Thin domains with a locally periodic highly oscillatory boundary

José M. Arrieta
Universidad Complutense de Madrid and
ICMAT (CSIC-UAM-UC3M-UCM)

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Joint work with
Manuel Villanueva-Pesqueira, UCM
Marcone C. Pereira - USP, Brazil

We are interested in analyzing the behavior of the solutions as the parameter $\epsilon \to 0$, of the following problem with Neumann boundary conditions:

$$(P_\epsilon) \quad \begin{cases} -\Delta u^\epsilon + u^\epsilon = f_\epsilon & \text{in } R^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial N^\epsilon} = 0 & \text{on } \partial R^\epsilon \end{cases}$$

where R^ϵ is a 2D-thin domain where the boundary presents oscillations.

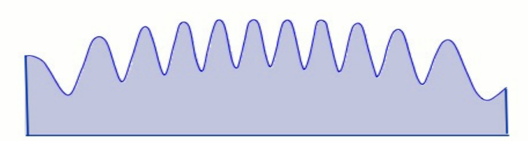
For instance:

$$R^\epsilon = \{(x, y) : 0 < x < 1;\ 0 < y < \epsilon g(x/\epsilon)\}, \quad g(\cdot)\ L-\text{periodic.}$$

We are trying to go beyond the purely periodic case:

We will deal with thin domains of the type:

Locally periodic oscillatory boundary



$$R_\epsilon = \{(x,y) : 0 < x < 1;\ 0 < y < \epsilon G(x, x/\epsilon)\}$$

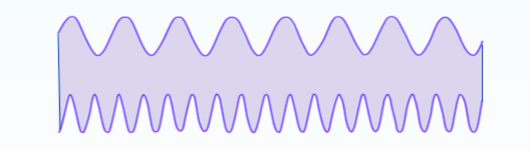
$$0 < G_0 \leq G(\cdot,\cdot) \leq G_1, \quad G(x,\cdot) \text{ is } l(x) - \textit{periodic}.$$

We will divide the analysis of this case in two:

- **Locally periodic with constant period**. That is $l(x) = L$
- **Locally periodic with varying period**

And also of the type:

Oscillations at both boundaries



$$R_\epsilon = \{(x, y) : 0 < x < 1;\ -\epsilon h(x/\epsilon^\alpha) < y < \epsilon g(x/\epsilon^\beta)\}$$

$$0 \le h_0 \le h(\cdot) \le h_1, \quad 0 < g_0 \le g(\cdot) \le g_1$$

$h(\cdot)$ L_1-perioidc and $g(\cdot)$ L_2- periodic.

Purely Periodic oscillations

$R^\epsilon = \{(x,y) : 0 < x < 1;\ 0 < y < \epsilon g(x/\epsilon)\},\quad g(\cdot)\ L-$periodic.

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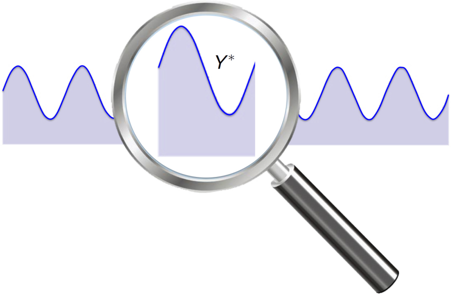

$$Y^* = \{(y_1, y_2) : 0 < y_1 < L, 0 < y_2 < g(y_1)\}$$

The limit problem is

$$(P_0) \quad \begin{cases} -q_0 u_{xx} + u = f(x), \quad x \in (0,1) \\ u'(0) = u'(1) = 0 \end{cases}$$

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$$(P_0) \quad \begin{cases} -q_0 u_{xx} + u = f(x), & x \in (0,1) \\ u'(0) = u'(1) = 0 \end{cases}$$

$$q_0 = \frac{1}{|Y^*|} \int_{Y^*} \left\{ 1 - \frac{\partial X}{\partial y_1}(y_1, y_2) \right\} dy_1 dy_2,$$

where X is the unique Y^*-periodic solution in y_1 (up to an additive constant) of:

$$\begin{cases} -\Delta X = 0 \text{ in } Y^* \\ \dfrac{\partial X}{\partial N} = 0 \text{ on } B_2 \\ \dfrac{\partial X}{\partial N} = N_1 \text{ on } B_1 \end{cases}$$

T.A. Mel'nyk, A.V. Popov *Asymptotic Analysis of BVP's in thin perforated domains with rapidly varying thickness,* Nonlinear Oscillations, Vol. 13, No. 1, (2010)

JA, A. Carvalho, M. Pereira, R. P. da Silva *Semilinear parabolic problems in thin domains with a highly oscillatory boundary,* Nonlinear Analysis T.M.A., Vol 74, 15 (2011), 5111-5132.

JA, M. Villanueva-Pesqueira *Thin domains with non-smooth oscillatory boundaries,* Journal of Math Anal and Appl 446, pp. 130-164 (2017) .

Convergence

In what sense is (P_0) the limit problem of (P_ϵ)?
Observe that

$$u_\epsilon = (-\Delta + I)^{-1} f_\epsilon \in L^2(R_\epsilon)$$
$$u_0 = (-q_0\Delta + I)^{-1} f \in L^2(0,1)$$

We have the extension map:

$$\begin{aligned} \mathcal{E} : L^2(0,1) &\to L^2(R_\epsilon) \\ \varphi(\cdot) &\to \mathcal{E}\,\varphi(x,y) \equiv \varphi(x) \end{aligned}$$

which trivially satisfies

$$c\|\varphi\|_{L^2(0,1)} \le \epsilon^{-1/2}\|\,\mathcal{E}\,\varphi\|_{L^2(R_\epsilon)} \le C\|\varphi\|_{L^2(0,1)}$$

We define "$\mathcal{E}$-convergence"

$$L^2(R_\epsilon) \ni f_\epsilon \xrightarrow{\mathcal{E}} f_0 \in L^2(0,1) \qquad \text{if} \qquad \epsilon^{-1/2}\|\,\mathcal{E}\,f_0 - f_\epsilon\|_{L^2(R_\epsilon)} \to 0$$

Convergence

Then, we can show the following convergence:

i) For any family $f_\epsilon \in L^2(R_\epsilon)$ s.t. $\exists\ f_0 \in L^2(0,1)$ with $f_\epsilon \xrightarrow{\mathcal{E}} f_0$ then $u_\epsilon \xrightarrow{\mathcal{E}} u_0$

ii) For any family of functions $f_\epsilon \in L^2(R_\epsilon)$ with $\epsilon^{-1/2}\|f_\epsilon\|_{L^2(R_\epsilon)} \leq M$ for some $M > 0$, there exists a sequence f_{ϵ_n} and $f_0 \in L^2(0,1)$ s.t. $u_{\epsilon_n} \xrightarrow{\mathcal{E}} u_0$.

This is called "Compact Convergence of $(-\Delta + I)^{-1}$ in $L^2(R_\epsilon)$ to $(q_0\Delta + I)^{-1}$ in $L^2(0,1)$". It is a kind of convergence in operator norm, when the operators are defined in different functional spaces (F. Stummel, G. Vainikko)

Convergence

The good news is that this "Compact Convergence" implies, among other things, the convergence of the eigenvalues and the eigenprojections. So if we consider the eigenvalue problems:

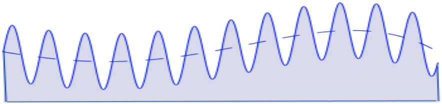
$$\begin{cases} -\Delta u^\epsilon + u^\epsilon = \lambda_\epsilon u^\epsilon \quad \text{in } R^\epsilon \\ \partial_N u^\epsilon = 0 \quad \text{on } \partial R^\epsilon \end{cases} \qquad \begin{cases} -q_0 u'' + u = \lambda u \quad \text{in } (0,1) \\ u'(0) = u'(1) = 0 \end{cases}$$

then,

$$\lambda_n^\epsilon \xrightarrow{\epsilon \to 0} \lambda_n$$

and similar statements for the eigenprojections.

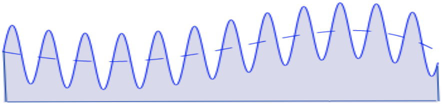
Locally Periodic with constant period



$$R_\epsilon = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1,\ 0 < y < \epsilon(a(x) + g(x/\epsilon))\}.$$

for a function $a \in C(0,1)$ and $g(\cdot)$ is L-periodic.

Locally Periodic with constant period



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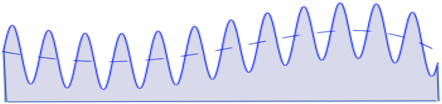


$$R_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1,\ 0 < y < \epsilon(a(x) + g(x/\epsilon))\}.$$

for a function $a \in C(0,1)$ and $g(\cdot)$ is L-periodic.

The most general case: $R_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1,\ 0 < y < \epsilon G(x, x/\epsilon)\}$. where $G(x, \cdot)$ is L-periodic is analyzed in:

JA, M. Pereira *Homogenization in a thin domain with an oscillatory boundary*, J. Math. Pures et Appl. 96 (1), pp. 29-57 (2011).

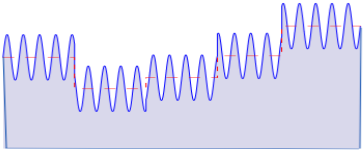
M. Mascarenhas, D. Polisevski *The warping, the torsion and the Neumann...*, Model. Math. Anal Numer. 28 (1994).

D. Chenais, M. Mascarenhas, L.Trabucho *On the optimization of non-periodic...*, Model. Math. Anal Numer. 31 (1997).

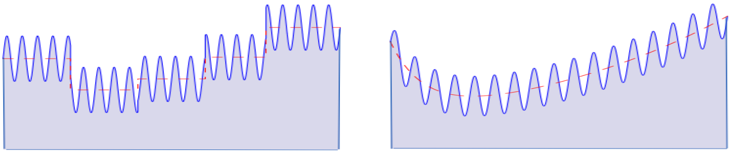
G.A. Chechkin, L. Piatnitski *Homogenization of boundary-value problems...*, Appl. Anal 71 (1999).

Piecewise constant amplitude

We approximate the function $a(\cdot)$ by piecewise constant functions $a_\delta(\cdot)$ so that $\|a - a_\delta\|_{L^\infty(0,1)} \leq \delta$. Then for a_δ we get the domain:



$$(\mathrm{Eq})_\delta : \quad \int_0^1 \left(q_\delta(x)|Y_\delta^*(x)|u_0'\varphi' + |Y_\delta^*(x)|u_0\varphi\right) dx = \int_0^1 |Y_\delta^*(x)| f\varphi dx$$

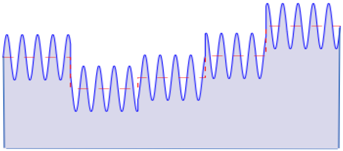


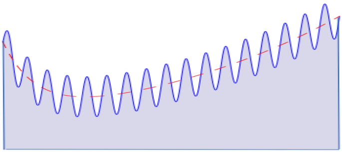
R_ϵ^δ $\qquad\qquad$ R_ϵ

$$R_\epsilon^\delta = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1,\ 0 < y < \epsilon(a_\delta(x) + g(x/\epsilon))\}.$$
$$R_\epsilon = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1,\ 0 < y < \epsilon(a(x) + g(x/\epsilon))\}.$$

where $a_\delta(x)$ is a piecewise constant function satisfying

$$\|a_\delta - a\|_{L^\infty(0,1)} \leq \delta.$$



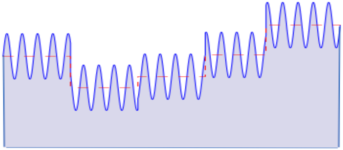


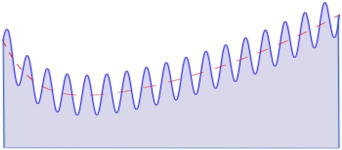
$$\downarrow (\epsilon \to 0)$$
$$\downarrow$$

$$(\text{Equation})_\delta \quad \xrightarrow{\delta \to 0} \quad (\text{Equation})_0$$

$$(\text{Eq})_\delta : \quad \int_0^1 \left(q_\delta |Y_\delta^*| u_0' \varphi' + |Y_\delta^*| u_0 \varphi\right) dx = \int_0^1 |Y_\delta^*| f \varphi dx$$

$$(\text{Eq})_0 : \quad \int_0^1 \left(q |Y^*| u_0' \varphi' + |Y^*| u_0 \varphi\right) dx = \int_0^1 |Y^*| f \varphi dx$$





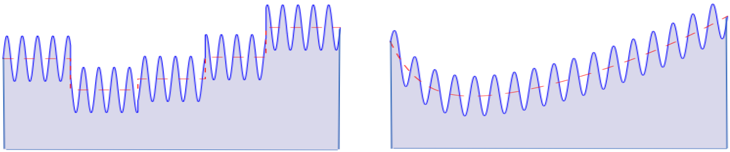
$$\downarrow (\epsilon \to 0)$$
$$\downarrow$$

$$(\text{Equation})_\delta \quad \xrightarrow{\delta\to 0} \quad (\text{Equation})_0$$

$$(\text{Eq})_\delta : \quad \int_0^1 (q_\delta |Y_\delta^*| u_0' \varphi' + |Y_\delta^*| u_0 \varphi)\, dx = \int_0^1 |Y_\delta^*| f \varphi dx$$

$$(\text{Eq})_0 : \quad \int_0^1 (q |Y^*| u_0' \varphi' + |Y^*| u_0 \varphi)\, dx = \int_0^1 |Y^*| f \varphi dx$$

$$q = q(x), \quad Y^* = Y^*(x) = \{(y_1, y_2), 0 < y_1 < L, 0 < y_2 < a(x) + g(y_1)\}$$



$$\downarrow (\epsilon \to 0) \qquad\qquad \downarrow ?$$

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$$(\text{Equation})_\delta \quad \xrightarrow{\delta \to 0} \quad (\text{Equation})_0$$

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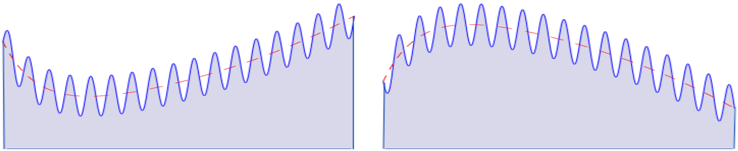
$$q = q(x), \quad Y^* = Y^*(x) = \{(y_1, y_2), 0 < y_1 < L, 0 < y_2 < a(x) + g(y_1)\}$$

We were able to show that the solutions depend continuously on the function $a(x)$ uniformly in ϵ:



R_ϵ $\qquad\qquad$ $\hat{R}_\epsilon$

$$R_\epsilon = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1,\, 0 < y < \epsilon(a(x) + g(x/\epsilon))\}.$$
$$\hat{R}_\epsilon = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1,\, 0 < y < \epsilon(\hat{a}(x) + g(x/\epsilon))\}.$$

with $\alpha_0 \le a(x), \hat{a}(x) \le \beta_0$ and $\|a - \hat{a}\|_{L^\infty(0,1)} \le \delta$. Then, there exists $\rho(\delta) \stackrel{\delta\to 0}{\longrightarrow} 0$

$$\|u_\epsilon - \hat{u}_\epsilon\|^2_{H^1(R_\epsilon \cap \hat{R}_\epsilon)} + \|u_\epsilon\|^2_{H^1(R_\epsilon \setminus \hat{R}_\epsilon)} + \|\hat{u}_\epsilon\|^2_{H^1(\hat{R}_\epsilon \setminus R_\epsilon)} \le \rho(\delta)\|f_\epsilon\|^2, \quad \forall \epsilon > 0$$

The homogenized limit problem

$$\begin{cases} -\dfrac{1}{|Y^*(x)|}\left(r(x)u_x\right)_x + u = f, \quad x \in (0,1) \\ u'(0) = u'(1) = 0 \end{cases}$$

where

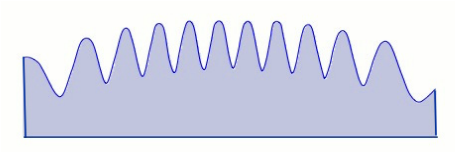
$$r(x) = \int_{Y^*(x)} \left\{1 - \frac{\partial X(x)}{\partial y_1}(y_1, y_2)\right\} dy_1 dy_2.$$

where $X(x)$ is the unique solution (up to an additive constant) which is L-periodic in the first variable, of the problem

$$\begin{cases} -\Delta X(x) = 0 \text{ in } Y^*(x) \\ \dfrac{\partial X(x)}{\partial N} = 0 \text{ on } B_2(x) \\ \dfrac{\partial X(x)}{\partial N} = N_1(x) \text{ on } B_1(x) \end{cases}$$

$$Y^*(x) = \{(y_1, y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 < L, \quad 0 < y_2 < a(x) + g(y_1)\}.$$

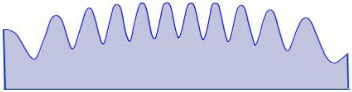
Locally periodic oscillations with varying period



J.A, M. Villanueva-Pesqueira *Locally periodic thin domains with varying period*", *C. R. Acad. Sci. Paris.* Serie I, 352, (2014) 397-403

J.A., M. Villanueva-Pesqueira *Unfolding operator method for thin domains with a locally periodic highly oscillatory boundary, SIAM J. Math. Anal. 48 - 3 (2016) pp 1634-1671*

$$R_\epsilon = \{(x,y) : 0 < x < 1; 0 < y < \epsilon G(x, x/\epsilon)\}$$



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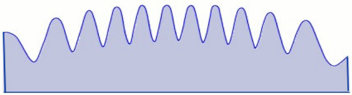


Hypothesis

- $G : (0,1) \times \mathbb{R} \longrightarrow (0,+\infty)$ is a smooth function, there exist G_0 and G_1 such that $0 < G_0 \leq G(x,y) \leq G_1$ and $G(x,\cdot)$ is $l(x)$-periodic.
- $l(\cdot)$ is a smooth function such that $0 < l_0 \leq l(x) < l_1$ and

$$xl'(x) < l(x) \quad \forall x \in I.$$

Example:

$$G(x,y) = 2 + \cos(\frac{2\pi y}{l(x)}), \quad \text{so that} \quad G(x, x/\epsilon) = 2 + \cos(\frac{2\pi x}{\epsilon l(x)})$$

Unfolding operator method

T. Arbogast, J. Douglas, U. Hornung *Derivation of the double porosity model....*, SIAM J. Math. Anal., 21 (1990), 823-836.

J. Casado-Díaz *Two scale convergence for nonlinear Dirichlet problems in ...*, Proc. Roy. Soc. Edinburgh 130 A (2000), 249-276.

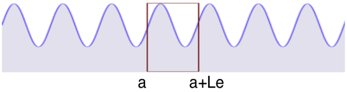
D. Cioranescu, A. Damlamian, G. Griso *Periodic unfolding and homogenization*, C. R. Acad. Sci. Paris, Série 1, 335 (2002), 99-104.

D. Cioranescu, A. Damlamian, G. Griso *The periodic unfolding method in hom...*, SIAM J. Math. Anal. Vol. 40, 4 (2008), 1585-1620

A. Damlamian and K. Pettersson *Homogenization of oscillating boundaries*, Discrete and Continuous Dynamical Systems 23, (2009), 197-219.

D. Blanchard, A. Gaudiello and G. Griso , *Junction of a periodic family of elastic rods with a 3d plate, Part I*, J. Math. Pures Appl., 88 (2007), 1-33.

$$R^{\epsilon} = \{(x,y) : 0 < x < 1;\ 0 < y < \epsilon g(x/\epsilon)\}$$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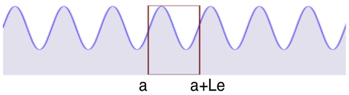
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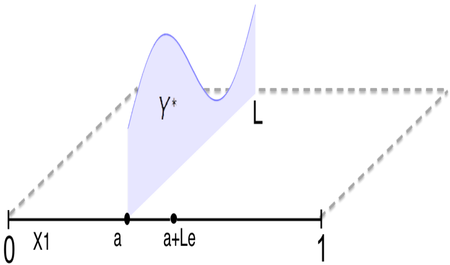
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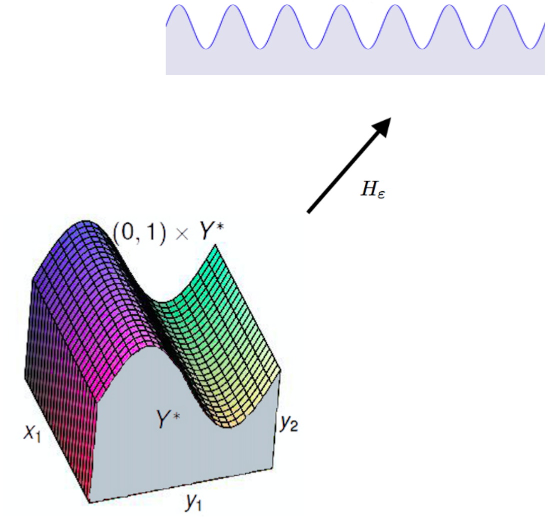
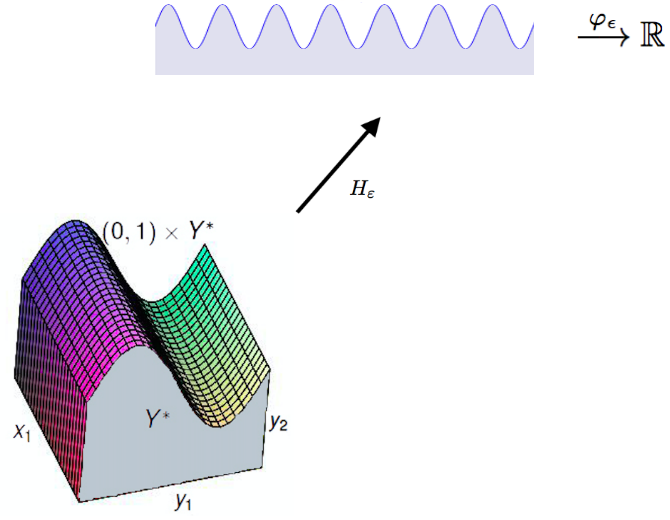
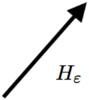


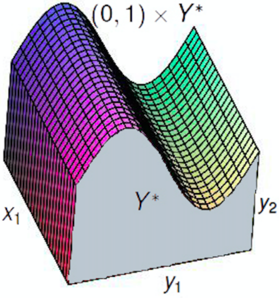
H_ε
$(0,1) \times Y^*$
Y^*
x_1
y_2
y_1


$\xrightarrow{\varphi_\epsilon} \mathbb{R}$
H_ε
$(0,1) \times Y^*$
Y^*
x_1
y_1
y_2

$\xrightarrow{\varphi_\epsilon} \mathbb{R}$



H_ε



$$\begin{aligned} \mathcal{T}_\varepsilon : L^1(R_\varepsilon) &\longrightarrow L^1((0,1) \times Y^*) \\ \varphi_\varepsilon &\longrightarrow \varphi_\varepsilon \circ H_\varepsilon \end{aligned}$$

Main Property of Unfolding Operator

Unfolding criterion for integrals (u.c.i.) :

$$\frac{1}{L}\int_{(0,1)\times Y^*} \mathcal{T}_\epsilon(\varphi)(x_1, y_1, y_2)dx_1 dy_1 dy_2 = \frac{1}{\epsilon}\int_{R^\epsilon} \varphi(x_1, x_2)dx_1 dx_2,$$

$\forall \varphi \in L^1(R^\epsilon)$.

- $\mathcal{T}_\epsilon$ is a continuous operator from $L^2(R^\epsilon)$ to $L^2\big((0,1)\times Y^*\big)$.

$$\frac{1}{L}\|\mathcal{T}_\epsilon(\varphi)\|^2_{L^2\left((0,1)\times Y^*\right)} = \frac{1}{\epsilon}||\varphi||^2_{L^2(R^\epsilon)}.$$

- $\nabla_{(y_1,y_2)}\mathcal{T}_\epsilon(\varphi)(x_1, y_1, y_2) = \epsilon\mathcal{T}_\epsilon(\nabla_{(x_1,x_2)}\varphi)(x_1, y_1, y_2)$, $\forall\varphi \in L^2(R^\epsilon)$.

Compactness Theorem

Let $\varphi^\epsilon \in H^1(R^\epsilon)$, with $\frac{1}{\epsilon}||\varphi^\epsilon||^2_{H^1(R^\epsilon)}$ uniformly bounded. Then,

1. There exists φ in $H^1(0,1)$ such that, up to subsequences:
   $\mathcal{T}_\epsilon(\varphi^\epsilon) \rightharpoonup \varphi$, $\mathrm{w} - L^2((0,1); H^1(Y^*))$.

2. There exists φ_1 in $L^2((0,1); H^1(Y^*))$ L-periodic in the second variable, such that, up to subsequences:

$$\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x_1}\Big) \rightharpoonup \frac{\partial \varphi}{\partial x_1}(x_1) + \frac{\partial \varphi_1}{\partial y_1}(x_1, y_1, y_2) \quad \mathrm{w} - L^2((0,1) \times Y^*).$$

$$\mathcal{T}_\epsilon\Big(\frac{\partial \varphi^\epsilon}{\partial x_2}\Big) \rightharpoonup \frac{\partial \varphi_1}{\partial y_2}(x_1, y_1, y_2) \quad \mathrm{w} - L^2((0,1) \times Y^*).$$

How to obtain the limit problem

The variational formulation: find $u^\epsilon \in H^1(R^\epsilon)$ such that

$$\int_{R^\epsilon} \Big\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \Big\} dx_1 dx_2 = \int_{R^\epsilon} f \varphi dx_1 dx_2, \ \forall \varphi \in H^1(R^\epsilon).$$

By the unfolding criterion for integrals we have:

$$\int_{(0,1)\times Y^*} \Big\{ \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x_1}\Big)\mathcal{T}_\epsilon\Big(\frac{\partial \varphi}{\partial x_1}\Big) + \mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x_2}\Big)\mathcal{T}_\epsilon\Big(\frac{\partial \varphi}{\partial x_2}\Big) + \mathcal{T}_\epsilon(u^\epsilon)\mathcal{T}_\epsilon(\varphi) \Big\} dx_1 dy_1 dy_2$$
$$= \int_{(0,1)\times Y^*} \mathcal{T}_\epsilon(f)\mathcal{T}_\epsilon(\varphi) dx_1 dy_1 dy_2, \ \forall \varphi \in H^1(R^\epsilon).$$

Taking $\varphi = u^\epsilon$:

$$\|\mathcal{T}_\epsilon(u^\epsilon)\|_{L^2((0,1)\times Y^*)}, \Big\|\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x_1}\Big)\Big\|_{L^2((0,1)\times Y^*)}, \Big\|\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x_2}\Big)\Big\|_{L^2((0,1)\times Y^*))} \le C \quad \forall \epsilon > 0.$$

Hence, there exist $u \in H^1(0,1)$ and $u_1 \in L^2((0,1); H^1(Y^*))$ such that

$$\mathcal{T}_\epsilon(u^\epsilon) \rightharpoonup u \quad \text{w- } L^2\big((0,1)\times Y^*\big),$$

$$\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x_1}\Big) \rightharpoonup \frac{\partial u}{\partial x_1} + \frac{\partial u_1}{\partial y_1} \quad \text{w- } L^2((0,1)\times Y^*),$$

$$\mathcal{T}_\epsilon\Big(\frac{\partial u^\epsilon}{\partial x_2}\Big) \rightharpoonup \frac{\partial u_1}{\partial y_2} \qquad \text{w- } L^2((0,1)\times Y^*).$$

Passing to the limit we obtain:

$$\int_{(0,1)\times Y^*} \Big\{ \Big(\frac{\partial u}{\partial x_1}(x_1) + \frac{\partial u_1}{\partial y_1}(x_1, y_1, y_2) \Big) \frac{\partial \phi}{\partial x_1}(x_1) + u(x_1)\phi(x_1) \Big\} dx_1 dy_1 dy_2$$
$$= \int_{(0,1)\times Y^*} f(x_1)\phi(x_1) dx_1 dy_1 dy_2 \quad \forall \phi \in H^1(0,1).$$

Using suitable test functions in the variational formulation we get:
$u_1(x_1, y_1, y_2) = -X(y_1, y_2)\frac{\partial u}{\partial x_1}(x_1), \quad \forall (x_1, y_1, y_2) \in (0,1) \times Y^*.$

$$\begin{cases} -q_0 w_{xx} + w = f(x), \quad x \in (0,1) \\ w'(0) = w'(1) = 0 \end{cases}$$

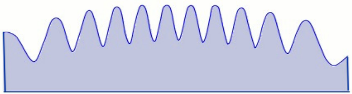
$$q_0 = \frac{1}{|Y^*|}\int_{Y^*} \left\{1 - \frac{\partial X}{\partial y_1}(y_1, y_2)\right\} dy_1 dy_2,$$

where X is the unique solution (up to additive constants) which is L-periodic in the first variable, of the problem:

$$\begin{cases} -\Delta X = 0 \text{ in } Y^* \\ \frac{\partial X}{\partial N} = 0 \text{ on } B_2 \\ \frac{\partial X}{\partial N} = N_1 \text{ on } B_1 \end{cases}$$

Locally periodic oscillations with varying period

$$R_\epsilon = \{(x, y) : 0 < x < 1; 0 < y < \epsilon G(x, x/\epsilon)\}$$



Locally periodic oscillations with varying period

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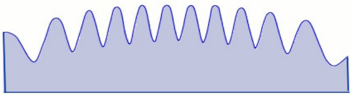
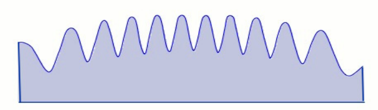


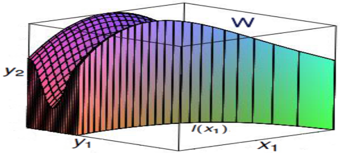
$$Y^*(x_1) = \{(y_1, y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 < l(x_1), \quad 0 < y_2 < G(x_1, y_1)\}$$

Locally periodic oscillations with varying period

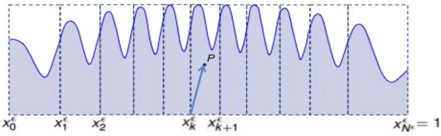
$$R_\epsilon = \{(x, y) : 0 < x < 1; 0 < y < \epsilon G(x, x/\epsilon)\}$$

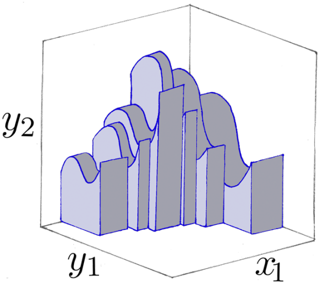


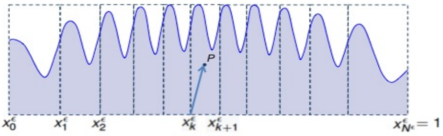
$$Y^*(x_1) = \{(y_1, y_2) \in \mathbb{R}^2 \ : \ 0 < y_1 < l(x_1), \quad 0 < y_2 < G(x_1, y_1)\}$$

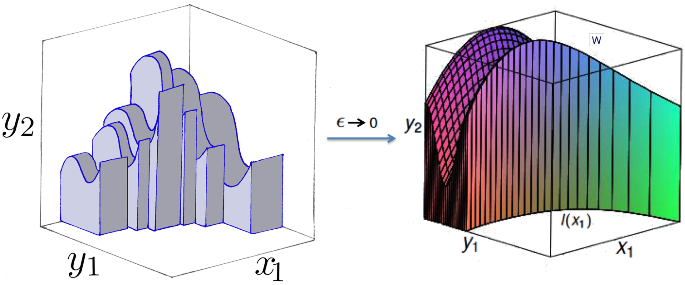


$$W = \{(x_1, y_1, y_2) \in \mathbb{R}^3 \ : \ x_1 \in (0, 1), (y_1, y_2) \in Y^*(x_1)\}.$$


P
x_0^ϵ
x_1^ϵ
x_2^ϵ
x_k^ϵ
x_{k+1}^ϵ
$x_{N^\epsilon}^\epsilon = 1$

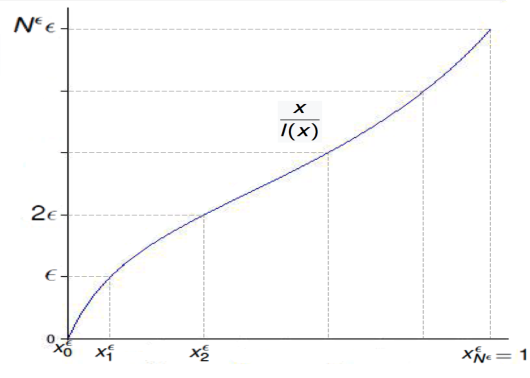

y_2
y_1
x_1


P
x_0^ϵ
x_1^ϵ
x_2^ϵ
x_k^ϵ
x_{k+1}^ϵ
$x_{N^\epsilon}^\epsilon = 1$


y_2
y_1
x_1
$\epsilon \to 0$
y_2
W
$l(x_1)$
y_1
x_1

We consider the following partition for the interval $[0, 1]$

$$x_0^\epsilon = 0 < x_1^\epsilon = \epsilon l(x_1^\epsilon) < x_2^\epsilon = 2\epsilon l(x_2^\epsilon) < \cdots < x_{N^\epsilon}^\epsilon = N^\epsilon \epsilon l(x_{N^\epsilon}^\epsilon) = 1.$$



The homogenized limit problem

$$\begin{cases} -\dfrac{l(x)}{|Y^*(x)|}\big(r(x)u_x\big)_x + u = f, \quad x\in(0,1) \\ u'(0)=u'(1)=0 \end{cases}$$

where

$$r(x)=\frac{1}{l(x)}\int_{Y^*(x)}\Big\{1-\frac{\partial X(x)}{\partial y_1}(y_1,y_2)\Big\}dy_1dy_2.$$

and $X(x)$ is the unique solution (up to an additive constant) which is $l(x)$-periodic in the first variable, of the problem

$$\begin{cases} -\Delta X(x) & = 0 \text{ in } Y^*(x) \\ \partial_N X(x) & = 0 \text{ on } B_2(x) \qquad (\text{ lower boundary}) \\ \partial_N X(x) & = N_1(x) \text{ on } B_1(x) \qquad (\text{ upper boundary}) \end{cases}$$

in the representative cell $Y^*(x)$ given by

$$Y^*(x)=\{(y_1,y_2)\in\mathbb{R}^2 \,:\, 0<y_1<l(x), \quad 0<y_2<G(x,y_1)\}.$$

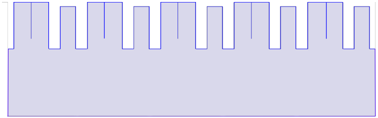
We also obtain the corrector result:

$$\epsilon^{-1}\|u_\epsilon - u\|^2_{L^2(\mathbb{R}^\epsilon)} + \epsilon^{-1}\|u_\epsilon - u + \epsilon\frac{\partial u}{\partial x}X^\epsilon\|^2_{H^1(\mathbb{R}^\epsilon)} \xrightarrow{\epsilon\to 0} 0$$

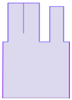
where $X^\epsilon(x,y) \equiv X(x)(x/\epsilon, y/\epsilon)$.

Remarks about the unfolding method

• The Unfolding method allows us to treat thin domains for more general functions g. In particular,



where the basic cell Y^* looks like



JA, M. Villanueva-Pesqueira *Thin domains with non-smooth periodic oscillatory boundaries*, J. Math. Anal Appl. 446 (2017) 130-164.

As a matter of fact, if we consider a thin domain:

$$R^\epsilon = \{(x,y) : 0 < x < 1;\ 0 < y < \epsilon g(x/\epsilon)\}$$

we just need that $g : [0,1] \to \mathbb{R}^+$ to be L-periodic, be defined at all points and satisfy

- $\exists\ g_0, g_1 > 0$ such that $g_0 \leq g(x) \leq g_1$
- g is lowersemicontinuous, that is $g(x_0) \leq \liminf_{x \to x_0} g(x)$, $\forall x_0 \in \mathbb{R}$.

The fact that g is lower-semicont implies that the basic cell

$$Y^* = \{(y_1, y_2) : 0 < y_1 < L,\ 0 < y_2 < g(y_1)\}$$

is an open set.

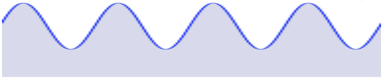
We may also consider the situation:

$$R^{\epsilon} = \{(x,y) : 0 < x < 1; 0 < y < \epsilon g(x/\epsilon^{\alpha})\}$$

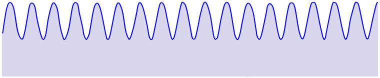
where $g(\cdot)$ $L-$periodic and the parameter $\alpha > 0$.

Observe that

• If $0 < \alpha < 1$ we have a weak oscillatory boundary.



• If $\alpha > 1$ we have a highly oscillatory boundary.



and the homogenized limit is different.

- If $0 < \alpha < 1$:

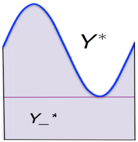
$$\begin{cases} -\dfrac{1}{\mathcal{M}(g)\mathcal{M}\left(\frac{1}{g}\right)} u_{xx} + u = f(x), \quad x \in (0,1), \\ u'(0) = u'(1) = 0. \end{cases}$$

- If $\alpha > 1$:

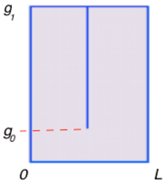
$$\begin{cases} -\dfrac{g_0}{\mathcal{M}(g)} u_{xx} + u = f, \quad x \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases}$$

where $g_0 = \min_{x \in [0,L]} g(x)$. As a matter of fact: $\dfrac{g_0}{\mathcal{M}(g)} = \dfrac{|Y_-^*|}{|Y^*|}$ where

$$Y_-^* = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \in (0, L), 0 < y_2 < g_0\}$$



Example

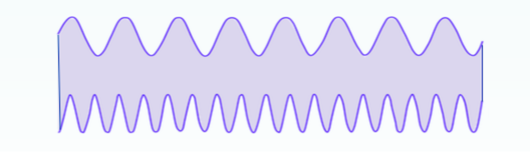


Then the limit equation is $-du_{xx} + u = f$, where

- For $0 < \alpha < 1$, $d = 1$. The limit equation does not see the crack.
- For $\alpha > 1$, $d = \dfrac{g_0}{g_1}$
- For $\alpha = 1$ $d = \frac{1}{|Y^*|}\displaystyle\int_{Y^*} \left\{1 - \frac{\partial X}{\partial y_1}(y_1, y_2)\right\} dy_1 dy_2$, where X is the solution ....

Oscillations in both boundaries

Let us consider now a domain of the type



$$R_\epsilon = \{(x, y) : 0 < x < 1;\ -\epsilon h(x/\epsilon^\alpha) < y < \epsilon g(x/\epsilon^\beta)\}$$

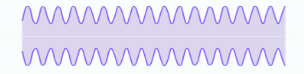
$h(\cdot)$ L_1-periodic and $g(\cdot)$ L_2- periodic.

$$0 \leq h(\cdot) \leq h_1, \quad 0 < g_0 \leq g(\cdot) \leq g_1$$

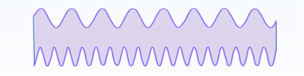
where

$$\min_{x\in[0,L_1]} h(x) = 0, \qquad \min_{x\in[0,L_2]} g(x) = g_0$$

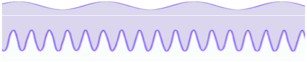
FF $\alpha > 1, \beta > 1$



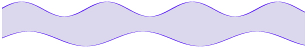
FR $\alpha > 1, \beta = 1$



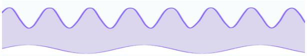
FW $\alpha > 1, \beta < 1$



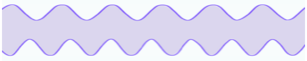
WW $\alpha < 1, \beta < 1$



WR $\alpha < 1, \beta = 1$

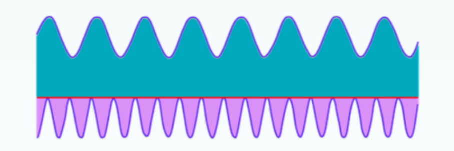


RR $\alpha = \beta = 1$



Cases *FF*, *FR*, *FW*

Whenever there is a "fast" oscillatory boundary ($\alpha > 1$) we can obtain the limit equation, since we can control the behavior of the solution in the "fast" boundary "independently" of the shape of other part of the thin domain.



As a matter of fact, we get the following limit problem:

$$\begin{cases} -q_0 u_{xx} + u = f(x), \quad x \in (0,1) \\ u'(0) = u'(1) = 0 \end{cases}$$

where

$$q_0 = \begin{cases} \dfrac{g_0}{\mathcal{M}(g) + \mathcal{M}(h)} & \text{if } \alpha > 1, \beta > 1 \\[2ex] \dfrac{\hat{q}}{\mathcal{M}(g) + \mathcal{M}(h)} & \text{if } \alpha > 1, \beta = 1 \\[2ex] \dfrac{\frac{1}{\mathcal{M}(1/g)}}{\mathcal{M}(g) + \mathcal{M}(h)} & \text{if } \alpha > 1, \beta < 1 \end{cases}$$

where

$$\hat{q} = \frac{1}{|Y^*|} \int_{Y^*} \left\{ 1 - \frac{\partial X}{\partial y_1}(y_1, y_2) \right\} dy_1 dy_2$$

and

$$Y^* = \{(y_1, y_2) : 0 < y_1 < L_2,\ 0 \leq y_2 \leq g(y_1)\}$$

Case *WW*

The domain is given by

$$R_\epsilon\{(x,y) : 0 < x < 1,\ -\epsilon h(x/\epsilon^\alpha) < y < \epsilon g(x/\epsilon^\beta)\}$$

with $\alpha, \beta < 1$. Since the function $x \to \epsilon h(x/\epsilon^\alpha)$ and $x \to \epsilon g(x/\epsilon^\beta)$ are C^1 functions, we can transform the domain in a rectangular one and after passing to the limit, we get

$$\begin{cases} -\dfrac{p}{\mathcal{M}(g)+\mathcal{M}(h)} u_{xx} + u = f(x), \quad x \in (0,1) \\ u'(0) = u'(1) = 0 \end{cases}$$

Case *WW*

The domain is given by

$$R_\epsilon\{(x,y) : 0 < x < 1,\ -\epsilon h(x/\epsilon^\alpha) < y < \epsilon g(x/\epsilon^\beta)\}$$

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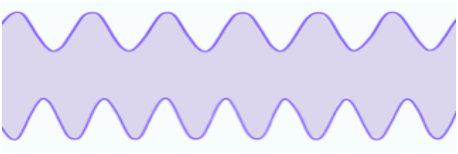
$$\begin{cases} -\dfrac{p}{\mathcal{M}(g)+\mathcal{M}(h)} u_{xx} + u = f(x), \quad x \in (0,1) \\ u'(0) = u'(1) = 0 \end{cases}$$

with

$$\frac{1}{p} = \begin{cases} \displaystyle\lim_{T\to+\infty} \frac{1}{T}\int_0^T \frac{1}{g(s)+h(s)} ds, & \alpha = \beta \\ \displaystyle\frac{1}{L_1 L_2}\int_0^{L_1}\int_0^{L_2} \frac{1}{g(y)+h(z)} dydz, & \alpha \neq \beta \end{cases}$$

Cases ***WR*** **and** ***RR***. (Work in progress).

Observe that the *RR* case contains the “quasiperiodic” resonant case:



$$R_\epsilon = \{(x,y) : 0 < x < 1, -\epsilon h(\frac{x}{\epsilon}) < y < \epsilon g(\frac{x}{\epsilon})\}$$

and such that L_1, L_2 are rationally independent.

THANKS