Warsaw University of Technology
Faculty of Mathematics and Information Science

Message hidden in the Independence of Matrix–Kummer and Wishart Matrices

Agnieszka Piliszek

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Plan

1. Setting
2. HV matrix characterization

Matrix setting

Ω_+ - cone of symmetric, positive definite matrices $n \times n$.

Two distributions

Random matrix Y has Wishart distribution with parameters $b > (r-1)/2$ and $\Sigma \in \Omega_+$ ($Y \sim \mathcal{W}(b, \Sigma)$) if it has the density

$$\mathcal{W}(b, \Sigma)(dy) = C_1 (\det y)^{b-(r+1)/2} e^{-\langle \Sigma, y \rangle} I_{\Omega_+}(y) dy.$$

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Random matrix X has Matrix–Kummer distribution with parameters $a > (r-1)/2, b \in \mathbb{R}, \Sigma \in \Omega_+$ ($X \sim \mathcal{MK}(a, b, \Sigma)$) if it has the density

$$\mathcal{MK}(a, b, \Sigma)(dx) = C_2 (\det x)^{a-\frac{r+1}{2}} (\det(I_n + x))^{-(a+b)} e^{-\langle \Sigma, x \rangle} I_{\Omega_+}(x) dx.$$

HV matrix characterization

Notation: $\circ$:

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Theorem 1 (2017+)

Let $\mathbf{X}$ and $\mathbf{Y}$ be independent random matrices in Ω_+ with positive and continuous densities.
Then

$$\mathbf{U} = (1+\mathbf{X})^{-1} \circ \mathbf{Y} \textit{ and } \mathbf{V} = [1 + (1+\mathbf{X})^{-1} \circ \mathbf{Y}] \circ \mathbf{X}$$

are independent if and only if $\mathbf{X} \sim \mathcal{MK}(a, b, c\mathbf{e})$ and $\mathbf{Y} \sim \mathcal{W}(a+b, c\mathbf{e})$, where $a > (r-1)/2$, $b > (r-1)/2 - a$, $c > 0$.

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where $a > (r-1)/2$*,* $b > (r-1)/2 - a$*,* $c > 0$*.*
Moreover $\mathbf{U} \sim \mathcal{MK}(a+b, -b, c\mathbf{e})$ *and* $\mathbf{V} \sim \mathcal{W}(a, c\mathbf{e})$*.*

Note:

- $\mathbf{U}, \mathbf{V} \in \Omega_+$

HV matrix characterization–remarks

We have

$$\mathbf{U} = (1+\mathbf{X})^{-1} \circ \mathbf{Y}, \tag{1}$$

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$$\mathbf{V} = [I_n + \mathbf{U}] \circ \mathbf{X} = [I_n + \mathbf{U}]^{1/2} \mathbf{X} [I_n + \mathbf{U}]^{1/2}$$

$$\Downarrow$$

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- $\mathbf{Y} = (I_n + \mathbf{X}) \circ \mathbf{U} = (I_n + [I_n + \mathbf{U}]^{-1} \circ \mathbf{V}) \circ \mathbf{U}.$

HV matrix characterization–remarks

Theorem 2 (2017+)

Let $\mathbf{X}$ *and* $\mathbf{Y}$ *be independent random matrices in* Ω_+ *with positive and continuous densities.*
Then $\mathbf{U} = (I_n + \mathbf{X})^{-1} \circ \mathbf{Y}$ *and* $\mathbf{V} = [I_n + (I_n + \mathbf{X})^{-1} \circ \mathbf{Y}] \circ \mathbf{X}$ *are independent if and only if* $\mathbf{X} \sim \mathcal{MK}(a, b, c\mathbf{e})$ *and* $\mathbf{Y} \sim \mathcal{W}(a + b, c\mathbf{e})$*, where* $a > (r-1)/2$*,* $b > (r-1)/2 - a$*,* $c > 0$*.*

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"$\Rightarrow$" Straightforward – just solve functional equations

$$a(x) + b(y) = c((I_n + x)^{-1} \circ y) + d([I_n + (I_n + x)^{-1} \circ y] \circ x) \quad x, y \in \Omega_+$$

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- use 1–dimensional theorem (P.,Wesołowski, 2016)

- Details on arXiv:1706.09718, A. Piliszek, Independence characterization for Wishart and Kummer matrices.
- A. Piliszek, J. Wesołowski, Kummer and gamma laws through independences on trees–Another parallel with the Matsumoto–Yor property. J. Multivar. Anal. **152** (2016), 15–27.

The end

Thank you!