The Brezis-Nirenberg Problem for the Laplacian with a singular drift in $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$.

Rafael D. Benguria
Instituto de Física, PUC
Santiago, Chile

"Shape Optimization and Isoperimetric and Functional Inequalities".

CIRM
Luminy, Marseille, November 23, 2016

This is joint work with

Soledad Benguria
Department of Mathematics
University of Wisconsin
Madison, WI, USA

Motivation (The Lane–Emden equation):

The equation

$$-\Delta u = u^p \tag{1}$$

for $u > 0$ in a ball of radius R in $\mathbb{R}^3$, with Dirichlet boundary conditions, is called, in physics, the Lane–Emden equation of index p. It was introduced in 1869 by Homer Lane, who was interested in computing both the temperature and the density of mass on the surface of the Sun. Unfortunately Stefan's law was unknown at the time (Stefan published his law in 1879). Instead, Lane used some experimental results of Dulong and Petit and Hopkins on the rate of emission of radiant energy by a heated surface, and he got the value of 30,000 degrees Kelvin for the temperature of the Sun, which is too big by a factor of 5. Then he used his value of the temperature together with the solution of (1) with $p = 3/2$, to estimate the density u near the surface.

Motivation (The Lane–Emden equation):

After the Lane–Emden equation was introduced, it was soon realized that it only had bounded solutions vanishing at R if the exponent is below 5. In fact, for $1 \leq p < 5$ there are bounded solutions, which are decreasing with the distance from the center. In 1883, Sir Arthur Schuster constructed a bounded solution of the Lane–Emden equation in the whole $\mathbb{R}^3$ vanishing at infinity. This equation on the whole $\mathbb{R}^3$, with exponent $p = 5$ plays a major role in mathematics. It is the Euler–Lagrange equation equation that one obtains when minimizing the quotient

$$\frac{\int_{\mathbb{R}^3} (\nabla u)^2 \, dx}{\left(\int_{\mathbb{R}^3} u^6 \, dx\right)^{1/3}}. \tag{1}$$

This quotient is minimized if $u(x) = 1/(|x|^2 + m^2)^{1/2}$. The minimizer is unique modulo multiplications by a constant, and translations. This function $u(x)$, is precisely the function determined by A. Schuster, up to a multiplicative constant. Inserting this function u back in (1), gives the classical Sobolev inequality (S. Sobolev 1938),

$$\frac{\int_{\mathbb{R}^3} (\nabla u)^2 \, dx}{\left(\int_{\mathbb{R}^3} u^6 \, dx\right)^{1/3}} \geq 3\left(\frac{\pi}{2}\right)^{4/3}, \tag{2}$$

for all functions in $\mathcal{D}^1(\mathbb{R}^3)$.

The Brezis–Nirenberg problem on $\mathbb{R}^N$

In 1983 Brezis and Nirenberg considered the nonlinear eigenvalue problem,

$$-\Delta u = \lambda u + |u|^{4/(n-2)} u,$$

with $u \in H_0^1(\Omega)$, where Ω is bounded smooth domain in $\mathbb{R}^n$, with $n \geq 3$. Among other results, they proved that if $n \geq 4$, there is a positive solution of this problem for all $\lambda \in (0, \lambda_1)$ where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Ω. They also proved that if $n = 3$, there is a $\mu(\Omega) > 0$ such that for any $\lambda \in (\mu, \lambda_1)$, the nonlinear eigenvalue problem has a positive solution. Moreover, if Ω is a ball, $\mu = \lambda_1/4$.

The Brezis–Nirenberg problem on $\mathbb{R}^N$

For positive radial solutions of this problem in a (unit) ball, one is led to an ODE that still makes sense when n is a real number rather than a natural number.

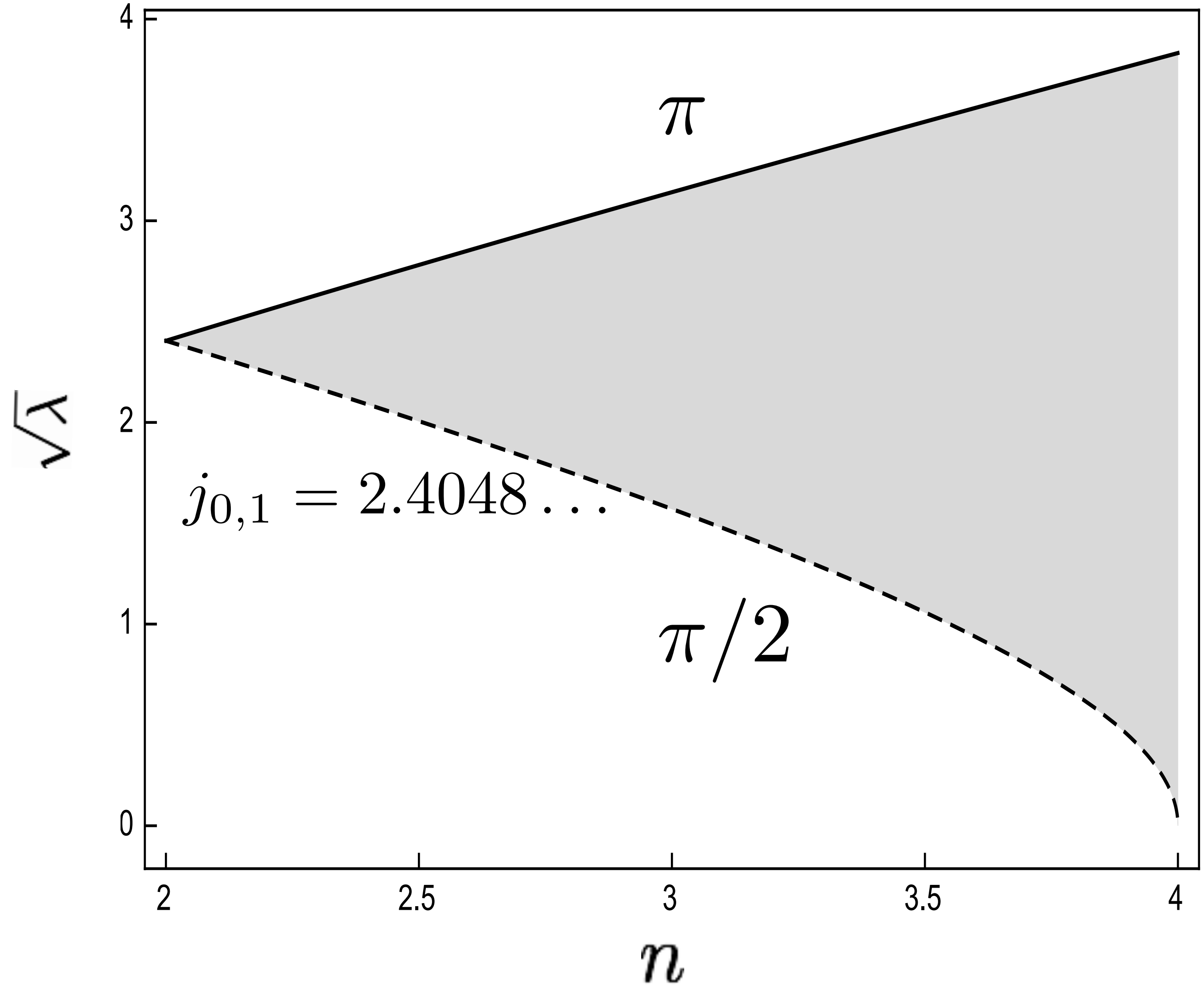
Precisely this problem with $2 \leq n \leq 4$, was considered by E. Jannelli, *The role played by space dimension in elliptic critical problems*, J. Differential Equations, **156** (1999), pp. 407–426.

Among other things Jannelli proved that this problem has a positive solution if and only if λ is such that

$$j_{-(n-2)/2,1} < \sqrt{\lambda} < j_{+(n-2)/2,1},$$

where $j_{\nu,k}$ denotes the k–th positive zero of the Bessel function J_ν.

The Brezis–Nirenberg problem on $\mathbb{R}^N$



Laplacian with a Singular Drift.

An interesting alternative to considering fractional dimension is to consider the Laplacian with a drift instead of the standard Laplacian. During the past decade there has been a growing interest in studying the spectral properties of Laplacians with drift (see, e.g., H. Berestycki, F. Hamel, N. Nadirashvili, CMP 2005, K. Bogdan and T. Jakubowski, CMP 2007, F. Hamel, N. Nadirashvili, and E. Russ, Annals of Math. 2011). Thus, instead of considering the Brezis-Nirenberg problem for the standard Laplacian in $\mathbb{R}^d$ for $d > 2$, one could consider the analogous problem

$$-\Delta u + \delta \frac{\vec{x}}{|x|^2} \cdot \nabla u = \lambda u + |u|^{4/(d-2-\delta)} u, \tag{1}$$

which involves the Laplacian with a singular drift. For positive radial solutions of (1) we are lead to our previous *fractional dimension formulation* provided we set $n = d - \delta$. Because of Hardy's inequality, the Laplacian with the singular drift one considers in the left side of (1) makes sense provided $\delta < (d-2)/2$. Notice that the critical Sobolev exponent on the right side of (1) depends on the parameter δ that characterizes the singular drift.

The Brezis–Nirenberg problem on $\mathbb{S}^N$

We consider the nonlinear eigenvalue problem,

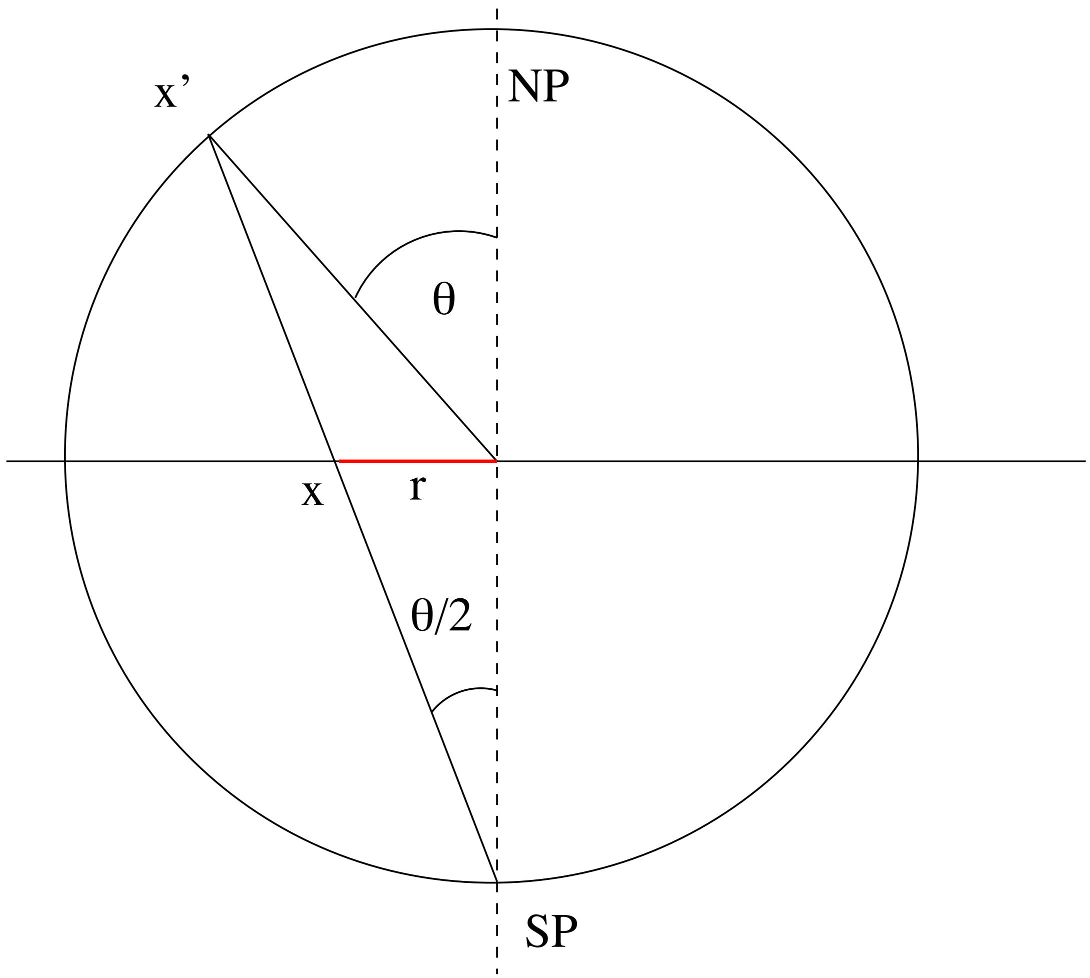
$$-\Delta_{\mathbb{S}^n} u = \lambda u + |u|^{4/(n-2)} u,$$

with $u \in H_0^1(\Omega)$, where Ω is a geodesic ball in $\mathbb{S}^n$. In dimension 3, Bandle and Benguria (JDE, 2002) proved that for $\lambda > -3/4$ this problem has a unique positive solution if and only if

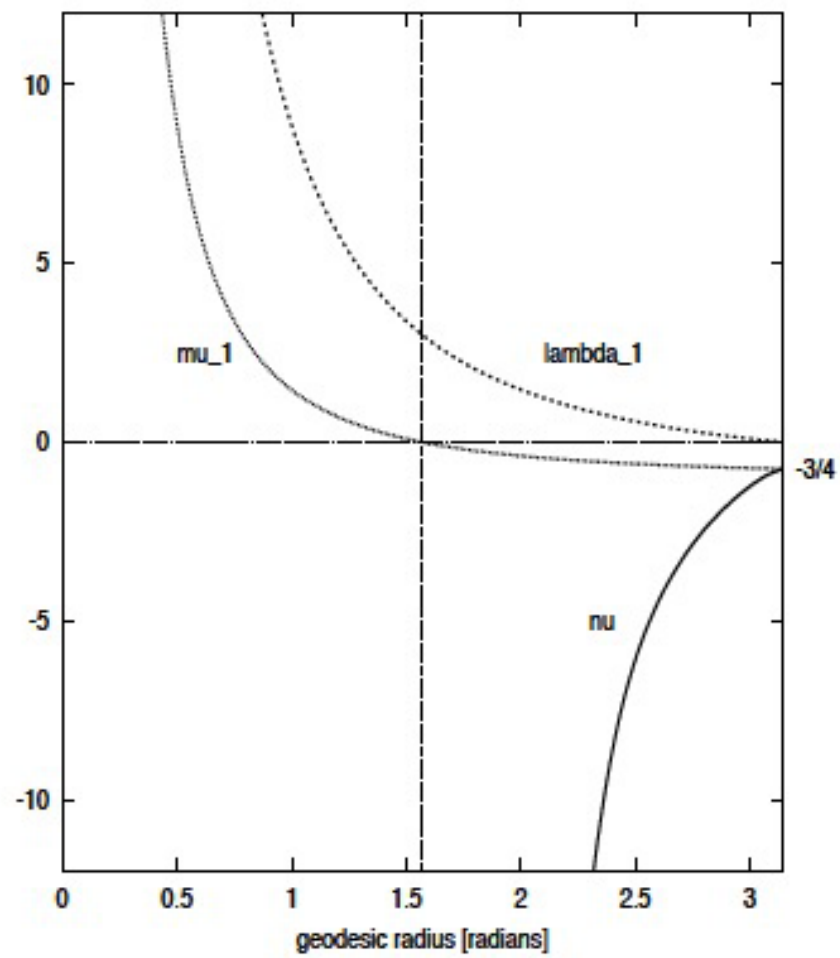
$$\frac{\pi^2 - 4\theta_1^2}{4\theta_1^2} < \lambda < \frac{\pi^2 - \theta_1^2}{\theta_1^2}$$

where θ_1 is the geodesic radius of the ball.

The Brezis–Nirenberg problem on $\mathbb{S}^N$



The Brezis–Nirenberg problem on $\mathbb{S}^3$



The Brezis–Nirenberg problem on $\mathbb{S}^N$

For positive radial solutions of this problem one is led to an ODE that still makes sense when n is a real number rather than a natural number.

Our main result is the following:

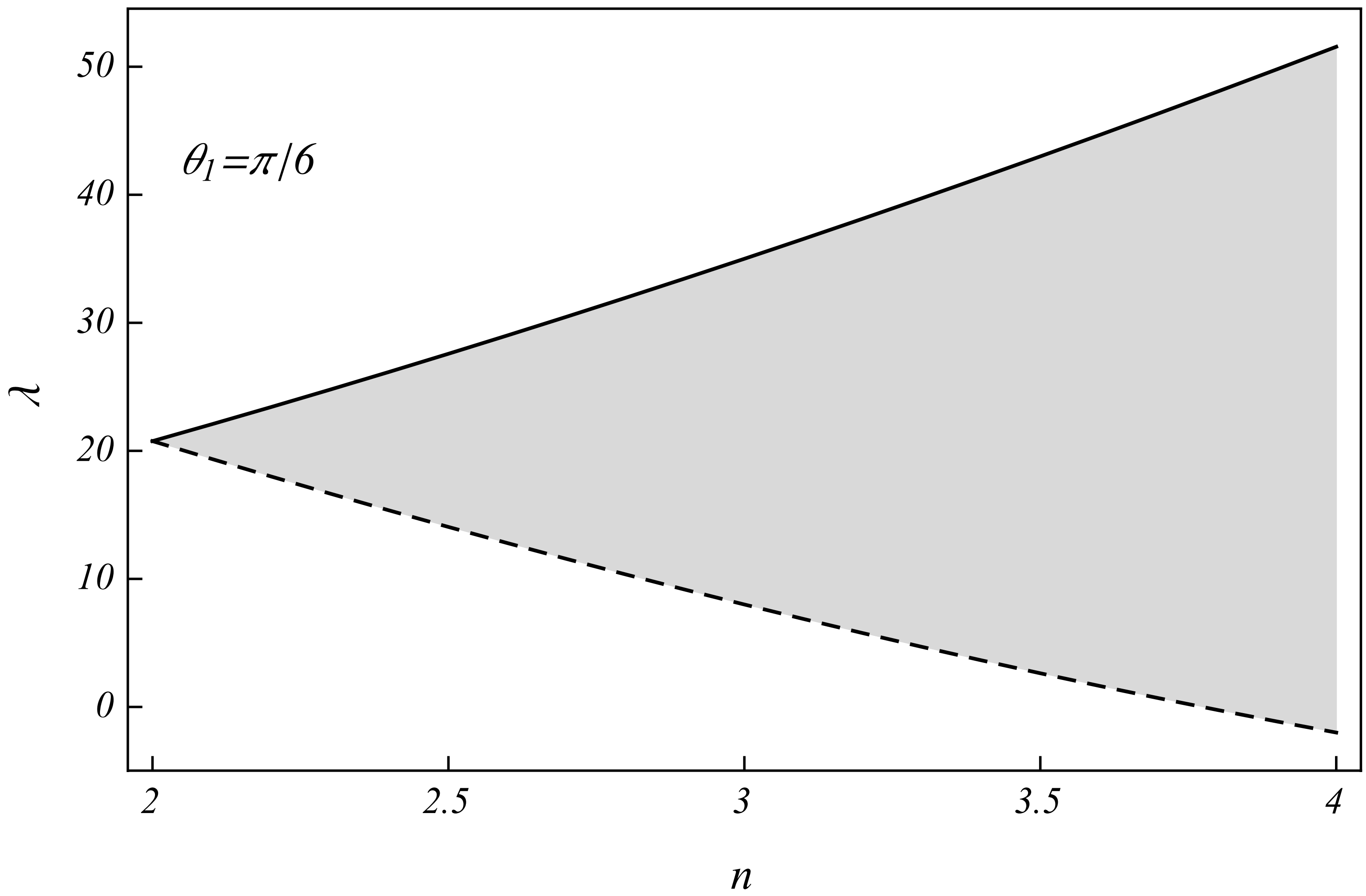
Theorem: For any $2 < n < 4$,
i) If $\lambda \geq -n(n-2)/4$ and $0 \leq \theta_1 \leq \pi$, the boundary value problem, in the interval $(0, \theta_1)$, with $u'(0) = u(\theta_1) = 0$ has a positive solution if and only if λ is such that

$$\frac{1}{4}[(2\ell_2 + 1)^2 - (n-1)^2] < \lambda < \frac{1}{4}[(2\ell_1 + 1)^2 - (n-1)^2]$$

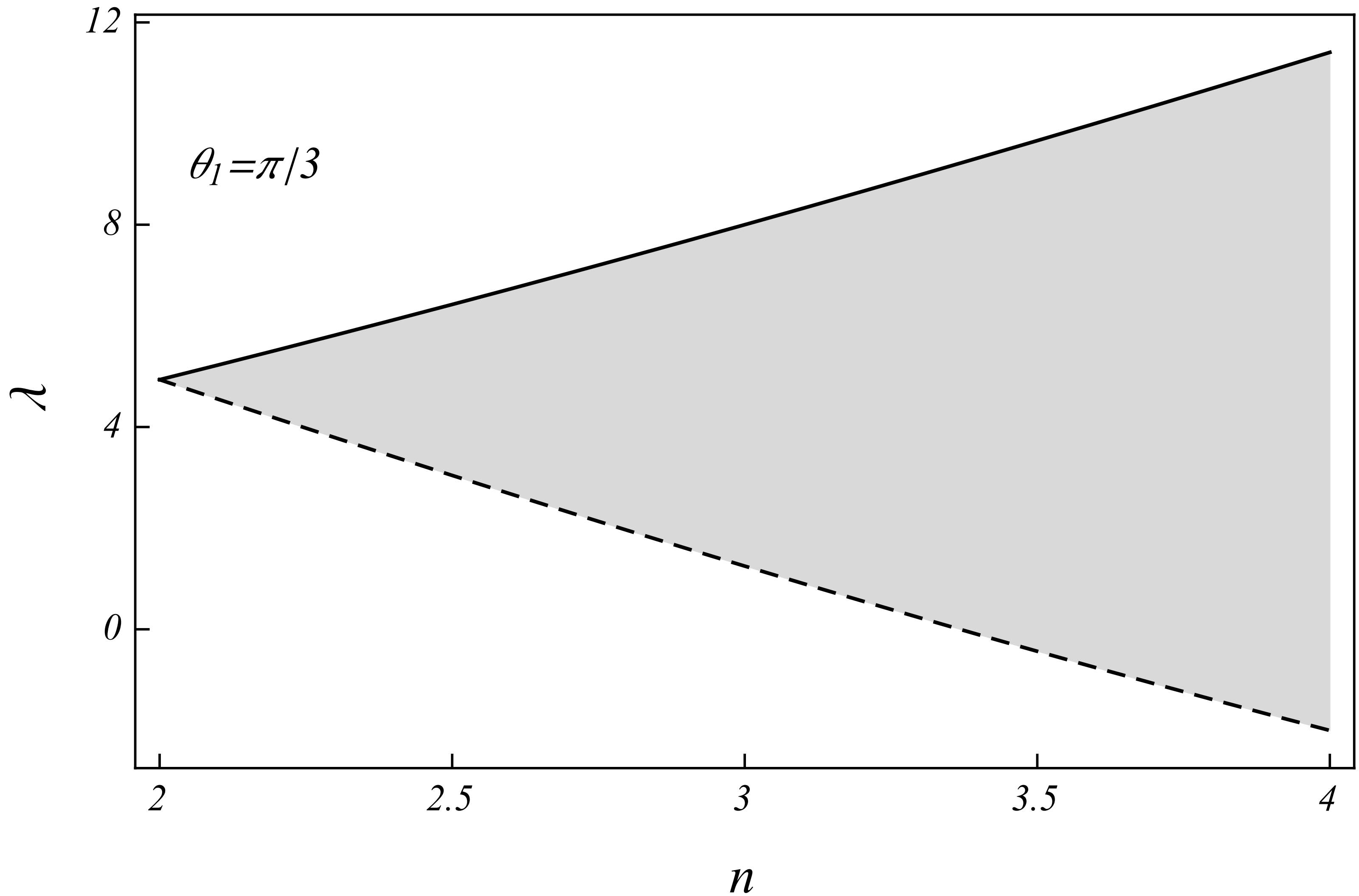
where ℓ_1 (respectively ℓ_2) is the first positive value of ℓ for which the associated Legendre function $\mathrm{P}_\ell^{(2-n)/2}(\cos\theta_1)$ (respectively $\mathrm{P}_\ell^{(n-2)/2}(\cos\theta_1)$) vanishes.

ii) If $\lambda \leq -n(n-2)/4$ and $0 \leq \theta_1 \leq \pi/2$, the boundary value problem, in the interval $(0, \theta_1)$, with $u'(0) = u(\theta_1) = 0$ does not have a positive solution.

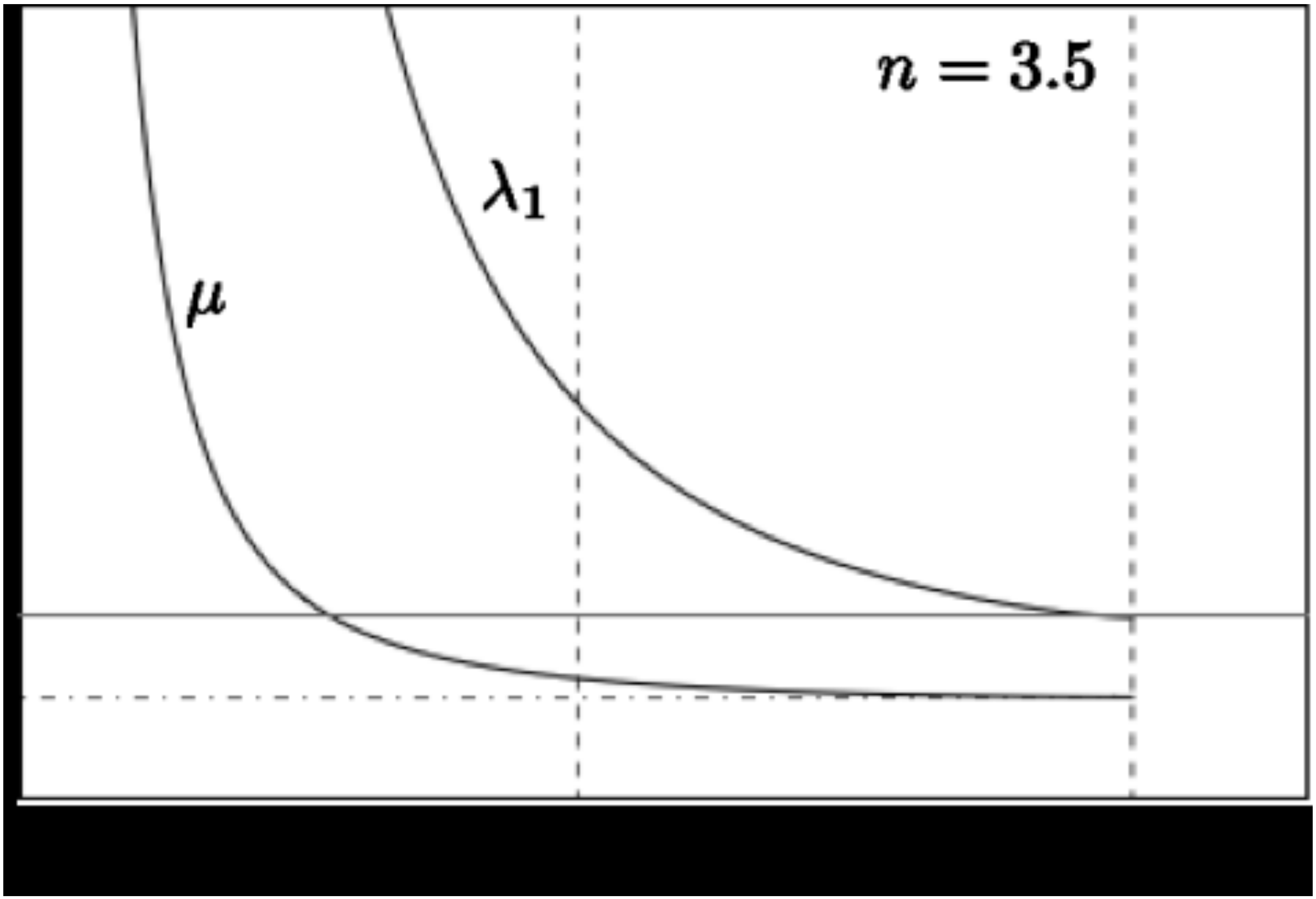
The Brezis–Nirenberg problem on $\mathbb{S}^N$



The Brezis–Nirenberg problem on $\mathbb{S}^N$



The Brezis–Nirenberg problem on $\mathbb{S}^N$



The Brezis–Nirenberg problem on $\mathbb{S}^N$

In the remaining sector, i.e., for $\lambda < -n(n-2)/4$ and $\pi/2 < \theta_1 \leq \pi$, for any $2 < n < 4$ one expects to have multiple solutions to this problem in a similar vein as in the case $n = 3$ studied in:

C. Bandle and J.-C. Wei, *Non–radial clustered spike solutions for semilinear elliptic problems on* $\mathbb{S}^N$, Journal d'Analyse Mathematique, 102 (2007), pp. 181–208.

C. Bandle and J.-C. Wei, *Multiple clustered layer solutions for semilinear elliptic problems on* $\mathbb{S}^n$, Communications in Partial Differential Equations, 33 (2008), pp. 613–635.

H. Brezis and L. A. Peletier, *Elliptic equations with critical exponent on* $\mathbb{S}^3$*: new non-minimising solutions*, Comptes Rendus Mathematique, 339 (2004), pp. 391–394.

H. Brezis and L. A. Peletier, *Elliptic equations with critical exponent on spherical caps of* $\mathbb{S}^3$, Journal d'Analyse Mathematique, 98 (2006), pp. 279–316.

The Brezis–Nirenberg problem on $\mathbb{S}^N$

Strategy of the Proof:

For the nonexistence of solutions:

i) Use a Rellich–Pohozaev's type argument for values of λ below the lower bound.

ii) Multiply the ODE by the first eigenfunction of the Dirichlet problem to rule out the values of λ larger than the upper bound.

For the Existence part, use a variational characterization of λ and a Brezis–Lieb lemma (or, alternatively, a concentration compactness argument).

Equation for the first Dirichlet Eigenvalue of a geodesic cap:

The equation that determines the first Dirichlet eigenvalue is given by,

$$u''(\theta) + (n-1)\frac{\cos\theta}{\sin\theta}u'(\theta) + \lambda\, u = 0, \tag{1}$$

with $u(\theta_1) = 0$, and $u(\theta) > 0$ in $0 \le \theta < \theta_1$ (here θ_1 is the radius of the geodesic ball in $\mathbb{S}^n$, and $0 < \theta_1 \le \pi$). For geodesic balls contained in a hemisphere, $0 < \theta_1 \le \pi/2$.

Let $\alpha = -(n-2)/2$, and set

$$u(\theta) = (\sin\theta)^{\alpha}\, v(\theta). \tag{2}$$

Then $v(\theta)$ satisfies the equation,

$$v''(\theta) + \frac{\cos\theta}{\sin\theta}v'(\theta) + \left(\lambda + \alpha(\alpha-1) - \frac{\alpha^2}{\sin^2\theta}\right) v = 0. \tag{3}$$

Equation for the first Dirichlet Eigenvalue of a geodesic cap:

In the particular case when $n = 3$, $\alpha = -1/2$ and this equation becomes,

$$v''(\theta) + \frac{\cos\theta}{\sin\theta} v'(\theta) + \left(\lambda + \frac{3}{4} - \frac{1}{4\sin^2\theta}\right) v = 0. \tag{1}$$

whose positive regular solution is given by,

$$v(\theta) = C\, \frac{\sin\left(\sqrt{1+\lambda}\,\theta\right)}{\sqrt{\sin\theta}} \tag{2}$$

hence, in this case,

$$u(\theta) = C\, \frac{\sin\left(\sqrt{1+\lambda}\,\theta\right)}{\sin\theta}. \tag{3}$$

Imposing the boundary condition $u(\theta_1) = 0$, in the case $n = 3$, we find that,

$$\lambda_1(\theta_1) = \frac{\pi^2 - \theta_1^2}{{\theta_1}^2}. \tag{4}$$

Equation for the first Dirichlet Eigenvalue of a geodesic cap:

The regular solution of the ODE for the first Dirichlet eigenvalue (for general n) is given by

$$v(\theta) = P_\ell^m(\cos\theta), \tag{1}$$

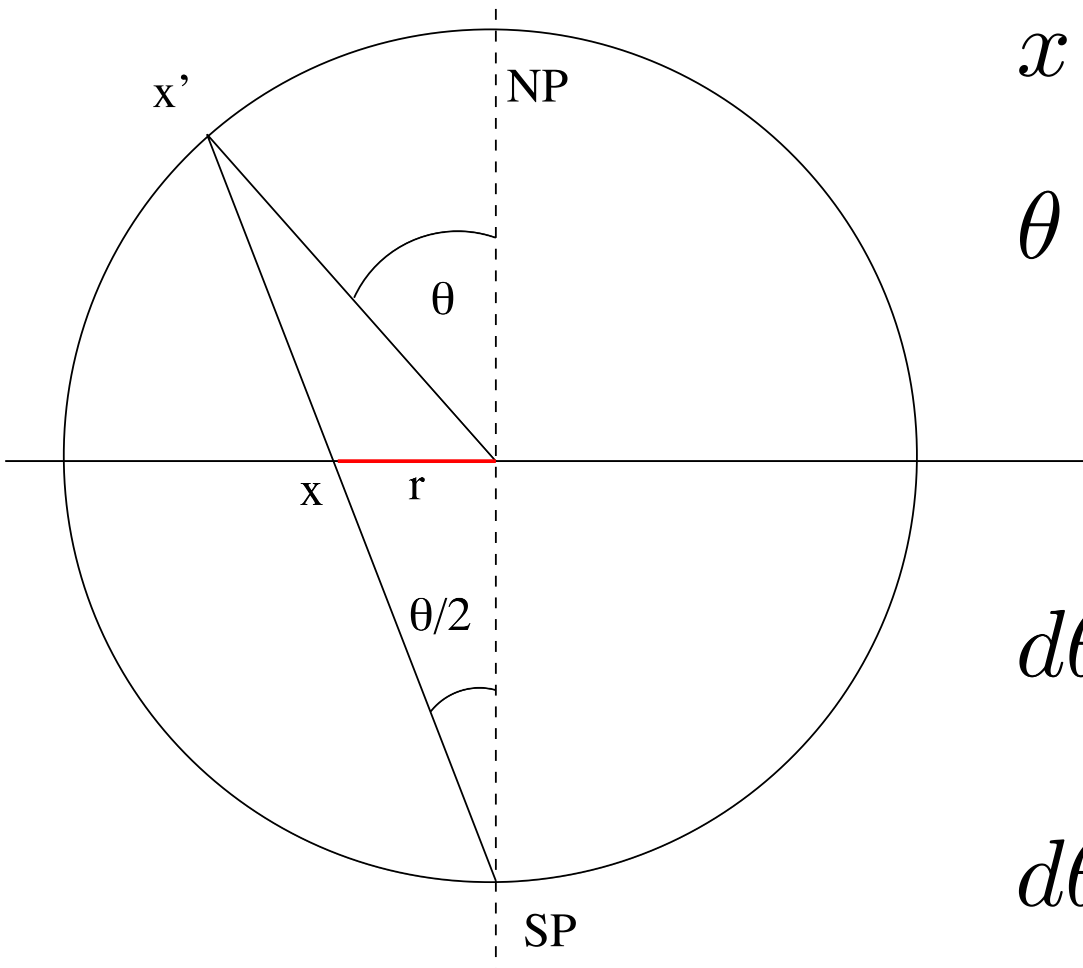
where $P_\ell^m(x)$ is an associated Legendre function, with indices,

$$m = \alpha = (2-n)/2, \tag{2}$$

and

$$\ell = \frac{1}{2}\left(\sqrt{1 + 4\lambda - 4\alpha + 4\alpha^2} - 1\right). \tag{3}$$

Existence of solutions (stereographic projection):



$$x = \tan\left(\frac{\theta}{2}\right)$$

$$\theta = 2\arctan(x)$$

$$d\theta = \frac{2}{1+|x|^2}\,dx$$

$$d\theta = q(x)\,dx$$

Existence of solutions:

Let D be a geodesic ball on $\mathbb{S}^n$. The solutions of

$$\begin{cases} -\Delta_{\mathbb{S}^n} u = \lambda u + u^p & \text{on} \quad D \\ \mathrm{u} > 0 & \text{on} \quad D \\ \mathrm{u} = 0 & \text{on} \quad \partial D, \end{cases}$$

where $p = \frac{n+2}{n-2}$ correspond to minimizers of

$$Q_\lambda(u) = \frac{\int (\nabla u)^2 q^{n-2}\, dx - \lambda \int u^2 q^n\, dx}{\left(\int u^{\frac{2n}{n-2}} q^n\, dx\right)^{\frac{n-2}{n}}}. \tag{1}$$

Here $q(x) = \frac{2}{1+|x|^2}$, so that the line element of S^n is proportional to the line element of the Euclidean space, i.e., $ds = q(x)dx$ through the standard stereographic projection.

Existence of solutions:

In 1999 Bandle and Peletier (Math. Annalen) proved that for domains contained in the hemisphere the infimum of the Rayleigh quotient of the Sobolev inequality on $\mathbb{S}^n$ is not attained, and the value of the sharp constant is precisely the same as in the Euclidean Space of the same dimension.

Thus, one can use the Brezis–Lieb classical lemma (1983) or alternatively a concentration compactness argument to show that if there is a function on the right space that satisfies $Q_\lambda(u) < S$, then the minimizer for Q_λ is attained. The minimiser is positive and satisfies the Brezis–Nirenberg equation.

To construct the desired function we use the *Schuster function* centred at the North Pole, multiplied by a cutoff function introduced to satisfy the Dirichlet boundary condition.

Existence of solutions:

Let φ be a smooth function such that $\varphi(0) = 1$, $\varphi'(0) = 0$ and $\varphi(1) = 0$. For $\epsilon > 0$, let

$$u_\epsilon(r) = \frac{\varphi(r)}{(\epsilon + r^2)^{\frac{n-2}{2}}}. \tag{1}$$

We claim that for ϵ small enough, $Q_\lambda(u_\epsilon) \leq S$. In the next three claims we compute $||\nabla u_\epsilon||_2^2$, $||u_\epsilon||_{p+1}^2$ and $||u_\epsilon||_2^2$.

$$\int (\nabla u_\epsilon)^2 q^{n-2}\, dx = \omega_n \int_0^R \varphi'(r)^2 r^{3-n} q^{n-2}\, dr - \omega_n (n-2)^2 \int_0^R \varphi(r)^2 r^{3-n} q^{n-1}\, dr$$
$$+ \omega_n n(n-2) 2^{n-2} D_n \epsilon^{\frac{2-n}{2}} + \mathcal{O}(\epsilon^{\frac{4-n}{2}}), \tag{2}$$

where

$$D_n = \frac{1}{2} \frac{\Gamma\left(\frac{n}{2}\right)^2}{\Gamma(n)}, \qquad \omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \tag{3}$$

Existence of solutions:

$$\int u_\epsilon^2 q^n \, dx = \omega_n \int_0^R q^n r^{3-n} \varphi^2 \, dr + \mathcal{O}(\epsilon)^{\frac{4-n}{2}}).$$

$$\left(\int u_\epsilon^{\frac{2n}{n-2}} q^n \, dx \right)^{\frac{n-2}{n}} = \omega_n^{\frac{n-2}{n}} 2^{n-2} \epsilon^{\frac{2-n}{2}} D_n^{\frac{n-2}{n}} + \mathcal{O}(\epsilon^{\frac{4-n}{2}}),$$

where

$$D_n = \frac{1}{2} \frac{\Gamma\left(\frac{n}{2}\right)^2}{\Gamma(n)}.$$

Existence of solutions:

$$Q_\lambda(u_\varepsilon) = n(n-2)(\omega_n\, D_n)^{\frac{2}{n}} + \epsilon^{\frac{n-2}{2}} C_n \left[\int_0^R r^{3-n} \left(q^{n-2}\varphi'^2 - (n-2)^2 q^{n-1}\varphi^2 - \lambda q^n \varphi^2 \right) \, dr \right] + \mathcal{O}(\epsilon), \tag{1}$$

where $C_n = \omega_n^{\frac{2}{n}} 2^{2-n} D_n^{\frac{2-n}{n}}$.

Notice that

$$n(n-2)(\omega_n D_n)^{\frac{2}{n}} = \pi n(n-2) \left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)} \right)^{\frac{2}{n}},$$

which is precisely the Sobolev critical constant S.

Existence of solutions:

Let

$$T(\varphi) = \int_0^R r^{3-n} \left(q^{n-2}\varphi'^2 - (n-2)^2 q^{n-1}\varphi^2 - \lambda q^n \varphi^2\right) dr.$$

It suffices to show that $T(\varphi)$ is positive. The associated Euler equation is

$$\varphi''(r) + (3-n)\frac{\varphi'(r)}{r} + (n-2)\frac{\varphi'(r)q'(r)}{q(r)} + (n-2)^2 q(r)\varphi(r) + \lambda q(r)^2\varphi(r) = 0.$$

Setting $r = \tan\theta/2$, and

$$\varphi = \sin^b \frac{\theta}{2}\, \sin^a \theta\, v,$$

where $b = 2n-4$ and $a = \frac{1}{2}(6-3n)$, and multiplying the equation through by $\sin^{-b}\frac{\theta}{2}\sin^{-a}\theta$ we obtain

$$\ddot{v}(\theta) + \cot\theta\, \dot{v}(\theta) + \left(\lambda + \frac{n(n-2)}{4} - \frac{(n-2)^2}{4\sin^2\theta}\right) v = 0.$$

Existence of Positive Solutions:

From here it follows that $T(\varphi) < 0$ provided

$$\lambda > \frac{1}{4}[(2\ell_2 + 1)^2 - (n-1)^2],$$

where ℓ_2 is the first positive value of ℓ for which the associated Legendre function $\mathrm{P}_\ell^{(2-n)/2}(\cos\theta_1)$ vanishes.

This concludes the proof of the existence of positive solutions.

Nonexistence of solutions (a Rellich–Pohozaev's argument):

For radial solutions, the original nonlinear eigenvalue problem,

$$-\Delta_{\mathbb{S}^n} u = \lambda u + u^p \tag{1}$$

where $u > 0$ on D, and $u = 0$ on ∂D can be written as

$$-\frac{(\sin^{n-1}\theta\, u')'}{\sin^{n-1}\theta} = u^p + \lambda u, \tag{2}$$

with initial conditions $u'(0) = 0$, and $u(\theta_1) = 0$.
Here D denotes a geodesic cap of geodesic radius θ_1, and $'$ denotes derivative with respect to θ.

Nonexistence of solutions (a Rellich–Pohozaev's argument):

Multiplying equation (1) by $g(\theta)u'(\theta)\sin^{2n-2}\theta$ we obtain

$$-\int_0^{\theta_1}(\sin^{n-1}\theta u')'u'g\sin^{n-1}\theta\,d\theta = \int_0^{\theta_1}\left(\frac{u^{p+1}}{p+1}\right)' g\sin^{2n-2}\theta\,d\theta + \lambda\int_0^{\theta_1}\left(\frac{u^2}{2}\right)' g\sin^{2n-2}\theta\,d\theta$$

Integrating by parts we have that

$$\int_0^{\theta_1} u'^2\left(\frac{g'}{2}\sin^{2n-2}\theta\right)d\theta + \int_0^{\theta_1}\frac{u^{p+1}}{p+1}\left(g'\sin^{2n-2}\theta + g(2n-2)\sin^{2n-3}\theta\cos\theta\right)d\theta$$
$$+\lambda\int_0^{\theta_1}\frac{u^2}{2}\left(g'\sin^{2n-2}\theta + g(2n-2)\sin^{2n-3}\theta\cos\theta\right)d\theta = \frac{1}{2}\sin^{2n-2}\theta_1 u'(\theta_1)^2 g(\theta_1). \tag{1}$$

Nonexistence of solutions (a Rellich–Pohozaev's argument):

On the other hand, setting $h = \frac{1}{2} g' \sin^{n-1}\theta$ and multiplying equation (1) by $h(\theta)\, u(\theta) \sin^{n-1}(\theta)$ we obtain

$$-\int_0^{\theta_1} (\sin^{n-1}\theta u')'\, h\, u\, d\theta = \int_0^{\theta_1} h u^{p+1} \sin^{n-1}\theta\, d\theta + \lambda \int_0^{\theta_1} h\, u^2 \sin^{n-1}\theta\, d\theta.$$

Integrating by parts we obtain

$$\int_0^{\theta_1} u'^2\, h\, \sin^{n-1}\theta\, d\theta = \int_0^{\theta_1} u^{p+1}\, h\, \sin^{n-1}\theta\, d\theta$$
$$+ \int_0^{\theta_1} u^2 \left(\lambda\, h\, \sin^{n-1}\theta + \frac{1}{2} h''\, \sin^{n-1}\theta + \frac{1}{2} h'(n-1) \sin^{n-2}\theta \cos\theta \right) d\theta.$$

Nonexistence of solutions (a Rellich–Pohozaev's argument):

$$\frac{1}{2}\sin^{2n-2}\theta_1 u'(\theta_1)^2\, g(\theta_1) = \int_0^{\theta_1} B\, u^{p+1}\, d\theta + \int_0^{\theta_1} A\, u^2\, d\theta, \qquad (1)$$

by hypothesis $g(\theta_1) \geq 0$, it follows that the left hand side is nonnegative. We will show that there exist a choice of g so that for appropriate values of λ, $A \equiv 0$, and B is negative, thus obtaining a contradiction.
Here,

$$A = \sin^{2n-2}\theta \left[\frac{g'''}{4} + \frac{3}{4} g''(n-1)\cot\theta \right.$$
$$\left. + \, g' \left(\frac{(n-1)(n-2)\cot^2\theta}{4} - \frac{n-1}{4} + \lambda \right) + \lambda g (n-1)\cot\theta \right].$$

and

$$B \equiv \frac{1}{2} g' \sin^{2n-2}\theta + \frac{g'\sin^{2n-2}\theta}{p+1} + \frac{(2n-2) g \sin^{2n-3}\theta\cos\theta}{p+1}. \qquad (2)$$

Nonexistence of solutions (a Rellich–Pohozaev's argument):

Setting $f = g\,\sin^2\theta$ and writing $m = n-3$, the equation $A = 0$ is equivalent to,

$$\sin^{2m+2}\theta\,\left[\frac{f'''}{4} + \frac{3}{4}m\cot\theta f'' + f'\left(\frac{m(2m-5)}{4}\cot^2\theta + \frac{4-m}{4} + \lambda\right)\right. \\ \left. + f\left(m(1-m)\cot^3\theta + 2m\cot\theta + \lambda m\cot\theta\right)\right] = 0 \tag{1}$$

An appropriate solution is given by,

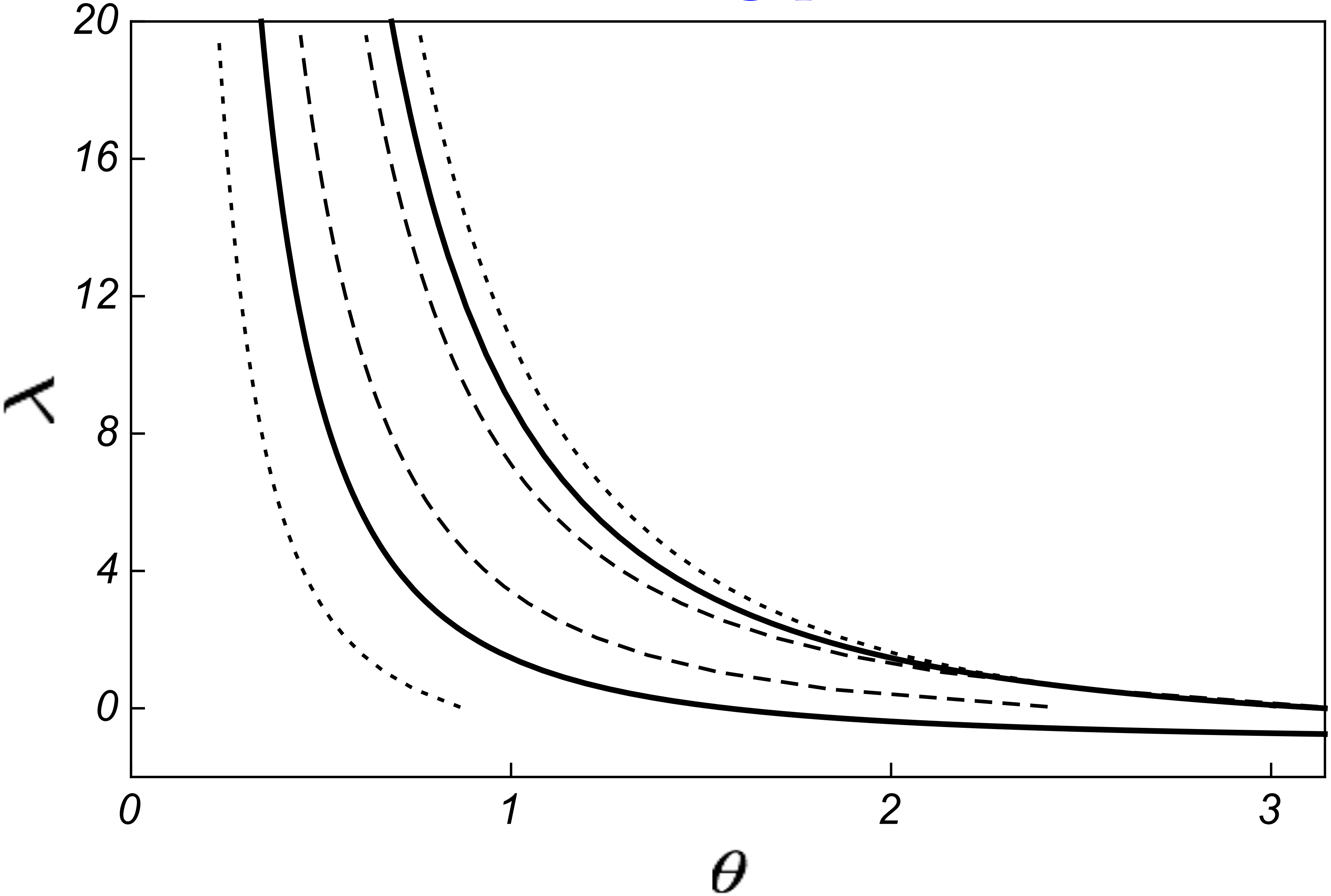
$$f(\theta) = \sin^{1-m}\theta P_\ell^{\nu}(\cos\theta)P_\ell^{-\nu}(\cos\theta),$$

where $\nu = \frac{m+1}{2}$ and $\ell = \frac{1}{2}\left(\sqrt{4\lambda + (m+2)^2} - 1\right)$.
Using the raising and lowering relations for the Associated Legendre functions and some work!, one can show that $B < 0$ for this choice of f, provided

$$\lambda < \frac{1}{4}[(2\ell_2+1)^2 - (n-1)^2],$$

where ℓ_2 is the first positive value of ℓ for which the associated Legendre function $\mathrm{P}_\ell^{(2-n)/2}(\cos\theta_1)$ vanishes.

The Brezis–Nirenberg problem on $\mathbb{S}^N$



The analogous problem on $\mathbb{H}^N$:

For $N = 3$, this was treated by Silke Stapelkamp on her Ph. D. Thesis (U. Basel, 2001).

For $2 \le N \le 4$, this was considered by Soledad Benguria, "The solution gap of the Brezis–Nirenberg problem on the hyperbolic space", Monatshefte für Mathematik **181** (2016) 537–559.

GENERAL HYPERBOLIC CASE

1. $a \in C^3[0, \infty]$;

2. $a''(0) = 0$;

3. $a'(x) > 0$ for all $x > 0$; and

4. $\lim_{x \to 0} \frac{a(x)}{x} = 1$.

Given $n \in (2, 4)$, we study the existence of positive solutions $u \in H_0^1(\Omega)$ of

$$-u''(x) - (n-1)\frac{a'(x)}{a(x)}u'(x) = \lambda u(x) + u(x)^p \qquad (1)$$

with boundary condition $u'(0) = u(R) = 0$. Here, as in the original problem, $p = (n+2)/(n-2)$ is the critical Sobolev exponent.

GENERAL HYPERBOLIC CASE (Existence)

For any $2 < n < 4$ and $0 < R < \infty$ the boundary value problem

$$-u''(x) - (n-1)\frac{a'}{a}u'(x) = \lambda u(x) + u(x)^{\frac{n+2}{n-2}} \tag{1}$$

with $u \in H_0^1(\Omega)$, $u'(0) = u(R) = 0$, and $x \in [0, R]$ has a positive solution if $\lambda \in (\mu_1, \lambda_1)$.

Here, λ_1 is the first positive eigenvalue of

$$y'' + \frac{a'}{a}y' + \left(\lambda - \alpha^2\left(\frac{a'}{a}\right)^2 + \alpha\frac{a''}{a}\right)y = 0 \tag{2}$$

with boundary conditions $\lim_{x\to 0} y(x)x^{\alpha} = 1$. And μ_1 is the first positive eigenvalue of (1) with boundary conditions $\lim_{x\to 0} y(x)x^{-\alpha} = 1$.

GENERAL HYPERBOLIC CASE (nonexistence)

There is no positive solution to problem (1) if $\lambda \geq \lambda_1$, or if $N^* \leq \lambda \leq \mu_1$, where

$$N^* = \sup\left\{\frac{\alpha^2}{a^2}(a'^2 - 1) - \frac{\alpha a''}{a}\right\}.$$

Moreover, then problem (1) has no solution if $\lambda \leq M^*$, where

$$M^* = \inf\left\{\alpha^2 \frac{a''}{a} - \frac{\alpha}{2}\left(\frac{a'''}{a'} + \frac{a''}{a}\right)\right\}.$$

Notice that in the cases that have already been studied, N^* and M^* coincide. In fact, in the Euclidean case, $N^* = M^* = 0$, in the spherical case $N^* = M^* = -n(n-2)/4$, and in the hyperbolic case, $N^* = M^* = n(n-2)/4$.

An improved bound on the nonexistence of solutions in the Hyperbolic case

Consider the Brezis–Nirenberg problem

$$-\Delta_{\mathbb{H}^n} u = \lambda u + |u|^{p-1} u, \tag{1}$$

on $\Omega \subset \mathbb{H}^n$, where Ω is smooth and bounded, with Dirichlet boundary conditions, i.e., $u = 0$ in $\partial\Omega$. After expressing the Laplace Beltrami operator $\Delta_{\mathbb{H}^n}$ in terms of the conformal Laplacian, Stapelkamp (2001) proved that (1) does not admit any regular solution for star-shaped domains Ω provided

$$\lambda \leq \frac{n(n-2)}{4}. \tag{2}$$

Here, we consider the BN problem (1) for radial solutions on geodesic balls of $\mathbb{H}^n$. We can prove a different bound, namely the problem for radial solutions on a geodesic ball Ω^* does not admit a solution if

$$\lambda \leq \frac{n^2(n-1)}{4(n+2)} \tag{3}$$

for $n > 2$. Our bound is better than (2) in the radial case, if $2 < n < 4$. Both bounds coincide when $n = 4$.

An improved bound on the nonexistence of solutions in the Hyperbolic case

R. D. Benguria, and S. Benguria, *An improved bound for the non–existence of radial solutions of the Brezis-Nirenberg problem in* $\mathbb{H}^n$ to appear in the book J. Dittrich, H. Kovarik, A. Laptev (Eds.): Functional Analysis and Operator Theory for Quantum Physics. A Festschrift in Honor of Pavel Exner (Europ. Math. Soc. Publ. House, 2016).