On an eigenvalue problem with infinitely many positive and negative eigenvalues.

Rayleigh-Faber-Krahn inequalities for the principal frequencies

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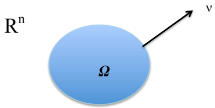

Ω is a bounded Lipschitz domain and ν is the outer normal.

Consider the parabolic problem with dynamical boundary conditions

$$(PP) \qquad \begin{aligned} u_t - \Delta u &= 0 \text{ in } \Omega \times \mathbb{R}^+, \\ \partial_\nu u + \sigma u_t &= 0 \text{ on } \partial\Omega \times \mathbb{R}^+ \qquad \sigma \in \mathbb{R}, \\ u(x,0) &= u_0(x). \end{aligned}$$

J. Escher, J. v. Below, J.L. Vazquez and E. Vitillaro

Separation of variables

$$u(x,t) = e^{-\lambda t}\varphi x$$

leads to the following eigenvalue problem

$$(EP) \qquad \begin{aligned} \Delta\varphi + \lambda\varphi &= 0 \text{ in } \Omega, \\ \partial_\nu \varphi &= \lambda\sigma\varphi \text{ on } \partial\Omega. \end{aligned}$$

Note that $\lambda_0 = 0$, $\varphi_0 =$const. is always an eigenpair.

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The corresponding *Rayleigh quotient* is

$$R[\varphi] = \frac{\int_\Omega |\nabla \varphi|^2 \, dx}{\int_\Omega \varphi^2 \, dx + \sigma \oint_{\partial\Omega} \varphi^2 \, ds}.$$

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Theorem

There exist infinitely many eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lim_{n\to\infty} \lambda_n = \infty.$$

The corresponding system of eigenfunctions $\{\varphi_j\}_0^\infty$ *is complete in* $H^1(\Omega)$.

Set

$$a(v,v) := \int_\Omega v^2\,dx + \sigma \oint_{\partial\Omega} v^2\,ds$$

and define

$$\mathcal{K}_{\pm j} := \left\{ v \in H^1(\Omega) : \int_\Omega |\nabla v|^2\,dx = 1, a(\varphi_{\pm k}, v) = 0, \pm k \in \{0, 1 \cdots j-1\} \right\}$$

Consequently, for $v \in \mathcal{K}_{\pm j}$

$$R[v] = \frac{\int_\Omega |\nabla v|^2\,dx}{\int_\Omega v^2\,dx + \sigma \oint_{\partial\Omega} v^2\,ds} = \frac{1}{a(v,v)}.$$

Theorem

The eigenvalues can be characterized by a variational principle.

$$\frac{1}{\lambda_j} = \sup_{\mathcal{K}_j} a(v,v).$$

Taking into account their multiplicity we have $0 = \lambda_0 < \lambda_1 \leq \cdots$.

$$\frac{1}{\lambda_{-j}} = \inf_{\mathcal{K}_{-j}} a(v,v).$$

Analogously we have $0 = \lambda_0 > \lambda_{-1} \geq \cdots$.

In particular

$$\frac{1}{\lambda_1} = \sup_{\mathcal{K}} a(v, v).$$

and

$$\frac{1}{\lambda_{-1}} = \inf_{\mathcal{K}} a(v, v).$$

where

$$\mathcal{K} = \left\{ v \in H^1(\Omega) : \int_\Omega |\nabla v|^2 \, dx = 1, a(1, v) = 0 \right\}.$$

We denote by $\{\varphi_{\pm j}\}_0^\infty$ the eigenfunctions which correspond to $\lambda_{\pm j}$.

- $\{\varphi_j\}_\infty^\infty$ is a complete orthogonal basis $(a(\varphi_j, \varphi_i) = \delta_{ij})$ in $H^1(\Omega)$.
- If $n > 1$, then $\lim_{m\to\infty} \lambda_{\pm m} = \pm\infty$.

Remark:

The case $n = 1$ is special. Let $\Omega = [0, L]$.

- If $\sigma < -L/2$, then there exists exactly one negative eigenvalue λ_{-1} with a sign-changing eigenfunction.
- If $-L/2 < \sigma < 0$, then there exist exactly two negative eigenvalues $\lambda_{-2} < \lambda_{-1} < 0$.
  λ_{-1} is simple.

Principal frequencies

Theorem

1. *If* $\sigma < \sigma_0 = -\frac{|\Omega|}{|\partial\Omega|}$, *then* λ_1 *is simple and*

$$\frac{1}{\lambda_1} = \sup_{H^1(\Omega)} \frac{\int_\Omega v^2\,dx + \sigma \oint_{\partial\Omega} v^2\,ds}{\int_\Omega |\nabla v|^2\,dx}.$$

2. *If* $\sigma_0 < \sigma < 0$, *then* λ_{-1} *is simple and*

$$\frac{1}{\lambda_{-1}} = \inf_{H^1(\Omega)} \frac{\int_\Omega v^2\,dx + \sigma \oint_{\partial\Omega} v^2\,ds}{\int_\Omega |\nabla v|^2\,dx}.$$

The role of σ_0

Observe that the condition $a(v,1) = 0$ is not required because it is automatically fulfilled.: Assume $\int_\Omega |\nabla v|^2\, dx = 1$ for some $v \in H^1(\Omega)$ and $a(v,1) \neq 0$. Define $c := -\frac{a(v,1)}{a(1,1)}$, this leads to

$$a(v+c,1) = a(v,1) + c\, a(1,1) = 0$$

and

$$a(v+c, v+c) = \ldots = a(v,v) - \frac{a(v,1)^2}{a(1,1)}.$$

Note: $a(1,1) = |\Omega| + \sigma|\partial\Omega|$

The role of σ_0

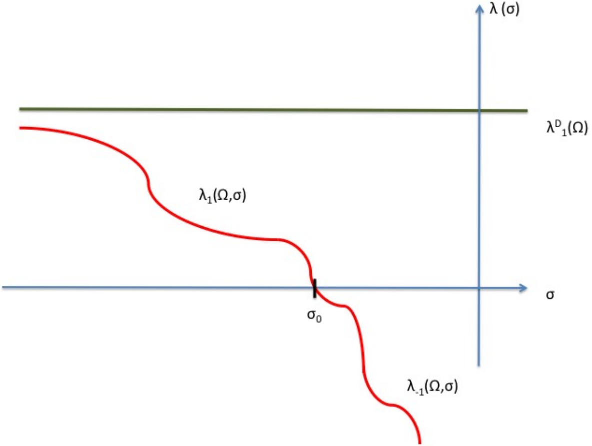
Thus, if $\sigma < \sigma_0 = -\frac{|\Omega|}{|\partial\Omega|}$

$$\frac{1}{\lambda_1(\sigma)} \geq a(v+c, v+c) \geq a(v,v)$$

and if $\sigma_0 < \sigma < 0$

$$\frac{1}{\lambda_{-1}(\sigma)} \leq a(v+c, v+c) \leq a(v,v).$$

$\lambda_{\pm 1}(\sigma)$



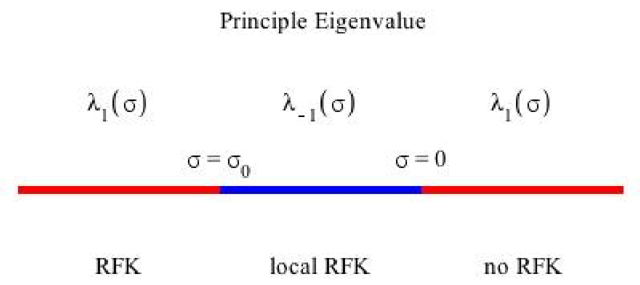
Definition

Ω^* = ball of the same volume as Ω, ($|\Omega^*| = |\Omega|$).

Theorem

1. *Let* $\sigma > 0$. *Then the ball is a critical domain. However it is neither a local minimizer nor a local maximizer.*
2. *Let* $\sigma < \sigma_0 < 0$ *Then* $\lambda_1(\Omega) > \lambda_1(\Omega^*)$.
3. *Let* $\sigma_0 < \sigma < 0$. *Then* $\lambda_{-1}(\Omega) > \lambda_{-1}(\Omega^*)$ *for nearly spherical domains* Ω.

Rayleigh-Faber-Krahn inequalities (C.B. and A.W.)



Proof (1), $\sigma > 0$

Recall

$$\lambda_1(\sigma) = R(\varphi) == \frac{\int_\Omega |\nabla \varphi|^2 \, dx}{\int_\Omega \varphi^2 \, dx + \sigma \oint_{\partial\Omega} \varphi^2 \, ds},$$

where φ denotes the corresponding eigenfunction.

Let $(\Omega_t)_t$ be a family of smoothly perturbed domains such that $\Omega_0 = \Omega^*$. Set

$$R(\tilde{\varphi}(t)) = \frac{\int_{\Omega_t} |\nabla \tilde{\varphi}(t)|^2 \, dx}{\int_{\Omega_t} \tilde{\varphi}(t)^2 \, dx + \sigma \oint_{\partial\Omega_t} \tilde{\varphi}(t)^2 \, ds},$$

where $\tilde{\varphi}(t) = \varphi(t; y_t)$ denotes the corresponding eigenfunction for Ω_t and $y_t \in \Omega_t$.

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Proof (1), $\sigma > 0$

Then

$$\frac{d}{dt} R(\tilde{\varphi}(t))|_{t=0} = 0$$

and

$$\begin{aligned} \frac{d^2}{dt^2} R(\tilde{\varphi}(t))|_{t=0} &= 2 \sum_{i=2}^{\infty} c_i^2 \mu_i^2 \left(-\frac{1}{\mu_2} + \frac{\lambda_1 \sigma \varphi(R)}{k(R)} \right) \\ &\quad - \lambda_1 \sigma \varphi^2(R) \ddot{S}(0) - k(R)\varphi(R)\ddot{V}(0). \end{aligned}$$

Any sign can occur.

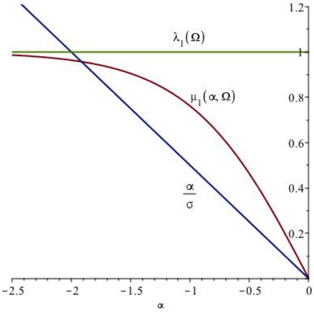
Proofs (2) $\lambda_1(\Omega) > \lambda_1(\Omega^*)$, $\sigma < \sigma_0$

Consider the Robin eigenvalue problem

$$\Delta\psi + \mu_1\psi = 0 \text{ in } \Omega, \quad \partial_\nu\psi = \alpha\psi \text{ on } \partial\Omega, \quad \alpha < 0.$$

The first eigenvalue $\alpha \to \mu_1(\alpha)$ is concave. $\lambda_1(\sigma)$ is the intersection of $\mu(\alpha)$ with the line α/σ. ($\sigma < \sigma_0$!) By the RFK inequality of Bossel $\mu^{\Omega}(\alpha) \geq \mu^{\Omega^*}(\alpha)$.

$$\Longrightarrow \text{For all } \sigma < \sigma_0 \text{ we have } \lambda_1(\Omega) \geq \lambda_1(\Omega^*).$$



We consider the Robin problem

$$\Delta\psi + \mu_1\psi = 0 \text{ in } \Omega, \quad \partial_\nu\psi = \alpha\psi \text{ on } \partial\Omega, \quad \alpha > 0.$$

In this case a RFK type inequality is not valid in general, s. Bareket, Ferone, Nitsch and Trombetti, Freitas and Krejcirik.

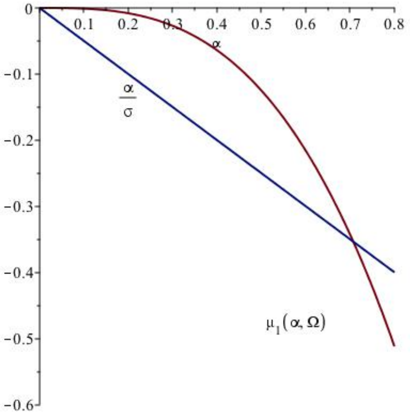
Lemma

C. B. and A. W. For nearly spherical domains

$$\mu_1^\Omega(\alpha) \leq \mu^{\Omega^*}(\alpha).$$

Proofs (3) $\lambda_{-1}(\Omega) > \lambda_{-1}(\Omega^*)$

The same arguments as before apply.



Global Lower bound for $\lambda_{-1}(\Omega)$ if $\sigma_0 < \sigma < 0$.

The lower bound given in the previous theorem holds only locally for domains near the ball. By means of the *harmonic transplantation* it is possible to construct global lower bounds.

Notation

$$G(x,y) = s_n(|x-y|) + h(x,y): \text{ Green's function}$$

$$\Delta_x h(\cdot, y) = 0 \text{ and } s_n(t) = \begin{cases} -\frac{1}{2\pi}\log(t) & \text{if } n = 2\,, \\ \frac{1}{(n-2)|\partial B_1|} t^{2-n} & \text{if } n > 2 \end{cases}$$

$$r(y) = \begin{cases} e^{-h(y,y)} & \text{if } n = 2\,, \\ h(y,y)^{-\frac{1}{n-2}} & \text{if } n > 2\,. \end{cases}$$

$$r_\Omega := max_\Omega r(y): \text{ harmonic radius.}$$

Global Lower bound for $\lambda_{-1}(\Omega)$ if $\sigma_0 < \sigma < 0$.

Isoperimetric inequality:

$$|B_{r_0}| \leq |\Omega|.$$

Note that $G_{B_R}(x, 0)$ is a monotone function in $r = |x|$. Consider any radial function $\phi : B_{r_\Omega} \to \mathbb{R}$ thus $\phi(x) = \phi(r)$. Then there exists a function $\omega : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(x) = \omega(G_{B_{r_\Omega}}(x, 0)).$$

To ϕ we associate the transplanted function $U : \Omega \to \mathbb{R}$ defined by $U(x) = \omega(G_\Omega(x, y_h))$. Then for any positive monotone function $f(s)$, the following inequalities hold true.

Global Lower bound for $\lambda_{-1}(\Omega)$ if $\sigma_0 < \sigma < 0$.

$$\int_\Omega |\nabla U|^2\,dx = \int_{B_{r_\Omega}} |\nabla \phi|^2\,dx \tag{1}$$

$$\int_\Omega f(U)\,dx \geq \int_{B_{r_\Omega}} f(\phi)\,dx. \tag{2}$$

$$\int_\Omega f(U)\,dx \leq \gamma^n \int_{B_{r_\Omega}} f(\phi)\,dx, \tag{3}$$

where $\gamma = \left(\frac{|\Omega|}{|B_{r_\Omega}|}\right)^{\frac{1}{n}}$. Moreover since U is constant on $\partial\Omega$ ($U = U(\partial\Omega)$) and since ϕ is radial we deduce

$$\begin{aligned} \int_{\partial\Omega} U^2\,dS &= U^2(\partial\Omega)\,|\partial\Omega| = \phi^2(r_\Omega)\,|\partial B_{r_\Omega}|\,\frac{|\partial\Omega|}{|\partial B_{r_\Omega}|} \\ &= \frac{|\partial\Omega|}{|\partial B_{r_\Omega}|} \int_{\partial B_{r_\Omega}} \phi^2\,dS. \end{aligned} \tag{4}$$

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Global Lower bound for $\lambda_{-1}(\Omega)$ if $\sigma_0 < \sigma < 0$.

Let u be a positive normalized radial eigenfunction for λ_{-1} in B_{r_Ω} corresponding to the eigenvalue $\lambda_1^{\sigma'}(B_{r_\Omega})$, where $\sigma' := \sigma\frac{|\partial\Omega||B_{r_\Omega}|}{|\Omega||B_{r_\Omega}|}$. Let U be the transplanted function of u in Ω. Then we get

$$\frac{1}{\lambda_{-1}^{\sigma}(\Omega)} \leq \int_{\Omega} U^2\, dx - |\sigma| \int_{\partial\Omega} U^2\, dS.$$

We apply (3) to the first integral on the right -hand side and (4) to the second one. Thus

$$\begin{aligned}
\frac{1}{\lambda_{-1}^{\sigma}(\Omega)} &\leq \gamma^n \int_{B_{r_\Omega}} u^2\, dx - |\sigma| \frac{|\partial\Omega|}{|\partial B_{r_\Omega}|} \int_{\partial B_{r_\Omega}} u^2\, dS \\
&= \gamma^n \left(\int_{B_{r_\Omega}} u^2\, dx + \sigma' \int_{\partial B_{r_\Omega}} u^2\, dS \right).
\end{aligned}$$

Thus

$$0 > \lambda_{-1}^{\sigma}(\Omega) \geq \frac{1}{\gamma^n} \lambda_{-1}^{\sigma'}(B_{r_\Omega})$$

We may rewrite this inequality as

$$|\Omega| \, \lambda_{-1}^{\sigma}(\Omega) \geq |B_{r_\Omega}| \, \lambda_{-1}^{\sigma'}(B_{r_\Omega}).$$

Global Lower bound for $\lambda_{-1}(\Omega)$ if $\sigma_0 < \sigma < 0$.

Theorem

Let $\sigma_0 < \sigma < 0$ *and let* $\sigma' := \sigma \frac{|\partial\Omega||B_{r_\Omega}|}{|\Omega||B_{r_\Omega}|}$. *Then*

$$|\Omega|\, \lambda^{\sigma}_{-1}(\Omega) \geq |B_{r_\Omega}|\, \lambda^{\sigma'}_{-1}(B_{r_\Omega}).$$

Equality holds if and only if $\Omega = B_{r_\Omega}$.