A Sharp Lower Bound for the First Eigenvalue of the Vibrating Clamped Plate Problem under Compression

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November, 2016

Luminy (Marseille), France
November 21, 2016

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Eigenvalue Problem

$\Omega \subset \mathbb{R}^n$ bounded open.
Eigenvalue problem for clamped plate under compression $\kappa > 0$

$$\left.\begin{aligned} \Delta^2 u + \kappa \Delta u = \ & \lambda u \quad \text{in } \Omega \\ u = |\nabla u| = \ & 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{2.1}$$

We take $\kappa < \lambda_{buckling}$ where

$$\lambda_{buckling} = \inf \left\{ \frac{\int_\Omega |\Delta\varphi|^2 \ dx}{\int_\Omega |\nabla\varphi|^2 \ dx} : \varphi \in H_0^2(\Omega) \setminus \{0\} \right\} \tag{2.2}$$

whereby, the operator $\Delta^2 + \kappa\Delta$ is uniformly elliptic and self-adjoint.
First eigenvalue $\lambda = \lambda(\Omega, \kappa)$ given by the variational characterization

$$\lambda(\Omega, \kappa) := \inf \left\{ \frac{\int_\Omega \left(|\Delta\varphi|^2 - \kappa|\nabla\varphi|^2 \right) \ dx}{\int_\Omega |\varphi|^2 \ dx} : \varphi \in H_0^2(\Omega) \setminus \{0\} \right\} . \tag{2.3}$$

Shape Optimization Problem

Do we have

$$\lambda(\Omega, \kappa) \geq \lambda(\Omega^*, \kappa)\,,$$

where Ω^* is a ball of the same volume as Ω?

History

- For the clamped plate problem ($\kappa = 0$) this is one of Rayleigh's conjectures.
  - Szegő [6, 7] showed the conjecture under the assumption that first eigenfunction u keeps same sign. Essentially by rearranging Δu in Ω.
  - But this hypothesis is not true as follows from the works of Duffin, Shaffer, Coffman.
  - For any n, Talenti [9] showed $\lambda(\Omega, \kappa = 0) \geq c_n \lambda(\Omega^*, \kappa = 0)$ for some constant $c_n \in (0, 1]$. c_n depends on the dimension.
  - For $n = 2$, Nadirashvili [5] shows the optimal result with $c_2 = 1$.
  - For $n = 3$ (and $n = 2$), Ashbaugh and Benguria [1] showed the optimal result with $c_3 = 1$.
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Result

Theorem (M. S. A.; R. Benguria; R. Mahadevan)

For $n = 2$, we have $\lambda(\Omega, \kappa) \geq \lambda(\Omega^*, \kappa)$ for $\kappa \in [0, a]$ for some $a < \lambda_{buckling}$.
Ω^* is a ball of the same volume as Ω (and whose radius we denote by L).

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Idea

- The idea of the proof is much the same as that in Ashbaugh and Benguria [1].
- Reduce to a two-ball optimization problem [1, 2, 3, 5, 9] using a rearrangement result of Talenti [8].
- Then study the two-ball optimization problem carefully using properties of Bessel functions.
- For $n = 2$, the analysis shows us that the solution of the two-ball problem corresponds to the situation where one ball degenerates to a point for $\kappa \in [0, a]$ for some value of $a < \lambda_{buckling}$.
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Reduction to the two ball problem

We shall use the following theorem, a slight variant of a result of Talenti.

Theorem (cf. Talenti [8], Theorem 1)

Let G be a domain in R^2, $F \in L^p(\Omega)$ for some $p > 1$ if $n = 2$. Let U be the solution of

$$\left.\begin{aligned} -\Delta U = F \quad & \text{in } G \\ U = 0 \quad & \text{on } \partial G \end{aligned}\right\} \tag{6.4}$$

and let Z be the solution of

$$\left.\begin{aligned} -\Delta Z = F^* \quad & \text{in } G^* \\ Z = 0 \quad & \text{on } \partial G^* \end{aligned}\right\} \tag{6.5}$$

where F^* is radially symmetric decreasing and equimeasurable with F on G^*. Then, if $U \geq 0$ on G, we have:

- $Z \geq U^* \geq 0$ *on* G^* *and therefore,* $\int_{G^*} |Z|^2 \, dx \geq \int_G |U|^2 \, dx$.
- $\int_{G^*} |\nabla Z|^2 \, dx \geq \int_G |\nabla U|^2 \, dx$

Note: If $n > 2$, the same result is true when $F \in L^{\frac{2n}{n+2}}(\Omega)$.

Reduction to the two ball problem

- Ω^* ball centered at the origin of radius L with volume $|\Omega|$.
- u any first eigenfunction for (2.1) in Ω. Since u may not keep the same sign, we take
  $\Omega_+ = \{x \in \Omega : u(x) > 0\}$ and $\Omega_- = \{x \in \Omega : u(x) < 0\}$
  $\Omega_\pm^*$ balls with the same volume as Ω_+ and Ω_- centered at the origin, with a and b their radii.
  We have $a^2 + b^2 = L^2$.
- $f(x) = (-\Delta u)^*(x), x \in \Omega_+^*$; $g(x) = (\Delta u)^*(x), x \in \Omega_-^*$. Let v and w solve

$$\left.\begin{array}{rl} -\Delta v = f & \text{in } \Omega_+^* \\ v = 0 & \text{on } \partial\Omega_+^* \end{array}\right\} \tag{6.6}$$

and

$$\left.\begin{array}{rl} -\Delta w = g & \text{in } \Omega_-^* \\ w = 0 & \text{on } \partial\Omega_-^* \,. \end{array}\right\} \tag{6.7}$$

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Reduction to the two-ball problem

- We have

$$\int_\Omega |\Delta u|^2\,dx = \int_{\Omega_+} |-\Delta u|^2\,dx + \int_{\Omega_-} |\Delta u|^2\,dx$$
$$= \int_{\Omega_+^*} f^2\,dx + \int_{\Omega_-^*} g^2\,dx$$
$$= \int_{\Omega_+^*} |\Delta v|^2\,dx + \int_{\Omega_-^*} |\Delta w|^2\,dx\,. \qquad (6.8)$$

- By Talenti's theorem, we also have

$$\int_{\Omega_+^*} |v|^2\,dx \geq \int_{\Omega_+^*} |u_+^*|^2\,dx \text{ and } \int_{\Omega_-^*} |w|^2\,dx \geq \int_{\Omega_-^*} |u_-^*|^2\,dx \quad (6.9)$$

and

$$\int_{\Omega_+^*} |\nabla v|^2\,dx \geq \int_{\Omega_+^*} |\nabla u_+^*|^2\,dx \text{ and } \int_{\Omega_-^*} |\nabla w|^2\,dx \geq \int_{\Omega_-^*} |\nabla u_-^*|^2\,dx\,. \quad (6.10)$$

Reduction to the two-ball problem

- We have

$$\begin{aligned}\int_\Omega |\Delta u|^2\,dx &= \int_{\Omega_+} |-\Delta u|^2\,dx + \int_{\Omega_-} |\Delta u|^2\,dx\\ &= \int_{\Omega_+^*} f^2\,dx + \int_{\Omega_-^*} g^2\,dx\\ &= \int_{\Omega_+^*} |\Delta v|^2\,dx + \int_{\Omega_-^*} |\Delta w|^2\,dx\,. \qquad (6.8)\end{aligned}$$

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Reduction to the two-ball problem

So, we conclude

$$\frac{\int_\Omega \left(|\Delta u|^2 - \kappa|\nabla u|^2\right)\ dx}{\int_\Omega |u|^2\ dx} \geq \frac{\int_{\Omega_+^*} \left(|\Delta v|^2 - \kappa|\nabla v|^2\right)\ dx + \int_{\Omega_-^*} \left(|\Delta w|^2 - \kappa|\nabla w|^2\right)}{\int_{\Omega_+^*} |v|^2\ dx + \int_{\Omega_-^*} |w|^2\ dx}$$

This gives

$$\lambda(\Omega, \kappa) \geq J_{a,b} \tag{6.11}$$

for some a, b with $a^2 + b^2 = 1$ where

$$J_{a,b} = \min_{v,w} \frac{\int_{|x|\leq a} \left(|\Delta v|^2 - \kappa|\nabla v|^2\right)\ dx + \int_{|x|\leq b} \left(|\Delta w|^2 - \kappa|\nabla w|^2\right)\ dx}{\int_{|x|\leq a} |v|^2\ dx + \int_{|x|\leq b} |w|^2\ dx} \tag{6.12}$$

minimum over $v \in H^2(B_a)$, $w \in H^2(B_b)$ radial and $a\frac{\partial v}{\partial r}\Big|_{\partial B_a} = b\frac{\partial w}{\partial r}\Big|_{\partial B_b}$.
Consequently,

$$\lambda(\Omega, \kappa) \geq \min_{\Omega} \lambda(\Omega, \kappa) \geq \min_{a,b} J_{a,b}\,. \tag{6.13}$$

We will be done if we can show

$$\min_{a,b} J_{a,b} \geq J_{L,0} = \lambda(\Omega^*, \kappa) \tag{6.14}$$

Variational equations for the two-ball problem

Before we analyze $\min_{a,b} J_{a,b}$, let us write the variational equations at the minimum for $J_{a,b}$ for fixed a, b.

$$\left.\begin{aligned} \Delta^2 v + \kappa \Delta v = \ \lambda v \quad &\text{in } B_a \\ v = 0 \quad &\text{on } \partial B_a \end{aligned}\right\} \tag{6.15}$$

and

$$\left.\begin{aligned} \Delta^2 w + \kappa \Delta w = \ \lambda w \quad &\text{in } B_b \\ w = 0 \quad &\text{on } \partial B_b\,. \end{aligned}\right\} \tag{6.16}$$

In addition,

$$a \frac{\partial v}{\partial r}\bigg|_{\partial B_a} = b \frac{\partial w}{\partial r}\bigg|_{\partial B_b} \tag{6.17}$$

and

$$\Delta v|_{\partial B_a} + \Delta w|_{\partial B_b} = 0\,. \tag{6.18}$$

Solution of the variational equations for the two-ball problem

For given a, b, want to obtain radial solutions v, w in the two-balls to

$$(\Delta^2 + \kappa\Delta)U = \lambda U\,.$$

Writing this as

$$(\Delta - \alpha^2)(\Delta + \beta^2)U = 0$$

with $(\alpha, \beta > 0)$

$$\alpha^2 = \sqrt{\lambda + \kappa^2/4} - \kappa/2\,, \qquad \beta^2 = \sqrt{\lambda + \kappa^2/4} + \kappa/2$$

we get v, w to be of the form

$$v(r) = AJ_0(\beta r) + BI_0(\alpha r)\,, \qquad w(r) = CJ_0(\beta r) + DI_0(\alpha r)\,. \tag{6.19}$$

$v(a) = 0 = w(b)$, $av'(a) = bw'(b)$ and $\Delta v|_{\partial B_a} + \Delta w|_{\partial B_b} = 0$ lead to a system of 4 homogeneous equations in A, B, C, D.

Solution of the variational equations for the two-ball problem

The above has a non-trivial solution iff λ is a zero of

$$F(\alpha, \beta, a) = f(\alpha, \beta, a) + f(\alpha, \beta, b), \qquad (a^2 + b^2 = L^2) \tag{6.20}$$

with

$$f(\alpha, \beta, a) = a\beta \frac{J_1(\beta a)}{J_0(\beta a)} + a\alpha \frac{I_1(\alpha a)}{I_0(\alpha a)}. \tag{6.21}$$

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