FREE PROBABILITY AND RANDOM MATRICES

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A simple example: random vectors in large dimensions

$v_1, \dots, v_k \in \mathbf{R}^N$.

$a_i = \|v_i\|^2$ fixed.

$v_1/\|v_1\|, \dots, v_k/\|v_k\| \in \mathbf{R}^N$ chosen independently uniformly on the sphere

As $N \to \infty$, with high probability v_i become orthogonal: for every $\epsilon > 0$,

$$P(|\langle v_i, v_j\rangle - a_i\delta_{ij}| > \epsilon) \to_{N\to\infty} 0$$

We will obtain analogous results for matrices

The geometry of matrices is more complex than that of vectors so the description will use more information

We will use the theory of *free probability*

Example:

Π_1 and Π_2 two $N \times N$ matrices
$=$ orthogonal projections on subspaces of dimensions $N/2$.

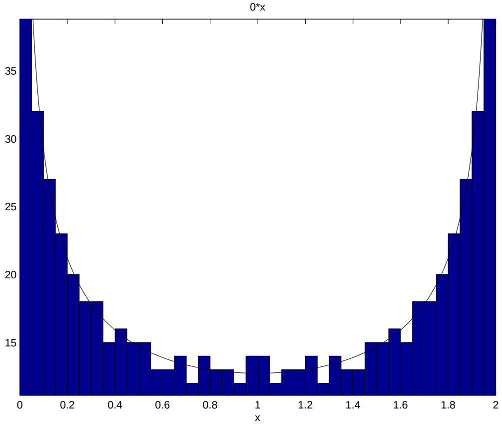
We chose the two subspaces independently and uniformly

$$\Pi_i = U_i \begin{pmatrix} I_{N/2} & 0 \\ 0 & 0 \end{pmatrix} U_i^*$$

U_i = independent unitary matrices chosen with the Haar measure on $U(N)$.

One computes the spectrum of $\Pi_1 + \Pi_2$.

Histogram of the spectrum of $\Pi_1 + \Pi_2 \qquad (N = 800)$



$$y = \frac{1}{\pi\sqrt{x(2-x)}}$$

BASIC INVARIANT THEORY FOR MATRICES

Two complex self-adjoint $N \times N$ matrices L and M have the same spectrum if and only if there exists a unitary U such that

$$L = UMU^*$$

Moreover L and M have the same spectrum if and only if for all $r \geq 0$

$$\frac{1}{N} Tr(L^r) = \frac{1}{N} Tr(M^r)$$

i.e. the matrices have the same moments
In the sequel I will use

$$tr = \frac{1}{N} Tr$$

Let $M_1, \ldots, M_n$ be $N \times N$ self-adjoint matrices.

Theorem (Procesi, 1978)

The "non-commutative joint moments"

$$tr(M_{i_1} \ldots M_{i_r}), \qquad r \geq 1,\ i_1, \ldots, i_r \in \{1, \ldots, n\}$$

form a complete set of invariants of the matrices up to conjugation.

This means that

$$tr(L_{i_1} \ldots L_{i_r}) = tr(M_{i_1} \ldots M_{i_r}) \qquad \text{for all } r, i_1, \ldots, i_r$$

if and only if there exists a unitary matrix U such that

$$L_i = U M_i U^* \qquad \text{for all } i$$

U **does not depend on** i**!**

Let $X_i = U_i D_i U_i^{-1}$
D_i=diagonal matrices
U_i= independent unitary random matrices taken with Haar measure.
The spectra of the X_i are fixed, their eigenvectors are chosen at random.

Theorem (Voiculescu, 1990)
As $N \to \infty$ the mixed moments

$$tr(X_{i_1} \dots X_{i_k})$$

are given by explicit polynomials in the $tr(D_i^k) = tr(X_i^k)$ (with high probability and with a small error)

Exemples:

$$tr(X_1X_2) \sim tr(X_1)tr(X_2)$$

$$tr(X_1^kX_2^l) \sim tr(X_1^k)tr(X_2^l)$$

$$tr(X_1X_2X_1X_2) \sim tr(X_1^2)tr(X_2)^2 + tr(X_1)^2tr(X_2^2) - tr(X_1)^2tr(X_2)^2$$

Here $\sim$ means that the difference is small with high probability.

Corollary

If we know the spectra of $X_1, \ldots, X_n$ we can compute, with a good approximation the spectrum of any polynomial in the X_i.

Example:

$$tr((X_1 + X_2)^n) = \sum_{i_1 \ldots i_n} tr(X_{i_1} \ldots X_{i_n})$$

can be computed in terms of the numbers

$$tr(X_1^k), tr(X_2^k), \qquad k = 1, 2, \ldots$$

The explicit computation of these polynomials can be done using the theory of free *random variables, free cumulants, noncrossing partitions*
This is a sophisticated algebraic and combinatorial theory

Cf Nica, Alexandru; Speicher, Roland
Lectures on the combinatorics of free probability.
London Mathematical Society Lecture Note Series, 335.
Cambridge University Press, Cambridge, 2006.

Example: the free convolution.
Suppose you know

$$tr(L^n), tr(M^n), n = 0, 1, 2, \dots$$

how do you compute $tr((L+M)^n)$?
The answer is given by *free cumulants*.

Introduce the generating functions of the moments of a matrix X

$$G_X(z) = tr((z-X)^{-1}) = \frac{1}{z} + \sum_{n=1}^{\infty} z^{-n-1} tr(X^n)$$

then

$$K_X(G_X(z)) = G_X(K_X(z)) = z; \qquad K_X(z) = \frac{1}{z} + \sum_{n=1}^{\infty} R_n(X) z^{n-1}$$

the $R_n(X)$ are the **free cumulants** of X.

Free cumulants and moments determine each other by a triangular polynomial system:

$$m_1 = R_1$$
$$m_2 = R_2 + R_1^2$$
$$m_3 = R_3 + 3R_1R_2 + R_1^3$$

etc.

Free cumulants and random matrices

Recall the model of random matrices

$$L = UDU^* \qquad M = VEV^*$$

D, E diagonal with given spectrum;
U, V independent Haar unitaries.

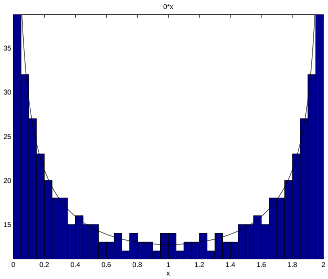
for N large one has

$$R_n(L+M) \sim R_n(L) + R_n(M)$$

Random matrix model: compute the spectrum of $\Pi_1 + \Pi_2$ where $\Pi_1, \Pi_2 =$ orthogonal projections on random subspaces of dimensions $N/2$.

The free cumulants computation gives the moments of an arcsine distribution:

$$\frac{dx}{\pi\sqrt{x(2-x)}} \qquad \text{arcsine distribution}$$



Free convolution

μ probability measure

$$G_\mu(z) = \int \frac{1}{z-x} d\mu(x) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^{-n-1}} \int x^n d\mu(x)$$

then

$$K_\mu(G_\mu(z)) = G_\mu(K_\mu(z)) = z; \qquad K_\mu(z) = \frac{1}{z} + \sum_{n=1}^{\infty} R_n(\mu) z^{n-1}$$

the $R_n(\mu)$ are the **free cumulants** of μ.
The free convolution of μ and ν is given by:

$$R_n(\mu \boxplus \nu) = R_n(\mu) + R_n(\nu)$$

This is the free version of convolution of measures

Free central limit theorem

$X_1, \ldots, X_n$ f.i.d

$$\tau(X_i) = 0 \qquad \tau(X_i^2) = \sigma^2$$

Theorem (Voiculescu, 1983)

$$\frac{X_1 + \ldots + X_n}{\sqrt{n}} \to_{n\to\infty}^{\text{(in law)}} \frac{1}{\pi\sigma}\sqrt{4\sigma^2 - x^2}dx \qquad x \in [-2\sigma, 2\sigma]$$

Semi-circle law with variance σ^2

$$w_{\sigma^2}(dx) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}dx; \quad x \in [-2\sigma, 2\sigma]$$

characterized by:

$$K_{w_{\sigma^2}}(z) = \frac{1}{z} + z\sigma^2$$

Remark

$$w_s \boxplus w_t = w_{s+t}$$

The semi-circle law is freely infinitely divisible
One can characterize freely infinitely divisible distributions, stable distributions, free Poisson distributions, etc.