Comparison of weak and strong moments for vectors with independent coordinates

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Wstęp

In many problems arising in probability theory and mathematical statistics one needs to study variables of the form

$$S = \sup_{t \in T} \Big| \sum_{i=1}^{n} t_i X_i \Big|,$$

where $X_1, \ldots, X_n$ are independent r.v's and T is a nonempty subset of $\mathbb{R}^n$.

In particular it is of interest to estimate tails of S (i.e. $\mathbb{P}(S \geq t)$, $t \geq 0$). Such estimates are strictly related to bounds for L_p-norms of S (i.e. $\|S\|_p := (\mathbb{E}|S|^p)^{1/p}$, $p \geq 1$).

There is a a trivial lower estimate:

$$\Big(\mathbb{E} \sup_{t \in T} \Big| \sum_{i=1}^{n} t_i X_i \Big|^p \Big)^{1/p} \geq \max \Big\{ \mathbb{E} \sup_{t \in T} \Big| \sum_{i=1}^{n} t_i X_i \Big|, \sup_{t \in T} \Big(\mathbb{E} \Big| \sum_{i=1}^{n} t_i X_i \Big|^p \Big)^{1/p} \Big\}. \tag{1}$$

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Gaussian case

Let $G = (g_1, \ldots, g_n)$, where g_i are i.i.d. $\mathcal{N}(0,1)$. Gaussian concentration states that for any L-Lipschitz function f,

$$\mathbb{P}(|f(G) - \mathbb{E}f(G)| \geq t) \leq \exp(-\frac{t^2}{2L^2})$$

Integrating by parts we get for $p \geq 1$,

$$(\mathbb{E}|f(G) - \mathbb{E}f(G)|^p)^{1/p} \leq C\sqrt{p}L.$$

Hence by the triangle inequality in L_p,

$$\|f(G)\|_p \leq |\mathbb{E}f(G)| + C\sqrt{p}L.$$

The function $x \mapsto \sup_{t\in T} |\sum_i t_i x_i|$ has the Lipschitz constant $\sup_{t\in T} |t|_2$, moreover $\|\sum_i t_i g_i\|_p = |t| \|g_1\|_p \sim |t|\sqrt{p}$, therefore

$$\left(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i g_i\Big|^p\right)^{1/p} \leq \mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i g_i\Big| + C\sup_{t\in T}\left(\mathbb{E}\Big|\sum_{i=1}^n t_i g_i\Big|^p\right)^{1/p}.$$

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Rademachers and variables with log-concave tails

In the case when X_i is the Rademacher sequence (i.e. sequence of i.i.d. symmetric ± 1-valued r.v's) Dilworth and Montgomery-Smith (1993) showed that

$$\left(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^{n} t_i X_i\Big|^p\right)^{\frac{1}{p}} \leq C_1 \mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^{n} t_i X_i\Big| + C_2 \sup_{t\in T}\left(\mathbb{E}\Big|\sum_{i=1}^{n} t_i X_i\Big|^p\right)^{\frac{1}{p}}.$$

This inequality was generalized (L'96) to the case when X_i are symmetric with log-concave tails (i.e. $t \mapsto \ln \mathbb{P}(|X_i| \geq t)$ is concave from $[0,\infty)$ to $[-\infty, 0]$).

Recently Strzelecka, Strzelecki and Tkocz showed that for symmetric variables with log-concave tails the inequality holds with $C_1 = 1$.

Estimates discussed above are strictly connected with concentration inequalities (two-level Talagrand's concetration, concentration for convex functions on discrete cube).

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Variables with sublinear growths of moments

One may show that for a r.v's X with log-concave tails $\|X\|_p \leq 2\frac{p}{q}\|X\|_q$ for $p \geq q \geq 1$.
L.-Tkocz'15 proved that if X_i are independent, centered and

$$\|X_i\|_p \leq \alpha\frac{p}{q}\|X_i\|_q \text{ for } p \geq q \geq 1,$$

then

$$\Big(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_iX_i\Big|^p\Big)^{\frac{1}{p}} \leq C_1(\alpha)\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_iX_i\Big| + C_2(\alpha)\sup_{t\in T}\Big(\mathbb{E}\Big|\sum_{i=1}^n t_iX_i\Big|^p\Big)^{\frac{1}{p}}.$$

Strzelecki, Strzelecka, Tkocz'17+ constructed an example showing that $C_1(\alpha) > 1$ for $\alpha \geq 3$.

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Main result

Theorem

Let $X_1, \ldots, X_n$ be centered, independent and

$$\|X_i\|_{2p} \le \alpha \|X_i\|_p \qquad \text{for } p \ge 2 \text{ and } i = 1, \ldots, n, \tag{2}$$

where α is a finite positive constant. Then for $p \ge 1$ and $T \subset \mathbb{R}^n$,

$$\left(\mathbb{E}\sup_{t\in T}\left|\sum_{i=1}^n t_i X_i\right|^p\right)^{\frac{1}{p}} \le C(\alpha)\left[\mathbb{E}\sup_{t\in T}\left|\sum_{i=1}^n t_i X_i\right| + \sup_{t\in T}\left(\mathbb{E}\left|\sum_{i=1}^n t_i X_i\right|^p\right)^{\frac{1}{p}}\right], \tag{3}$$

where $C(\alpha)$ is a constant depending only on α.

Remark Symmetric r.v's such that $\mathbb{P}(|X_i| \ge t) = \exp(-t^r)$, $r \in (0,1)$ satisfy the asumptions, but do not have exponential moments, so there are no dimension-free concentration inequalities for $(X_1, \ldots, X_n)$.

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Ideas of the proof

Standard symmetrization argument shows that we may assume that X_i are symmetric.

We have $\|X_i\|_{2^k p} \leq \alpha^k \|X_i\|_p$ so $\|X_i\|_q \leq \alpha(q/p)^{\log_2 \alpha}\|X_i\|_p$ for $q \geq p \geq 1$.

Variables $Y_i = \mathrm{sgn}(X_i)|X_i|^r$ have sublinear growth of moments for $r = 1/\log_2 \alpha$. This way we may get the assertion for unconditional sets T.

The main tool to go from unconditional to the general case is Talagrand's contraction principle.

Theorem

Let φ_i be 1-Lipschitz functions on $\mathbb{R}$ such that $\varphi_i(0) = 0$. Then for any set T

$$\mathbb{E} \sup_{t \in T} \sum_{i=1}^{n} \varphi_i(t_i)\varepsilon_i \leq \mathbb{E} \sup_{t \in T} \sum_{i=1}^{n} t_i \varepsilon_i.$$

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Optymality of the assumptions

It turns out that in the i.i.d case the main theorem may be reversed.

Theorem

Let $X_1, X_2, \ldots$ be i.i.d. random variables. Assume that there exists a constant L such that for every $p \geq 1$, every n and every non-empty set $T \subset \mathbb{R}^n$ we have

$$\left(\mathbb{E}\sup_{t\in T}\left|\sum_{i=1}^{n} t_i X_i\right|^p\right)^{1/p} \leq L\left[\mathbb{E}\sup_{t\in T}\left|\sum_{i=1}^{n} t_i X_i\right| + \sup_{t\in T}\left(\mathbb{E}\left|\sum_{i=1}^{n} t_i X_i\right|^p\right)^{1/p}\right]. \tag{4}$$

Then

$$\|X_1\|_{2p} \leq \alpha(L)\|X_1\|_p \qquad \text{for } p \geq 2, \tag{5}$$

where $\alpha(L)$ is a constant which depends only on $L \geq 1$.

Optymality of the assumptions - idea of the proof

Comparison of weak and strong moments for $T = \{e_1, \ldots, e_n\}$ gives

$$(\mathbb{E}\max_{i\le n}|X_i|^p)^{1/p} \le L\left(\mathbb{E}\max_{i\le n}|X_i| + (\mathbb{E}|X_i|^p)^{1/p}\right). \qquad (6)$$

Fix $p \ge 2$ and set $n := \lfloor (4L)^{2p} \rfloor + 1$, $A := n^{1/p}\|X_1\|_p$. Then $\mathbb{P}(|X_1| \ge A) \le 1/n$.

Standard bound on the tail of maxima

$$\frac{1}{3}\min\{1, n\mathbb{P}(|X_1|\ge t)\} \le \mathbb{P}\Big(\max_{i\le n}|X_i| \ge t\Big) \le \min\{1, n\mathbb{P}(|X_1|\ge t)\}.$$

together with integration by parts yield

$$\mathbb{E}\max_{i\le n}|X_i| \le A + n^{\frac{1}{p}}\|X_1\|_p, \quad (\mathbb{E}\max_{i\le n}|X_i|^{2p})^{\frac{1}{2p}} \ge (\frac{n}{3})^{\frac{1}{2p}}(\|X_1\|_{2p} - A).$$

Simple calculations show that (6) with $2p$ instead of p implies

$$\|X_1\|_{2p} \le \left(4 + \frac{1}{2L}\right)(16L^2 + 1)\|X_1\|_p.$$

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Standard bound on the tail of maxima

$$\frac{1}{3}\min\{1,n\mathbb{P}(|X_1|\geq t)\}\leq \mathbb{P}\Big(\max_{i\leq n}|X_i|\geq t\Big)\leq \min\{1,n\mathbb{P}(|X_1|\geq t)\}.$$

together with integration by parts yield

$$\mathbb{E}\max_{i\leq n}|X_i|\leq A+n^{\frac{1}{p}}\|X_1\|_p,\quad (\mathbb{E}\max_{i\leq n}|X_i|^{2p})^{\frac{1}{2p}}\geq (\frac{n}{3})^{\frac{1}{2p}}(\|X_1\|_{2p}-A).$$

Simple calculations show that (6) with $2p$ instead of p implies

$$\|X_1\|_{2p}\leq \left(4+\frac{1}{2L}\right)(16L^2+1)\|X_1\|_p.$$

Optymality of the assumptions - idea of the proof

Comparison of weak and strong moments for $T = \{e_1, \ldots, e_n\}$ gives

$$(\mathbb{E}\max_{i\le n}|X_i|^p)^{1/p} \le L\left(\mathbb{E}\max_{i\le n}|X_i| + (\mathbb{E}|X_i|^p)^{1/p}\right). \qquad (6)$$

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Weak and strong tails

Under the assumptions of the main theorem one may also compare weak and strong tails.

Corollary

Assume that X_i, $1 \leq i \leq n$ *are centered, independent and* $\|X_i\|_{2p} \leq \alpha \|X_i\|_p$ *for* $p \geq 1$. *Then for* $u \geq 0$ *and* $T \subset \mathbb{R}^n$,

$$\mathbb{P}\left(\sup_{t \in T}\left|\sum_{i=1}^n t_i X_i\right| \geq C_1(\alpha)\left[\mathbb{E}\sup_{t \in T}\left|\sum_{i=1}^n t_i X_i\right| + u\right]\right) \leq C_2(\alpha)\sup_{t \in T}\mathbb{P}\left(\left|\sum_{i=1}^n t_i X_i\right| \geq u\right), \quad (7)$$

where constants $C_1(\alpha)$ *and* $C_2(\alpha)$ *depend only on* α.

From comparison of moments to comparison of tails

Chebyshev's inequality implies

$$\mathbb{P}(|Y| \geq e\|Y\|_p) \leq e^{-p} \text{ for } p \geq 1.$$

One may reverse this inequality for regular random variables. Using the Paley-Zygmund inequality

$$\mathbb{P}(Z \geq \frac{1}{2}\mathbb{E}Z) \geq \frac{(\mathbb{E}Z)^2}{4\mathbb{E}Z^2}$$

for $Z = |Y|^q$ and choosing in a right way q one may show that if $\|Y\|_{2p} \leq \alpha\|Y\|_p$ for $p \geq 1$, then

$$\mathbb{P}(|Y| \geq c(\alpha)\|Y\|_p) \geq e^{-p} \text{ for } p \geq C(\alpha).$$

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Khinchine-Kahane type inequalities

Corollary

Assume that independent, centered random variables X_i satisfy $\|X_i\|_{2p} \le \alpha \|X_i\|_p$ for $p \ge 1$. Then for $p \ge q \ge 2$ and $T \subset \mathbb{R}^n$ we have

$$\Big(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big|^p\Big)^{\frac{1}{p}} \le C_3(\alpha)\Big(\frac{p}{q}\Big)^{\max\{\frac{1}{2},\log_2\alpha\}}\Big(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_i X_i\Big|^q\Big)^{\frac{1}{q}},$$

where a constant $C_3(\alpha)$ depends only on α.

Remark. *Exponent* $\max\{1/2, \log_2 \alpha\}$ *is optimal*. Indeed, since $\|g\|_p \sim \sqrt{p/e}$ as $p \to \infty$ one cannot go below $1/2$ by the Central Limit Theorem. Moreover, symmetric r.v's with tails $\exp(-t^r)$ have moments of order $p^{1/r}$ and one may check that satisfy the assumptions with $\alpha = 2^{1/r}$, so exponent cannot be lower than $\log_2 \alpha$.

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Proof of the Khinchine-Kahane inequality

Comparison of weak and strong moments implies that it is enough to show for $t \in \mathbb{R}^n$ and $p \geq q \geq 2$,

$$\Big\|\sum_{i=1}^{n} t_i X_i\Big\|_p \leq C_3(\alpha)\Big(\frac{p}{q}\Big)^{\beta}\Big\|\sum_{i=1}^{n} t_i X_i\Big\|_q,$$

where $\beta = \max\{1/2, \log_2 \alpha\}$.

It is enough to establish the bound in the case when $p = 2k$, $q = 2l$, where $k \geq l$ are positive integers and X_i are symmetric.

By hypercontractivity method it is enough to show that for any $u \in \mathbb{R}$ and all i,

$$\|1 + \sigma_{k,l}(\alpha) u X_i\|_{2k} \leq \|1 + u X_i\|_{2l},$$

where $\sigma_{k,l}(\alpha)^{-1} \leq C(\alpha)(k/l)^{\beta}$

One may show it with $C(\alpha) = 2\sqrt{2}e\alpha$ expanding even moments and using the standard estimates.

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Thank you for your attention!