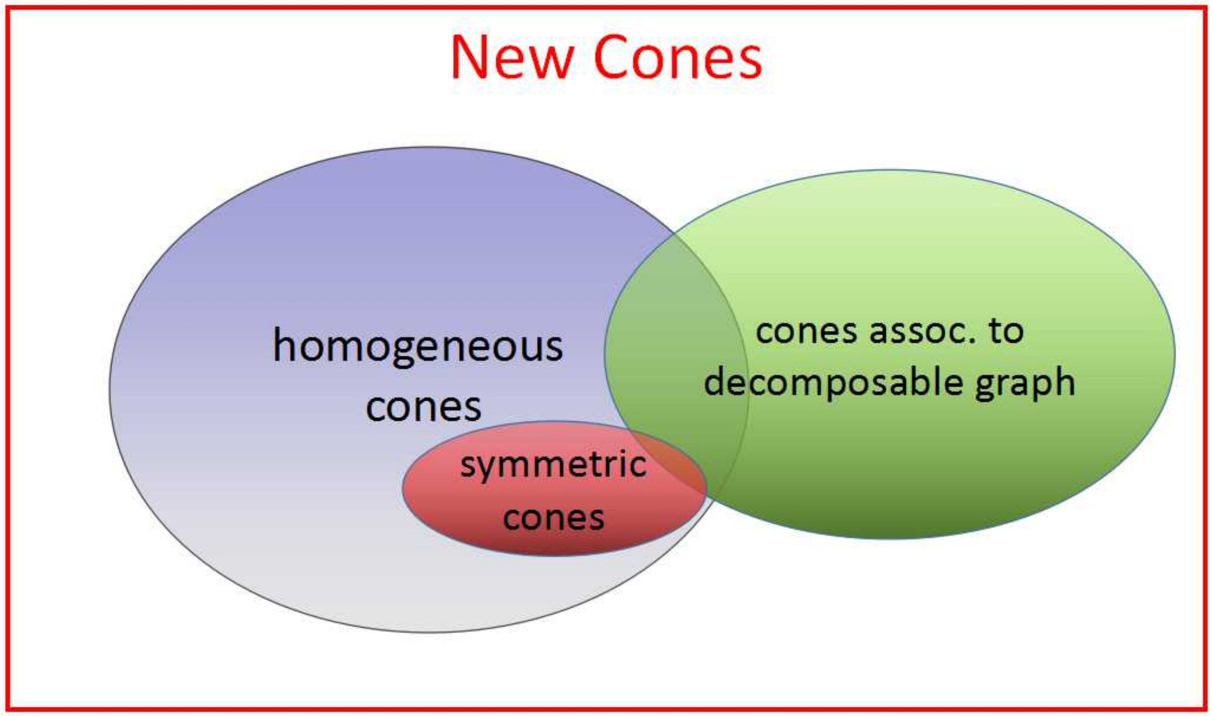
Wishart laws for a wide class of regular convex cones

Hideyuki ISHI
(Nagoya University)

Joint work with Piotr Graczyk, Bartosz Kołodziejek,
and Yoshihiko Konno


New Cones
homogeneous
cones
symmetric
cones
cones assoc. to
decomposable graph

§ Introduction

$\mathcal{P}_N := \{ x \in \mathrm{Sym}(N, \mathbb{R}) \mid x \text{ is positive definite} \}.$

Siegel integral formula (Wishart, Ingham, Siegel)

$$\int_{\mathcal{P}_N} e^{-\mathrm{tr}\, x\xi} (\det x)^{\alpha-(N+1)/2}\, dx = \Gamma_N(\alpha)(\det \xi)^{-\alpha}$$

$$(\xi \in \mathcal{P}_N,\ \alpha > \frac{N-1}{2}),$$

where $\Gamma_N(\alpha) := \pi^{N(N-1)/4} \prod_{j=1}^{N} \Gamma(\alpha - \frac{j-1}{2})$.

Setting $\alpha = p/2$, $\xi = \Sigma^{-1}/2$, we get the Wishart law:

$$\frac{e^{-\mathrm{tr}\, x\xi} (\det x)^{\alpha-(N+1)/2}}{\Gamma_N(\alpha)(\det \xi)^{-\alpha}} 1_{\mathcal{P}_N}(x)\, dx$$

$$= \frac{e^{-\mathrm{tr}\, x\Sigma^{-1}/2} (\det x)^{(p-N+1)/2}}{\Gamma_N(p/2)(\det 2\Sigma)^{p/2}} 1_{\mathcal{P}_N}(x)\, dx =: W_{p,\Sigma}(dx).$$

Group action behind the Siegel integral

The group $GL(N,\mathbb{R})$ of invertible matrices acts on $\mathcal{P}_N$ linearly by

$$\rho(A)x := Ax\,{}^{\mathrm{t}}\!A \quad (A \in GL(N,\mathbb{R}),\ x \in \mathcal{P}_N).$$

The action ρ is transitive, i.e.

$$\forall x, y \in \mathcal{P}_N\ \exists A \in GL(N,\mathbb{R}) \text{ s.t. } y = \rho(A)x.$$

Moreover, $\det x$ is relatively invariant under ρ:

$$\det(\rho(A)x) = (\det A)^2 \det x.$$

Siegel integrals and Wishart laws on homogeneous cones

$\mathbb{R}^n \supset \Omega$: open convex cone containing no straight line
(regular cone)

If Ω is homogeneous, i.e. Ω admits a group action which is linear and transitive, then we have Siegel type integral formulas for relative invariant functions on Ω (Gindikin).

Based on the integral formulas, Wishart laws on homogeneous cones are defined.
(Andersson-Wojnar, Hassairi-Boutouria, Graczyk-Ishi,...)

Siegel integrals for decomposable graphical models

In the graphical model theory, a simple undirected graph corresponds to a vector space of symmetric matrices. For instance, the graph $\mathcal{G} : 1 - 2 - 3 - 4$ corresponds to the space $\mathcal{Z}_{\mathcal{G}}$ consisting of $x \in \mathrm{Sym}(4, \mathbb{R})$ of the form

$$x = \begin{pmatrix} x_{11} & x_{21} & 0 & 0 \\ x_{21} & x_{22} & x_{32} & 0 \\ 0 & x_{32} & x_{33} & x_{43} \\ 0 & 0 & x_{43} & x_{44} \end{pmatrix}.$$

Let $\mathcal{P}_{\mathcal{G}} := \mathcal{Z}_{\mathcal{G}} \cap \mathcal{P}_4$. Then $\mathcal{P}_{\mathcal{G}}$ is NOT a homogeneous cone, but Siegel-type integral formulas hold for $\mathcal{P}_{\mathcal{G}}$!

Let $\mathcal{Q}_{\mathcal{G}}$ be the subset

$$\left\{ \begin{pmatrix} \xi_{11} & \xi_{21} & 0 & 0 \\ \xi_{21} & \xi_{22} & \xi_{23} & 0 \\ 0 & \xi_{32} & \xi_{33} & \xi_{43} \\ 0 & 0 & \xi_{43} & \xi_{44} \end{pmatrix} \mid \xi_{\{1,2\}}, \xi_{\{2,3\}}, \xi_{\{3,4\}} \in \mathcal{P}_2 \right\}$$

of $\mathcal{Z}_{\mathcal{G}}$, where $\xi_A := (\xi_{ij})_{i,j \in A}$ for $A \subset \{1,2,3,4\}$.
For $\xi \in \mathcal{Q}_{\mathcal{G}}$, define

$$\varphi_{\mathcal{G}}(\xi) := |\xi_{\{1,2\}}|^{-3/2} |\xi_{\{2,3\}}|^{-3/2} |\xi_{\{3,4\}}|^{-3/2} \xi_{\{2\}} \xi_{\{3\}},$$
$$\delta_{\mathcal{G}}(\xi) := |\xi_{\{1,2\}}| |\xi_{\{2,3\}}| |\xi_{\{3,4\}}| \xi_{\{2\}}^{-1} \xi_{\{3\}}^{-1}.$$

Integral formulas:

$$\int_{\mathcal{P}_{\mathcal{G}}} e^{-\mathrm{tr}\, x\xi} (\det x)^{\alpha}\, dx = \gamma_{\mathcal{G}}(\alpha)\delta_{\mathcal{G}}(\xi)^{-\alpha}\varphi_{\mathcal{G}}(\xi) \quad (\xi \in \mathcal{Q}_{\mathcal{G}},\, \alpha > -1),$$

where $\gamma_{\mathcal{G}}(\alpha) = \pi^{3/2}\Gamma(\alpha+1)\Gamma(\alpha+3/2)^3$.

$$\int_{\mathcal{Q}_{\mathcal{G}}} e^{-\mathrm{tr}\, x\xi} \delta_{\mathcal{G}}(\xi)^{\alpha}\varphi_{\mathcal{G}}(\xi)\, d\xi = \Gamma_{\mathcal{G}}(\alpha)(\det x)^{-\alpha} \quad (x \in \mathcal{P}_{\mathcal{G}},\, \alpha > \frac{1}{2}),$$

where $\Gamma_{\mathcal{G}}(\alpha) = \pi^{3/2}\Gamma(\alpha)\Gamma(\alpha - \frac{1}{2})^3$.

We have such formulas for any decomposable graph. Further generalizations with multi-parameter α are considered by Letac-Massam and Andersson-Klein, and applied to certain local zeta integrals by Sato.

§1. New cones

Let $N = n_1 + \cdots + n_r$ be a partition, and $\mathcal{V}_{lk} \subset M(n_l, n_k, \mathbb{R})$ $(1 \le k < l \le r)$ be vector spaces. Define

$$\mathcal{Z}_{\mathcal{V}} := \left\{ \begin{pmatrix} X_{11} & {}^{\mathrm{t}}X_{21} & \dots & {}^{\mathrm{t}}X_{r1} \\ X_{21} & X_{22} & & {}^{\mathrm{t}}X_{r2} \\ \vdots & & \ddots & \\ X_{r1} & X_{r2} & \dots & X_{rr} \end{pmatrix} \;\middle|\; \begin{matrix} X_{kk} \in \mathbb{R}E_k \ (k = 1, \dots, r) \\ X_{lk} \in \mathcal{V}_{lk} \ (1 \le k < l \le r) \end{matrix} \right\},$$

$\mathcal{P}_{\mathcal{V}} := \mathcal{Z}_{\mathcal{V}} \cap \mathcal{P}_n$, and

$$H_{\mathcal{V}} := \left\{ \begin{pmatrix} T_{11} & & & \\ T_{21} & T_{22} & & \\ \vdots & & \ddots & \\ T_{r1} & T_{r2} & \dots & T_{rr} \end{pmatrix} \;\middle|\; \begin{matrix} T_{kk} = t_{kk}E_k, \, t_{kk} > 0 \ (k = 1, \dots, r) \\ T_{lk} \in \mathcal{V}_{lk} \ (1 \le k < l \le r) \end{matrix} \right\}.$$

We shall assume

(V1) $A \in \mathcal{V}_{lk} \Rightarrow A\,{}^{\mathrm{t}}A \in \mathbb{R}E_{n_l}$ $(1 \le k < l \le r)$,

(V2) $A \in \mathcal{V}_{li}, B \in \mathcal{V}_{ki} \Rightarrow A\,{}^{\mathrm{t}}B \in \mathcal{V}_{lk}$ $(1 \le i < k < l \le r)$.

Theorem 1 (Cholesky decomposition)

One has a bijection $H_{\mathcal{V}} \ni T \mapsto T\,{}^{\mathrm{t}}T \in \mathcal{P}_{\mathcal{V}}$.

For $\underline{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$, define $\Delta_{\underline{s}} : \mathcal{P}_{\mathcal{V}} \to (0, +\infty)$ by

$$\begin{aligned}\Delta_{\underline{s}}(x) &:= (\det x)^{s_r/n_r} \prod_{k=1}^{r-1} (\det x^{[N_k]})^{s_k/n_k - s_{k+1}/n_{k+1}} \\ &= \prod_{k=1}^{r} (t_{kk})^{2s_k} \quad (x = T\,{}^{\mathrm{t}}T \in \mathcal{P}_{\mathcal{V}},\ T \in H_{\mathcal{V}}),\end{aligned}$$

where $N_k := \sum_{i=1}^{k} n_i$ and $x^{[N_k]} := (x_{\alpha\beta})_{\alpha,\beta \le N_k} \in \mathrm{Sym}(N_k, \mathbb{R})$.

Homogeneous case

Besides (V1) and (V2), we assume

(V3) $A \in \mathcal{V}_{lk},\ B \in \mathcal{V}_{ki} \Rightarrow AB \in \mathcal{V}_{li}$ $(1 \le i < k < l \le r)$.

In this case, $H_{\mathcal{V}}$ forms a group, and $H_{\mathcal{V}}$ acts transitively on $\mathcal{P}_{\mathcal{V}}$ by

$$\rho(T)x := Tx{}^{\mathrm{t}}T \quad (T \in H_{\mathcal{V}},\ x \in \mathcal{P}_{\mathcal{V}}).$$

Moreover, $\Delta_{\underline{s}}$ is relatively invariant under the action of $H_{\mathcal{V}}$.

Theorem 2 (I. 2006)

Every homogeneous cone is isomorphic to $\mathcal{P}_{\mathcal{V}}$ satisfying (V1)–(V3).

Example.
Let $\mathcal{V}_{21} = \left\{ \begin{pmatrix} a & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\}$, $\mathcal{V}_{31} = \left\{ \begin{pmatrix} 0 & b \end{pmatrix} \mid b \in \mathbb{R} \right\}$, and $\mathcal{V}_{32} = \{0\}$. Then $\mathcal{P}_{\mathcal{V}}$ is the set of positive definite matrices of the form

$$x = \begin{pmatrix} x_{11} & 0 & a & 0 \\ 0 & x_{11} & 0 & b \\ a & 0 & x_{22} & 0 \\ 0 & b & 0 & x_{33} \end{pmatrix}.$$

The cone $\mathcal{P}_{\mathcal{V}}$ is homogeneous, called the Vinberg cone.

Relation to graphical model theory

Consider the case $n_1 = n_2 = \cdots = n_r = 1$. Then $\mathcal{V}_{lk} = \mathbb{R}$ or $\{0\}$ for all $1 \leq k < l \leq r$. Let $\mathcal{G}$ be a simple undirected graph with vertices $\{1, 2, \ldots, r\}$ defined by

$$k \sim l \text{ if } \mathcal{V}_{lk} = \mathbb{R}, \quad k \not\sim l \text{ if } \mathcal{V}_{lk} = \{0\}$$

for $k < l$. Then (V2) implies

$$i \sim k, i \sim l \Rightarrow k \sim l$$

for $1 \leq i < k < l \leq r$, which means that the natural order "$\leq$" is an eliminating order of the vertices for the graph $\mathcal{G}$. (In other word, "$\leq$" induces a moral DAG on $\mathcal{G}$).

Proposition 3.
The cone $\mathcal{P}_{\mathcal{G}} \subset \mathcal{P}_r$ associated to a decomposable graph $\mathcal{G}$ is realized as $\mathcal{P}_{\mathcal{V}}$ satisfying (V1) and (V2) with $n_1 = \cdots = n_r = 1$.

§4. Dual cones

For matrices $A, B \in \mathrm{Mat}(p, q; \mathbb{R})$, we define the inner product by $(A|B) := \mathrm{tr}\, A^{\mathrm{t}} B$.

Let $\mathcal{Q}_{\mathcal{V}} \subset \mathcal{Z}_{\mathcal{V}}$ be the dual cone of $\mathcal{P}_{\mathcal{V}}$, i.e.

$$\mathcal{Q}_{\mathcal{V}} := \left\{ \xi \in \mathcal{Z}_{\mathcal{V}} \,|\, (x|\xi) > 0 \text{ for all } x \in \overline{\mathcal{P}_{\mathcal{V}}} \setminus \{0\} \right\}.$$

For $k = 1, \ldots, r$, let $\mathcal{W}_k$ be the vector space consisting of $w \in \mathrm{Mat}(n, n_k; \mathbb{R})$ of the form

$$w = \begin{pmatrix} 0_{N_{k-1}, n_k} \\ T_{kk} \\ \vdots \\ T_{rk} \end{pmatrix} \quad (T_{kk} \in \mathbb{R} I_{n_k},\ T_{lk} \in \mathcal{V}_{lk} \text{ for } l > k).$$

The k-th (block) column of $T \in H_{\mathcal{V}}$ belongs to $\mathcal{W}_k$. We have $\mathcal{W}_k \simeq \mathbb{R} \oplus \sum_{l>k}^{\oplus} \mathcal{V}_{lk}$. Taking an ONB of $\mathcal{W}_k$, we introduce $\mathcal{W}_k \ni w \overset{\sim}{\mapsto} \mathrm{vec}(w) \in \mathbb{R}^{1+q_k}$, where
$q_k := \sum_{l>k} \dim \mathcal{V}_{lk} \geq 0$.
Define a linear map $\phi_k : \mathcal{Z}_{\mathcal{V}} \to \mathrm{Sym}(1 + q_k, \mathbb{R})$ in such a way that

$$(w^{\mathsf{t}} w | \xi) = {}^{\mathsf{t}}\mathrm{vec}(w) \phi_k(\xi) \mathrm{vec}(w) \quad (w \in \mathcal{W}_k,\ \xi \in \mathcal{Z}_{\mathcal{V}}).$$

If $\xi \in \mathcal{Q}_{\mathcal{V}}$, then $\phi_k(\xi)$ is positive definite because $w^{\mathsf{t}} w \in \overline{\mathcal{P}_{\mathcal{V}}} \setminus \{0\}$ for $w \in \mathcal{W}_k \setminus \{0\}$.

Proposition 4.

$\mathcal{Q}_{\mathcal{V}} = \{\, \xi \in \mathcal{Z}_{\mathcal{V}} \,|\, \det \phi_k(\xi) > 0 \text{ for } k = 1, \ldots, r \,\}$.

Let $\check{\mathcal{W}}_k := \{\, w \in \mathcal{W}_k \,|\, T_{kk} = 0 \,\} \simeq \sum_{l>k}^{\oplus} \mathcal{V}_{lk} \simeq \mathbb{R}^{q_k}$, and define a linear map $\psi_k : \mathcal{Z}_{\mathcal{V}} \to \mathrm{Sym}(q_k, \mathbb{R})$ similarly, whereas we set $\psi_k(\xi) \equiv 1$ if $q_k = 0$.

Define $\delta_{\underline{s}} : \mathcal{Q}_{\mathcal{V}} \to (0, +\infty)$ for $\underline{s} \in \mathbb{R}^r$ and $\varphi_{\mathcal{V}} : \mathcal{Q}_{\mathcal{V}} \to (0, +\infty)$ by

$$\delta_{\underline{s}}(\xi) := \prod_{k=1}^{r} \left(\frac{\det \phi_k(\xi)}{\det \psi_k(\xi)} \right)^{s_k},$$

$$\varphi_{\mathcal{V}}(\xi) := \prod_{k=1}^{r} \frac{(\det \phi_k(\xi))^{-1-q_k/2}}{(\det \psi_k(\xi))^{-1/2-q_k/2}}.$$

Theorem 5. (I. Entropy 18 (2016))
(i) If $s_k > -1 - q_k/2$ for $k = 1, \ldots, r$, one has

$$\int_{\mathcal{P}_{\mathcal{V}}} e^{-\mathrm{tr}\, x\xi} \Delta_{\underline{s}}(x)\, dx = \gamma_{\mathcal{V}}(\underline{s}) \delta_{-\underline{s}}(\xi) \varphi_{\mathcal{V}}(\xi) \quad (\xi \in \mathcal{Q}_{\mathcal{V}}),$$

where $\gamma_{\mathcal{V}}(\underline{s}) := C_{\mathcal{V}} \prod_{k=1}^{r} (n_k^{-s_k} \Gamma(s_k + 1 + q_k/2))$ with some $C_{\mathcal{V}} > 0$.
(ii) If $s_k > q_k/2$ for $k = 1, \ldots, r$, one has

$$\int_{\mathcal{Q}_{\mathcal{V}}} e^{-\mathrm{tr}\, x\xi} \delta_{\underline{s}}(\xi) \varphi_{\mathcal{V}}(\xi)\, d\xi = \Gamma_{\mathcal{V}}(\underline{s}) \Delta_{-\underline{s}}(x) \quad (x \in \mathcal{P}_{\mathcal{V}}),$$

where $\Gamma_{\mathcal{V}}(\underline{s}) := C'_{\mathcal{V}} \prod_{k=1}^{r} (n_k^{-s_k} \Gamma(s_k - q_k/2))$ with some $C'_{\mathcal{V}} > 0$.
(iii) For $x \in \mathcal{P}_{\mathcal{V}}$ and $\underline{s} \in \mathbb{R}^r_{>0}$, one has

$$\delta_{\underline{s}}(\pi_{\mathcal{V}}(x^{-1})) = A_{\mathcal{V}} \Delta_{-\underline{s}}(x)$$

with some $A_{\mathcal{V}} > 0$, where $\pi_{\mathcal{V}} : \mathrm{Sym}(N, \mathbb{R}) \to \mathcal{Z}_{\mathcal{V}}$ is the orthogonal projection.

(iv) For $\xi \in \mathcal{Q}_{\mathcal{V}}$ and $\underline{s} \in \mathbb{R}^r_{>0}$, one has

$$\delta_{\underline{s}}(\xi) = A'_{\underline{s}} \sup_{x \in \mathcal{P}_{\mathcal{V}}} e^{-\mathrm{tr}(x\xi)} \Delta_{\underline{s}}(x)$$

with some $A'_{\underline{s}} > 0$.
(v) The map $\mathcal{I}_{\underline{s}} : \mathcal{P}_{\mathcal{V}} \ni x \mapsto \mathrm{grad}\log \Delta_{\underline{s}}(x) \in \mathcal{Q}_{\mathcal{V}}$ is bijective. The inverse map is given by

$$\mathcal{I}_{\underline{s}}^{-1}(\xi) = \mathrm{grad}\log \delta_{\underline{s}}(\xi) \in \mathcal{P}_{\mathcal{V}} \quad (\xi \in \mathcal{Q}_{\underline{s}}).$$

Example 1. When $\mathcal{Z}_{\mathcal{V}} = \mathcal{Z}_{\mathcal{G}}$ with $\mathcal{G} : 1 - 2 - 3 - 4$, we have for $\xi = \begin{pmatrix} \xi_{11} & \xi_{21} & 0 & 0 \\ \xi_{21} & \xi_{22} & \xi_{32} & 0 \\ 0 & \xi_{32} & \xi_{33} & \xi_{43} \\ 0 & 0 & \xi_{43} & \xi_{44} \end{pmatrix} \in \mathcal{Q}_{\mathcal{G}}$,

$$\delta_{\underline{s}}(\xi) = \left(\frac{\begin{vmatrix} \xi_{11} & \xi_{21} \\ \xi_{21} & \xi_{22} \end{vmatrix}}{\xi_{22}} \right)^{s_1} \left(\frac{\begin{vmatrix} \xi_{22} & \xi_{32} \\ \xi_{32} & \xi_{33} \end{vmatrix}}{\xi_{33}} \right)^{s_2} \left(\frac{\begin{vmatrix} \xi_{33} & \xi_{43} \\ \xi_{43} & \xi_{44} \end{vmatrix}}{\xi_{44}} \right)^{s_3} (\xi_{44})^{s_4}$$

and

$$\varphi_{\mathcal{V}}(\xi) = \begin{vmatrix} \xi_{11} & \xi_{21} \\ \xi_{21} & \xi_{22} \end{vmatrix}^{-3/2} \begin{vmatrix} \xi_{22} & \xi_{32} \\ \xi_{32} & \xi_{33} \end{vmatrix}^{-3/2} \begin{vmatrix} \xi_{33} & \xi_{43} \\ \xi_{43} & \xi_{44} \end{vmatrix}^{-3/2} \xi_{22}\xi_{33}.$$

When $\underline{s} = (1,1,1,1)$, we have $\Delta_{(1,1,1,1)}(x) = \det x$ for $x \in \mathcal{P}_{\mathcal{V}}$, so that

$$\mathcal{I}_{(1,1,1,1)}(x) = \operatorname{grad} \log \det x = \pi_{\mathcal{V}}(x^{-1}) \in \mathcal{Q}_{\mathcal{V}}.$$

The inverse map is given by

$$\operatorname{grad} \log \delta_{(1,1,1,1)}(\xi)$$
$$= (\xi_{\{1,2\}})_0^{-1} + (\xi_{\{2,3\}})_0^{-1} + (\xi_{\{3,4\}})_0^{-1} - (\xi_{22})_0^{-1} - (\xi_{33})_0^{-1},$$

where $(A)_0$ denotes the "zero-completion" of a matrix A of smaller size.

Example 2. When $\mathcal{Z}_{\mathcal{V}} = \mathcal{Z}_{\mathcal{G}}$ with $\mathcal{G} : 1 - 2 - 4 - 3$, we have for $\xi = \begin{pmatrix} \xi_{11} & \xi_{21} & 0 & 0 \\ \xi_{21} & \xi_{22} & 0 & \xi_{42} \\ 0 & 0 & \xi_{33} & \xi_{43} \\ 0 & \xi_{42} & \xi_{43} & \xi_{44} \end{pmatrix} \in \mathcal{Q}_{\mathcal{G}}$,

$$\delta_{\underline{s}}(\xi) = \left(\frac{\begin{vmatrix} \xi_{11} & \xi_{21} \\ \xi_{21} & \xi_{22} \end{vmatrix}}{\xi_{22}} \right)^{s_1} \left(\frac{\begin{vmatrix} \xi_{22} & \xi_{42} \\ \xi_{42} & \xi_{44} \end{vmatrix}}{\xi_{44}} \right)^{s_2} \left(\frac{\begin{vmatrix} \xi_{33} & \xi_{43} \\ \xi_{43} & \xi_{44} \end{vmatrix}}{\xi_{44}} \right)^{s_3} (\xi_{44})^{s_4}$$

and

$$\varphi_{\mathcal{V}}(\xi) = \begin{vmatrix} \xi_{11} & \xi_{21} \\ \xi_{21} & \xi_{22} \end{vmatrix}^{-3/2} \begin{vmatrix} \xi_{22} & \xi_{42} \\ \xi_{42} & \xi_{44} \end{vmatrix}^{-3/2} \begin{vmatrix} \xi_{33} & \xi_{43} \\ \xi_{43} & \xi_{44} \end{vmatrix}^{-3/2} \xi_{22}\xi_{44}.$$

The two examples correpond to two DAGs $\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet$ and $\bullet \leftarrow \bullet \leftarrow \bullet \rightarrow \bullet$.

Example 3. Let

$$\mathcal{Z}_{\mathcal{V}} = \left\{ \begin{pmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_1 & 0 & x_5 \\ x_4 & 0 & x_2 & x_6 \\ 0 & x_5 & x_6 & x_3 \end{pmatrix} \mid x_1, \dots x_6 \in \mathbb{R} \right\}.$$

Then $\mathcal{Q}_{\mathcal{V}}$ is the set of $\xi \in \mathcal{Z}_{\mathcal{V}}$ satisfying

$$\begin{vmatrix} \xi_1 & \xi_4 & \xi_5 \\ \xi_4 & \xi_2 & 0 \\ \xi_5 & 0 & \xi_3 \end{vmatrix} > 0, \quad \begin{vmatrix} \xi_2 & \xi_6 \\ \xi_6 & \xi_3 \end{vmatrix} > 0, \quad \xi_3 > 0.$$

We have

$$\varphi_{\mathcal{V}}(\xi) = \begin{vmatrix} \xi_1 & \xi_4 & \xi_5 \\ \xi_4 & \xi_2 & 0 \\ \xi_5 & 0 & \xi_3 \end{vmatrix}^{-2} \begin{vmatrix} \xi_2 & \xi_6 \\ \xi_6 & \xi_3 \end{vmatrix}^{-3/2} \xi_2^{3/2} \xi_3^{3/2} \quad (\xi \in \mathcal{Q}_{\mathcal{V}}).$$

For $\underline{s} = (s_1, s_2, s_3) \in \mathbb{R}^3$, we have

$$\delta_{\underline{s}}(\xi) = \left(\frac{\begin{vmatrix} \xi_1 & \xi_4 & \xi_5 \\ \xi_4 & \xi_2 & 0 \\ \xi_5 & 0 & \xi_3 \end{vmatrix}}{\xi_2 \xi_3} \right)^{s_1} \left(\frac{\begin{vmatrix} \xi_2 & \xi_6 \\ \xi_6 & \xi_3 \end{vmatrix}}{\xi_3} \right)^{s_2} \xi_3^{s_3}$$

$$= \begin{vmatrix} \xi_1 & \xi_4 & \xi_5 \\ \xi_4 & \xi_2 & 0 \\ \xi_5 & 0 & \xi_3 \end{vmatrix}^{s_1} \begin{vmatrix} \xi_2 & \xi_6 \\ \xi_6 & \xi_3 \end{vmatrix}^{s_2} \xi_2^{-s_1} \xi_3^{s_3 - s_1 - s_2} \quad (\xi \in \mathcal{Q}_{\mathcal{V}}),$$

$$\Delta_{\underline{s}}(x) = x_1^{s_1 - s_2 - s_3} \begin{vmatrix} x_1 & x_4 \\ x_4 & x_2 \end{vmatrix}^{s_2 - s_3} (\det x)^{s_3} \quad (x \in \mathcal{P}_{\mathcal{V}}).$$

The cone $\mathcal{P}_{\mathcal{V}}$ is NOT homogeneous, but corresponding to a colored graph.

§5. Wishart laws on $\mathcal{Q}_{\mathcal{V}}$

For $x \in \mathcal{P}_{\mathcal{V}}$ and $\underline{s} \in \mathbb{R}^r$ with $s_k > q_k/2$ $(k = 1, \ldots, r)$, define

$$W_{\underline{s},x}(d\xi) := \frac{e^{-\mathrm{tr}\, x\xi}\delta_{\underline{s}}(\xi)\varphi_{\mathcal{V}}(\xi)}{\Gamma_{\mathcal{V}}(\underline{s})\Delta_{-\underline{s}}(x)} 1_{\mathcal{Q}_{\mathcal{V}}}(\xi)\, d\xi.$$

Let $Y \sim W_{\underline{s},x}$.

Proposition 6.

(i) $\mathbb{E}(Y)$ equals $\mathcal{I}_{\underline{s}}(x) = \mathrm{grad}\,\log \Delta_{\underline{s}}(x) \in \mathcal{Q}_{\underline{s}}$.

(ii) For $\eta \in \mathcal{Z}_{\mathcal{V}}$, one has

$$\mathbb{E}(\mathrm{tr}\, Y\eta) = \frac{s_r}{n_r}\mathrm{tr}(x^{-1}\eta) + \sum_{k=1}^{r-1}\{(\frac{s_k}{n_k} - \frac{s_{k+1}}{n_{k+1}})\mathrm{tr}\left((x^{[N_k]})^{-1}\eta^{[N_k]}\right)\}.$$

(iii) If $\xi_0 = \mathbb{E}(Y)$, then

$$x = \arg\max_{x \in \mathcal{P}_{\mathcal{V}}} e^{-\mathrm{tr} x\xi_0}\Delta_{\underline{s}}(x) = \mathrm{grad}\,\log \delta_{\underline{s}}(\xi_0).$$

Proposition 7

For $\theta \in \mathcal{Z}_{\mathcal{V}}$ s.t $x - \theta \in \mathcal{P}_{\mathcal{V}}$, one has

$$\mathbb{E}(e^{\operatorname{tr} \theta x}) = \Delta_{-\underline{s}}(x - \theta)\Delta_{\underline{s}}(x).$$

Theorem 8 (cf. Graczyk-I. 2014)

For $\eta_1, \eta_2, \ldots, \eta_d \in \mathcal{Z}_{\mathcal{V}}$, one has

$$\mathbb{E}((\operatorname{tr} Y\eta_1)(\operatorname{tr} Y\eta_2)\ldots(\operatorname{tr} Y\eta_d))$$
$$= \sum_{\sigma \in \mathfrak{S}_d} \prod_{c \in C(\sigma)} \left\{ \sum_{k=1}^{r} \left(\frac{s_k}{n_k} - \frac{s_{k+1}}{n_{k+1}}\right)^{\sharp c} \operatorname{tr}\left(\prod_{j \in c} (x^{[N_k]})^{-1} \eta_j^{[N_k]})\right) \right\},$$

where $C(\sigma)$ denotes the set of cycles in a permutation $\sigma \in \mathfrak{S}_d$, and $s_{r+1}/n_{r+1} := 0$.

Statistical interpretation of $W_{\underline{s},x}$.

Let $\Sigma := x^{-1}/2 \in \mathrm{Sym}(N, \mathbb{R})$ with $x \in \mathcal{P}_{\mathcal{V}}$, and $\underline{p} = (p_1, \dots, p_r) \in \mathbb{Z}^r_{\geq 0}$ with $p_r > 0$. For $k = 1, \dots, r$ and $1 \leq j \leq p_k$, let $U^{kj} = {}^{\mathrm{t}}(U^{kj}_1, \dots, U^{kj}_N)$ be $\mathbb{R}^N$-valued, mutually independent random vectors s.t.

$$U^{kj} \sim N(0, \Sigma) | U_{1+N_k} = \cdots = U_N = 0.$$

for $k = 1, \dots, r$. Let

$$Y := \pi_{\mathcal{V}} \left(\sum_{k=1}^{r} \sum_{j=1}^{p_k} U^{kj}\, {}^{\mathrm{t}}U^{kj} \right).$$

If $\frac{s_k}{n_k} - \frac{s_{k+1}}{n_{k+1}} = \frac{p_k}{2}$ with $s_k > q_k/2$ for $k = 1, \dots, r$, then

$$Y \sim W_{\underline{s},x}.$$