential-type conduction laws?
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Newtonian p-Capacity:

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Anisotropic spaces (i)

Euclidean balls: $B = \{|x| < t\}$ $\rightsquigarrow$ anisotropic balls: $B_H = \{H(x) < t\}$

NOT anisotropic balls:

Let $H : \mathbb{R}^N \to \mathbb{R}$ be a regular norm in $\mathbb{R}^N_*$:

- **(i)** H is convex;
- **(ii)** $H(\xi) \geq 0$ for $\xi \in \mathbb{R}^N$ and $H(\xi) = 0$ if and only if $\xi = 0$;
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NOTICE: B_{H_0} is still uniformly convex.

Let $H : \mathbb{R}^N \to \mathbb{R}$ be a regular norm in $\mathbb{R}^N_*$ its **dual norm** is

$H_0(x) = \sup_{\xi \neq 0} \dfrac{x \cdot \xi}{H(\xi)}$, for $x \in \mathbb{R}^N$.

Notice: $x \in \Omega \subseteq \mathbb{R}^N \rightsquigarrow H_0$ norm of $\mathbb{R}^N$.

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Wulff shapes:

Def. Ω is Wulff shape of H if $\Omega = rB_{H_0}$ for some $r > 0$.

that is Ω is an anisotropic ball for the dual norm H_0.

Main results: looking at the anisotropic capacity problem
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References on anisotropic results

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- CB, L. Brasco, G. Ciraolo: preprint ⇝ anisotropic p-capacity, p-Serrin interior problem.

Ingredients

Newton's inequality: If $W = BC$, with $B, C \in \mathbb{R}^{N\times N}$ symmetric, B positive definite, then

$$S_2(W) \le \frac{N-1}{2N}\, tr(W)^2.$$

Moreover, if $tr(W) \neq 0$ and equality holds, then $W = \gamma\, I$,.

[A. Cianchi, P. Salani, Math. Ann. 2009]

where $S_2(A)$ is the elementary symmetric function of A of order 2.

Anisotropic Aleksandrov Theorem: If $\mathrm{M}_H(\Omega)$ is constant then Ω is Wulff shape.

[He, Li, Ma, Ge, Indiana Univ, 2009]

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Wulff shape characterizations

Theorem. Let H be a regular strictly convex norm of $\mathbb{R}^N_*$; assume Ω to be convex and let u be a solution to:

$$(Pb): \begin{cases} \Delta_p^H u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega, \\ u \to 0 & \text{if } |x| \to \infty, \end{cases}$$

for $p < N$. The following are equivalent:
$H(Du) = C$ on $\partial\Omega$, $\Leftrightarrow$ equality holds in Newton's inequality for $\nabla_H^2 v$
$\Leftrightarrow$ Ω is Wulff shape of H.

Idea of the proof:

$\rightsquigarrow$*same spirit of:*
[Brandolini, Nitsch, Salani, Trombetti, 08] for classical interior Serrin pb;
[Cianchi Salani, 09] for interior anisotropic Serrin pb ;
[Colesanti, Reichel, Salani, in progress] for classical exterior Serrin pb.

AIM: $H(Du) = C$ on $\partial\Omega$, $\Rightarrow$ equality holds in Newton $\Rightarrow$ Ω Wulff shape.

- auxiliary pb for $v = u^{\frac{p}{p-N}}$;
- $W = \nabla^2 V_p D^2 v$, anisotropic Hessian where $V_p(\xi) = \frac{1}{p} H^p(\xi)$ so that $\Delta_H v = tr(W)$;
- via some integral inequalities $\rightsquigarrow$ the equality sign in the generalized Newton's ineq: $S_2(W) \leq \frac{N-1}{2N}\, tr(W)^2$ holds
- hence $D^2 v = \lambda(\nabla^2_\xi V(Dv))^{-1} \rightsquigarrow M_H(\Omega)$ is constant

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- hence $D^2 v = \lambda(\nabla^2_\xi V(Dv))^{-1}$ ⇝ $M_H(\Omega)$ is constant

Idea of the proof:

$\rightsquigarrow$ *same spirit of:*
[Brandolini, Nitsch, Salani, Trombetti, 08] for classical interior Serrin pb;
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(1) **Original problem:** solution u

$$\begin{cases} \Delta_p^H u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega, \\ H(Du) = C & \text{on } \partial\Omega, \\ u \to 0 & \text{if } |x| \to \infty. \end{cases}$$

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let $V(\xi) = \frac{1}{p} H^p(\xi)$. $W = \nabla_\xi^2 V(Dv)\, D^2 v \rightsquigarrow \Delta_p^H v = Tr(W)$.

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Step II: Newton's Inequality for W: $2\, S_2(W) \leq \frac{N-1}{N}(Tr(W))^2$, where

$$2S_2(W) = S^2_{ij}(W) V_{\xi_i\xi_k} v_{kj} \quad \text{and} \quad S^2_{ij} W = \begin{cases} -V_{\xi_j\xi_k} v_{ki} & \text{if } i \neq j \\ -V_{\xi_j\xi_k} v_{ki} + \Delta^H_p v & \text{if } i = j. \end{cases}$$

recall: $tr(W) = \Delta^H_p v \rightsquigarrow$ by Newton ineq:

$$0 \geq 2v^\gamma S_2(W) - \frac{N-1}{N} v^\gamma (\Delta^H_p v)^2 = div(*) - v^{\gamma-2} H^{2p}(Dv)(P(\gamma)).$$

$\rightsquigarrow$ for $\gamma = 1 - N$ we have $P(\gamma) = 0$ and hence $div(*) \leq 0$ that is

$$div(v^{1-N} S^2_{ij}(W) V_{\xi_i} - (N-1)(p-1) v^{-N} V(Dv) \nabla_\xi V(Dv)) \leq 0,$$

$\forall x \in \mathbb{R}^N \setminus \Omega.$

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Step III: By the Divergence Theorem, the fact that $\nu = \frac{Dv}{|Dv|}$ on $\partial\Omega$, the definition of $S^2_{ij}(W)$, the homogeneity of $H(\xi)$ and the fact that

$$\Delta_p^H v = (p-1)H^{p-2}H_{\xi_k}H_{\xi_i}v_{ki} + H^{p-1}\mathrm{M}_H,$$

it holds $\int_{\partial\Omega} H(\nu)H^{2(p-1)}(Dv)\left(\frac{\mathrm{M}_H(\partial\Omega)}{N-1} - \frac{p-1}{p}\frac{H(Dv)}{v}\right) d\sigma \geq 0.$

$\rightsquigarrow$NOTICE: with $H(Du) = C$ on $\partial\Omega$ and Step I, we have

$$\int_{\partial\Omega} H(\nu)\frac{\mathrm{M}_H(\partial\Omega)}{N-1} \geq \frac{P_H^2(\Omega)}{N|\Omega|}.$$

This is the reverse of Minkowski inequality $\rightsquigarrow$ equality holds.

Step IV: $\rightsquigarrow$equality holds in Newton's inequality for $W = \nabla^2_\xi V(Dv(x))\, D^2 v(x)$ that is $W = \gamma(x)I$.

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Step VI: by the definition of the anisotropic mean curvature we obtain:

$$\mathrm{M}_H(\partial\Omega) = H_{\xi_i\xi_j}(Dv)v_{ij} = \lambda H_{\xi_i\xi_j}[(\nabla^2_\xi V)^{-1}]_{ij} = \tilde{c}H^{2-p}(Dv)$$

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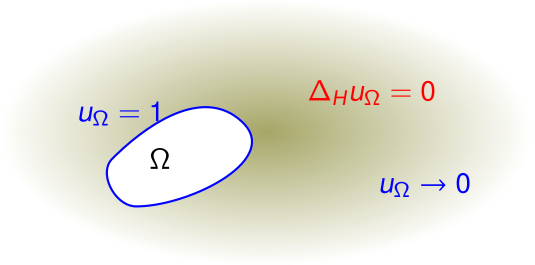
Anisotropic overdetermined capacitary problem:

Ω ⇝ conductor

u_Ω ⇝ electrostatic potential

$H(Du_\Omega)$ ⇝ norm of the field;



⇝ *An electrical conductor Ω while embedded in an anisotropic dielectric and maintaining a given potential energy has constant intensity of the electrostatic field on its boundary if and only if it is an anisotropic ball.. The same holds in the case of power-type conductivity laws.*

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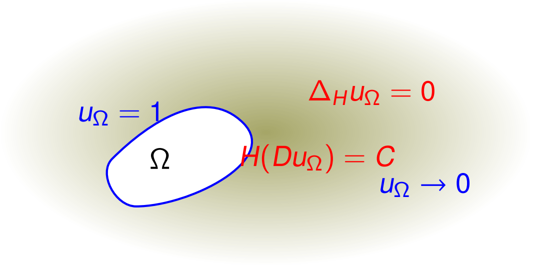

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Further investigations

This technique has been adapted to other situations:

- in the case of other integral constraints
  (involving geometric quantities as M_H, the capacity...);
- for Serrin interior problem for the Finsler p-Laplacian.

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