Some dynamical applications of Carathéodory's prime ends theory

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If $G \subset \mathbb{R}^2$ is a topological open disc (typically the basin of attraction of an asymptotically stable fixed point of a homeomorphism), $\partial(G)$ can be very complicated.

Roughly speaking, prime ends theory allows to "give" a nice boundary to G. One can add a circle to G to obtain a compactification of G which is a topological closed disc.

1.1 Prime ends theory

A *Jordan curve* γ is a non-self-intersecting continuous loop in the plane. Every Jordan curve decompose the plain into two components. A *Jordan domain* is the bounded component of the complement of a Jordan curve.

Let $B \subset \mathbb{C}$ be the unit open disc and let $f : B \to G \subset \mathbb{C} \cup \{\infty\}$ be a conformal mapping. The problem whether f admits an extension to $cl(B) = B \cup S^1$, by defining $f(z) = \lim_{x \to z} f(x)$ for $z \in S^1$, has a topological answer : f admits and extension iff $\partial(G)$ is locally connected. The problem whether f has an injective extension has also an answer of topological nature: f has an injective extension iff $\partial(G)$ is a Jordan curve (Carathéodory's Theorem).

If $\partial(G)$ is locally connected but not a Jordan curve there are points of $\partial(G)$ that have several pre-images. The situation becomes much more complicated if $\partial(G)$ is not locally connected.

Carathéodory introduced the notion of prime end to describe this setting.

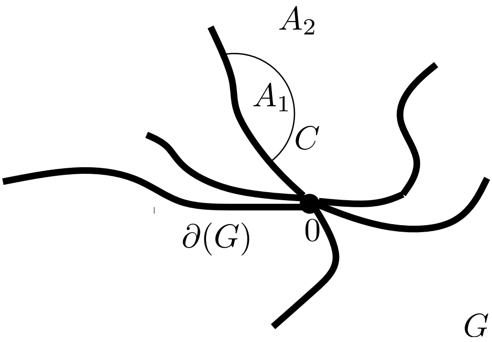
The points $z \in S^1$ correspond one-to-one to the prime ends of G and the limit $f(z)$ exists if and only if the prime end "has" only one point (Prime End Theorem).

We will recall here the main definitions of the theory of ends that we will need.

Let $G \subset \mathbb{R}^2$ be a simply connected open domain containing the point at infinity such that $\partial(G)$ contains more than one point. Then $\partial(G)$ is bounded.

A *cross-cut* is a simple arc, C, lying in G, except in the end points, with different extremities.

If C is a cross-cut of G then $G \setminus C$ has exactly two components A_1 and A_2 such that $G \cap \partial(A_1) = G \cap \partial(A_2) = C \setminus \{ \text{ end points } \}$.



A sequence $\{C_n\}$ of mutually disjoint cross-cuts and such that each C_n separates C_{n-1} and C_{n+1} is called a *chain*.

A chain of cross-cuts induces a nested chain of domains (bounded by each C_n) $\ldots D_{n+1} \subset D_n \ldots$.

Each chain of cross-cut defines an *end*. Two chains of cross-cuts, $\{C_n\}$ and $\{C'_n\}$, are *equivalent* if for any $n \in \mathbb{N}$ there is $m(n)$ such that $D_m \subset D'_n$ and $D'_m \subset D_n$ for every $m > m(n)$.

Equivalent chains of cross-cuts are said to induce the same end.

If P and Q are ends represented by chains of cross-cuts $\{C(P)_n\}$ and $\{C(Q)_n\}$ such that for every n, $D(P)_m \subset D(Q)_n$ for $m > m(n)$ we say that P *divides* Q.

A *prime end* P is an end which can not be divided by any other (i.e. $diam(C_n) \to 0$).

Let P be a prime end. The *set of points* of P (or the impression of P) is the intersection $I(P) = \bigcap_{n \in \mathbb{N}} cl(D(P)_n)$ where $\{D(P)_n\}$ is the sequence of domains bounded by any sequence of cross-cuts representing P.

A *principal point* of P is a limit point of a chain of cross-cuts representing P tending to a point. The set $\Pi(P) \subset E$ of principal points of a prime end P is a continuum (compact connected set).

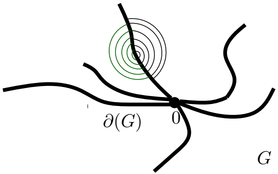
$$\Pi(P) = \{y : \text{ there is } \{C_n\} \text{ with } P = [\{C_n\}] \text{ and } \lim C_n \to y\}.$$

A point $y \in \partial(G)$ is *accessible* if there exists an arc $\gamma : [0,1] \subset cl(G)$ such that $\gamma(1) = y$ and $\gamma([0,1)) \subset G$. Each accessible point determines a prime end.

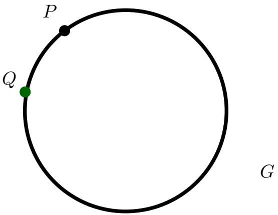
Each chain of cross-cuts inducing a prime end P determines a basis of neighborhoods of P. We obtain in this way a topology in the set of prime ends of G.

More precisely, if $\mathbb{P}$ is the set of prime ends of G and G^* is the disjoint union of G and $\mathbb{P}$, we can introduce a topology in G^* in such a way that it becomes homeomorphic to the closed disc and the boundary being composed by the prime ends.

It is enough to define a basis of neighborhoods of a prime end $P \in \mathbb{P}$. Given the sequence of domains $\{D(P)_n\}$, we produce a basis of neighborhoods $\{U_n\}$ of P in G^*. Each U_n is composed by the points in $D(P)_n$ and by the prime ends Q such that $D(Q)_m \subset D(P)_n$ for m large enough.



Chains of cross-cuts defining prime ends P and Q

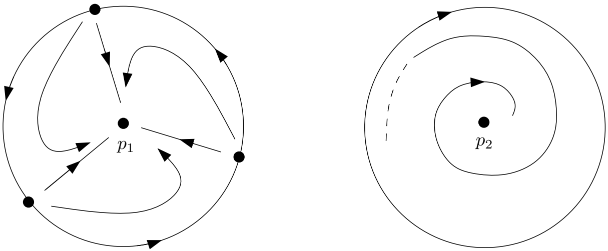


Prime ends P and Q in the set of prime ends, $\mathbf{P}$, of G

If S^2 is the 2-sphere $\mathbb{R}^2 \cup \{\infty\}$ and $\infty \in G \subset S^2$ is a simply connected open domain, the natural compactification, due to Carathéodory, of G obtained by attaching to G a set homeomorphic to the one-dimensional sphere S^1 is called *the prime ends compactification of* G.

We identify $\mathbb{R}^2 = \mathbb{C}$ and we consider a conformal homeomorphism $g : G \to S^2 \setminus B$ (where B is the disc $B = \{z \in \mathbb{C} : |z| \leq 1\}$). Now a one-dimensional sphere S^1 is attached to G using g. Each point of S^1 corresponds to a prime end of G.

Rotation numbers can be assigned to attractors in two dimensions. This is illustrated by the figures below



The idea of associating a rotation number to an attractor was fully developed by Cartwright and Littlewood. More recently Alligood and Yorke have used these ideas to explore fractal boundaries.

Our approach has many points in common them but our goal is somehow different. We place the emphasis on results about global attraction and their applicability to differential equations.

To be more precise, we consider an orientation-preserving homeomorphism of the plane, denoted by h, and a fixed point $p = h(p)$ that is asymptotically stable.

The region of attraction

$$U = \{q \in \mathbb{R}^2 : \lim_{n \to +\infty} h^n(q) = p\}$$

is an open and simply connected subset of $\mathbb{R}^2$.

While this implies that U is homeomorphic to the open unit disc the boundary of U can have a complicated structure.

Assuming that $U \neq \mathbb{R}^2$, the theory of prime ends associates a copy of $\mathbb{S}^1$ to the boundary of U and the map h induces an orientation-preserving homeomorphism of $\mathbb{S}^1$.

The corresponding rotation number will be denoted by $\rho = \rho(h, U)$.

We showed that if $\rho = 0$, h is dissipative and U is unbounded then there exists a fixed point lying in $\mathbb{R}^2 \setminus U$.

As a corollary one obtains a criterion for global attraction when p is the unique fixed point.

Dissipativity means that ∞ is a repeller for h. The assumption on the unboundedness of the region of attraction is satisfied as soon as h is area-contracting.

These are typical assumptions motivated by the theory of nonlinear oscillations.

For maps h coming from differential equations (h to be the Poincaré map associated to a periodic differential system) it is not easy to determine the rotation number. This fact was pointed out by Cartwright and Littlewood when they were dealing with the forced Van der Pol equation.

In order to make our results applicable we need to obtain some criteria for the computation of the rotation number directly from the equation.

From here one can derive consequences for orientation-reversing maps (applicable to periodic equations where the Poincaré map is invariant by symmetries), extinction in population dynamics or global attraction in nonlinear oscillators.

Asymptotic stability and prime ends

The class of homeomorphisms of the plane is denoted by $\mathcal{H}$. The notation $\mathcal{H}_+$ will be employed for the subclass of orientation-preserving homeomorphisms. Similarly $\mathcal{H}_-$ is employed for orientation-reversing maps in $\mathcal{H}$.

Assume now that p is an asymptotically stable fixed point for $h \in \mathcal{H}$, its region of attraction U is an open and simply connected subset of the plane.

When $U \neq \mathbb{R}^2$ and h is orientation-preserving, it is possible to assign a rotation number to the fixed point. To this end we must enter into Carathéodory's theory of prime ends applied to $\partial_{\mathbb{S}^2} U$.

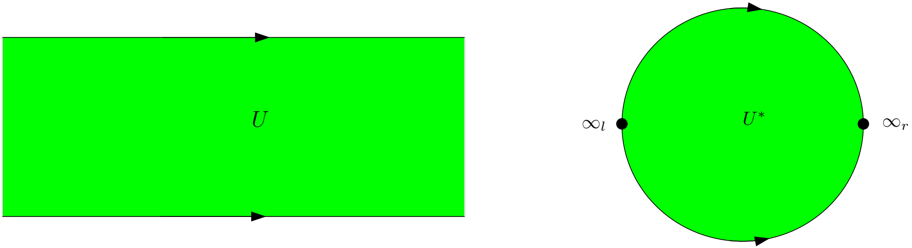
Theorem

(Ortega, R.P.) Assume that $h \in \mathcal{H}_+$ is dissipative and U is a simply connected open subset of the plane that is unbounded and proper, $\emptyset \neq U \subsetneq \mathbb{R}^2$. In addition,

$$h(U) = U, \quad \rho(h, U) = 0.$$

Then h has a fixed point in $\mathbb{R}^2 \setminus U$.

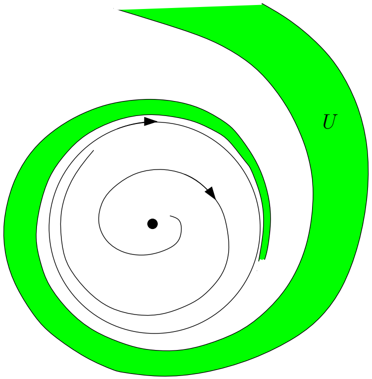
The dissipativity of h is essential. This can be shown by considering the translation
$h(x_1, x_2) = (x_1 + 1, x_2)$ and the set $U = \mathbb{R}\times]0, 1[$.
This map has no fixed points and the rotation number on U vanishes.
To justify that $\rho(h, U) = 0$ it is enough to describe the dynamics of $h^\star$ and observe that it has two fixed points.



A result by Barge and Gillette leads to a result similar but for bounded domains.

It can happen that the fixed point found by the Theorem does not lie on the boundary of U.

As an example we can consider the time 1 map associated to the van der Pol flow. The region U is determined by two orbits emanating from infinity and attracted by the closed orbit.



The previous result can be modified for area-contracting maps.

Corollary

Assume that all the conditions of the last theorem holds. In addition h is area-contracting and there exists a topological disc $D \subset U$ such that

$$\mathrm{Fix}(h) \cap U \subset \mathrm{int}(D) \quad \textit{and} \quad i(h, D) = 1.$$

Then h has a fixed point lying on ∂U.

Computing the rotation number

To make the theorem useful for applications we need some conditions on h and U implying that the rotation number vanishes.

We list some of these conditions in the next result.

Theorem

(Ortega, R.P.) Assume that $h \in \mathcal{H}_+$ *and* $U = h(U)$ *is a simply connected open subset of the plane that is unbounded and proper,* $\emptyset \neq U \subsetneq \mathbb{R}^2$. *Then*

$$\rho(h, U) = 0$$

if any of the conditions below holds

(i) ∂U *is connected and* ∞ *is accessible from* U

(ii) $h = r \circ r$ *with* $r \in \mathcal{H}_-$ *and* $r(U) = U$

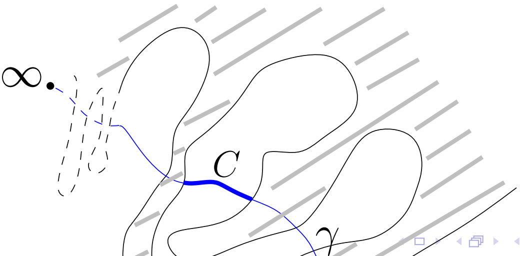
(iii) *There exists an arc* $\gamma \subset \mathbb{S}^2$ *having* ∞ *as one of the end points with* $\gamma \setminus \{\infty\} \subset U$ *and* $h(\gamma) \subset \gamma$

(iv) *There exists a sector* $K = \{\rho e^{i\theta} : \rho \geq 0,\ \theta \in [\Theta_-, \Theta_+]\}$, $\Theta_- < \Theta_+$, *and a disc* $D = \{|z| \leq R\}$ *such that* $h(K \setminus D) \subset K$ *and* $K \setminus D \subset U$.

Irrational rotation number case

Theorem

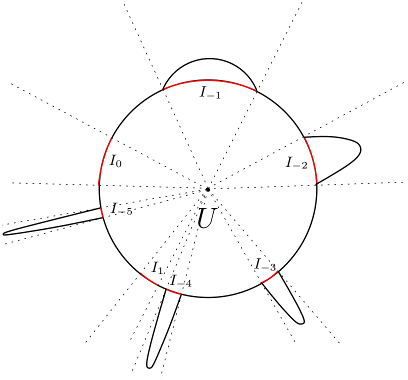
(Hdez-Corbato, Ortega, R.P.) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be an orientation-preserving, dissipative homeomorphism of the plane with* $f(0) = 0$*. Let* $0 \in U$ *be a proper, unbounded and simply connected open set such that* $f(U) = U$*. Let* $\mathbb{P}$ *be the circle of prime ends of* U *and* ρ *the rotation number of the homeomorphism of* $\mathbb{P}$ *induced by* f*,* $\bar{f}_{|\mathbb{P}}$*. Then, if* $\rho \notin \mathbb{Q}$*,* $\bar{f}_{|\mathbb{P}} : \mathbb{P} \to \mathbb{P}$ *is a Denjoy homeomorphism.*



Corollary

Under the conditions of the above theorem, if $Per(f) \cap (\mathbb{R}^2 \setminus U) = \emptyset$ *then* $\bar{f}_{|\mathbb{P}} : \mathbb{P} \to \mathbb{P}$ *is a Denjoy homeomorphism.*

In order to complete the above theorem, note that explicit examples of homeomorphisms satisfying the hypothesis and inducing a Denjoy type dynamics in the circle of prime ends of U, can be given (even with $\partial(U)$ connected).



A natural question is if this is a characterization of $\rho(h, U) \in \mathbb{R} \setminus \mathbb{Q}$ in the class $\mathcal{H}_+$.
Matsumoto gave a counterexample:
There exist an orientation preserving planar homeomorphism H such that:
- 0 is an asymptotically stable fixed point.
-The unit open disk is contained in the region of attraction of 0, say U.
- U is unbounded as a region of the plane.
- $Fix(H) \cap (\mathbb{R}^2 \setminus U) \neq \emptyset$.
- H is dissipative.

The next result is a converse of the last corollary, imposing ∞ is an accessible point.

Theorem

(Hdez.-Corbato, Ortega, R.P.)
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving dissipative homeomorphism. Let $0 \in Fix(f)$ be a local attractor and U its basin of attraction. Assume $U \subsetneq \mathbb{R}^2$ and let $\mathbb{P}$ be the Carathéodory's circle of prime ends of $S^2 \setminus U$. Let $\bar{f} : \mathbb{P} \to \mathbb{P}$ be the induced homeomorphism. If the rotation number $\rho = \rho(\bar{f}) \notin \mathbb{Q}$, then $Per(f) \cap (\mathbb{R}^2 \setminus U) = \emptyset$, provided ∞ is accessible by an arc in $U \cup \{\infty\} \subset S^2$.

Questions.

- It is still possible to construct Matsumoto's examples if in addition h is area-contracting?
- (Less precise) Given a one-parameter family, h_λ, of dissipative homeomorphisms is $\rho(h_\lambda)$ continuous?
- Given a one-parameter family, F_λ, of periodic planar differential systems such that the corresponding Poincaré map P_λ is well defined and dissipative, is $\rho(P_\lambda)$ continuous?

The previous result (iii) is applicable to systems with two populations. In these systems the coordinate axes are invariant and they produce the invariant ray. More precisely we consider the system

$$\dot{u} = uF(t, u, v), \quad \dot{v} = vG(t, u, v), \qquad u, v \geq 0, \tag{1}$$

where $F, G : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are of class C^1 and 1-periodic in time. The periodicity in time reflects the seasonal effects. We also assume that F and G are such that there is global existence for the associated initial value problem on the first quadrant.

We think of $u(t)$ and $v(t)$ as the sizes of two species and say that there is extinction for (1) if

$$u(t) \to 0, \quad v(t) \to 0 \quad \text{as } t \to +\infty$$

for each solution $(u(t), v(t))$.
We shall assume that the system is dissipative and this is a very natural assumption in population dynamics, particularly when logistic effects are involved. We also employ the condition

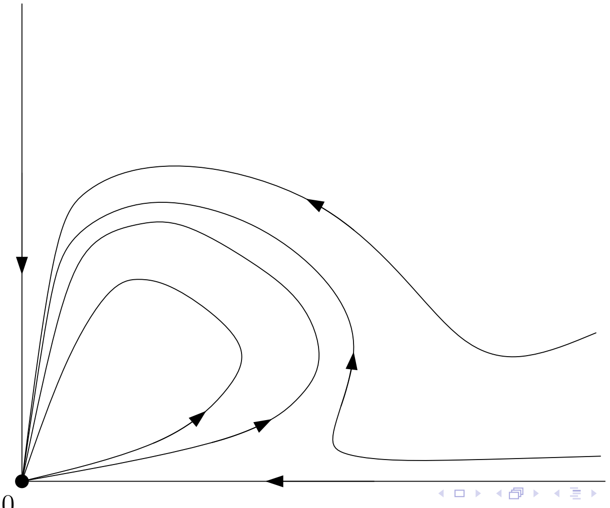
$$\int_0^1 F(t,0,0)dt < 0, \quad \int_0^1 G(t,0,0)dt < 0. \tag{2}$$

This condition says that the averaged balance between birth and death is negative when the populations are very small.

Theorem

Assume that the system (1) is dissipative and the condition (2) holds. Then there is extinction if and only if $u = v = 0$ is the unique 1-periodic solution.

Remark. The condition (2) is necessary to exclude cases where the origin is an unstable attractor. We illustrate this situation in the following phase portrait



Sectorial attraction in forced oscillators. (Application of (iv)).

Consider the equation

$$\ddot{x} + c\dot{x} + g(x) = p(t) \tag{3}$$

where $c > 0$, $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz-continuous and p is continuous and 2π-periodic. In the terminology of Pliss this equation is *convergent* if it has a 2π-periodic solution that is globally asymptotically stable.

We say that the equation has the property of Σ-*uniqueness* it has a unique 2π-periodic solution and this solution is asymptotically stable. This property is weaker than convergence but we want to show that they are equivalent as soon as the solutions starting at some angular sector are attracted.

Let us assume that g has finite limits at $\pm\infty$ and

$$g(-\infty) < \overline{p} < g(+\infty), \tag{4}$$

where $\overline{p} = \frac{1}{2\pi}\int_0^{2\pi} p(t)dt$ is the mean value of p. This condition has a mechanical meaning, it says that the averaged force $-g(x) + \overline{p}$ points towards the origin, at least in a neighborhood of infinity. In agreement with this intuition it is known that (4) is sufficient to guarantee the dissipativity of the first order system associated to (3). (Yoshizawa pages 70 and 71). From now on $\varphi(t)$ will denote the unique 2π-periodic solution of (3). Given another solution $x(t)$ we say that it is attracted by φ if

$$|x(t) - \varphi(t)| + |\dot{x}(t) - \dot{\varphi}(t)| \to 0 \quad \text{as } t \to +\infty.$$

Theorem

Assume that (4) holds and there is Σ-uniqueness for (3). In addition there are positive numbers ρ and ϵ such that all the solutions satisfying

$$x(0) \geq \rho, \quad |\dot{x}(0)| \leq \epsilon x(0)$$

are attracted by φ. Then (3) is convergent.

Estimation of basins of attraction without Lyapunov functions
(Work in progress.)

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-contracting and orientation preserving homeomorphism such that M is a local attractor (asymptotically stable).

$U \subset \mathbb{R}^2$ denotes again the basin of attraction of M.

For simplicity we shall assume $M = \{0\}$, a fixed point.

A necessary condition, for a positively invariant set C, to be contained in U is that $Per(h) \cap C = \emptyset$. When this property is also sufficient?

Theorem

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-contracting and orientation preserving homeomorphism such that 0 is a local attractor. Denote by U the basin of attraction of $\{0\}$. Let $0 \in K$ be a nontrivial invariant continuum such that $\rho(h, \mathbb{R}^2 \setminus K) \in \mathbb{Q}$. Then, $K \subset U$ if and only if $Per(h) \cap K = \{0\}$.

Lemma

(Mather) Let X be a connected topological space and let A and B be open and connected subsets of X such that the boundaries ∂A and ∂B are connected and satisfy

$$\partial A \neq \emptyset, \ \ \partial B \neq \emptyset, \ \ \partial A \cap \partial B = \emptyset.$$

Then one of the following alternatives holds:

(i) $X = A \cup B$ *(ii)* $cl_X A \subset B$ *(iii)* $cl_X B \subset A$ *(iv)* $cl_X A \cap cl_X B = \emptyset$.

Idea of the proof.

We can take a topological disc B, a neighborhood of 0, such that $h(B) \subset int(B)$. The continuum K decomposes ∂B in components.

Each of these components with its endpoints determines a cross-cut, say α, in $S^2 \setminus K$ (note that if $K \cap \partial B = \emptyset$ then $K = \{0\}$).

Note also that $h(\alpha) \cap \alpha = \emptyset$ for every cross-cut $\alpha \subset \partial B$.

Working with a certain h^k instead of h we can assume $\rho(h) = 0$.

Case I.

If $0 \notin \Pi(P_0)$ for a $P_0 \in Fix(h^*)$ one has that $\Pi(P_0)$ is an invariant and non-separating continuum with empty interior.

Using Cartwright-Littlewood theorem there exists a fixed point of h in $\Pi(P_0)$ different from 0, then $Fix(h) \cap (K \setminus \{0\}) \neq \emptyset$.

Case II.

If $0 \in \Pi(P)$ for every $P \in Fix(h^*)$ the components of $\partial B \setminus K$ determine a family of cross-cuts which define an open covering of $Fix(h^*)$.

Then there exists a finite family of arcs in ∂B such that $Fix(h^*) \subset D(C_1) \cup \cdots \cup D(C_r) \subset U^*$ and $Fix(h^*) \cap D(C_j) \neq \emptyset$ for every $j \in \{1, \ldots r\}$.

Since $h(C_j) \cap C_j = \emptyset$ and h contracts area, we have that $h(D(C_j)) \subset D(C_j)$ for every $j \in \{1, \dots r\}$ (Mather + h^* is not a Denjoy).

As a consequence, each $D(C_j) \cap \mathbb{P}$ contains a maximal interval which is an attractor in $\mathbb{P}$.

Since our cross-cuts are disjoint, unless maybe in the end points, there must be repelling fixed points (or intervals) in $\mathbb{P}$ and we arrive to a contradiction.

Therefore $0 \notin \Pi(P)$ for some $P \in Fix(h^*)$ and $Per(h) \cap (K \setminus \{0\}) \neq \emptyset$.

Corollary

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-contracting and orientation preserving homeomorphism such that $\{0\}$ is a local attractor. Denote by U the basin of attraction of $\{0\}$. Let C be a continuum containing 0 such that:
a) $h(C) \subset C$,
b) There exists a topological disc $B \subset U$, a neighborhood of 0, such that $h(B) \subset int(B)$ and there are $m \in \mathbb{N}$ and a component S of $int(B) \setminus C$ with $h^m(S) \cap S \neq \emptyset$.
Then, $C \subset U$ if and only if $Per(h) \cap C = \{0\}$.

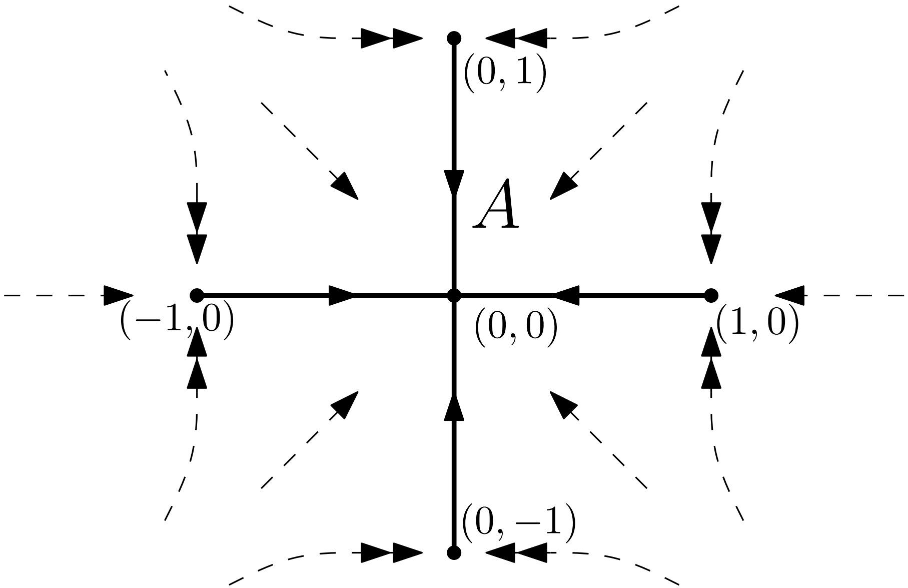
Remark 1.

Note that in the last theorem one obtains the same conclusion if one replace b) by b') "There exists a topological disc B, a neighborhood of 0, such that $h(B) \subset int(B)$ and there are $m \in \mathbb{N}$ and a component S of $int(B) \setminus K$ with $h^m(S) \cap S \neq \emptyset$".

Example.

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-contracting homeomorphism having a global attractor $A = [-1,1] \times 0 \cup 0 \times [-1,1]$ (see figure). h has a fixed point, 0, which is a local attractor and a periodic orbit of period four $(\{(1,0),(0,1),(-1,0),(0,-1)\})$. Given $r > 0$ one has that $K = \bigcap_{i \in \mathbb{N}} h^i(B(0,r))$ is $\{0\}$ if $r < 1$ and A for $r \geq 1$.

Note that for small $\epsilon > 0$ there are four components S_j of $int(B(0,\epsilon)) \setminus A$ and for all of them $h^4(S_j) \cap S_j \neq \emptyset$. A is not contained in the basin of attraction of $\{0\}$ and A contains a periodic orbit of period 4.


$(0,1)$
A
$(-1,0)$
$(0,0)$
$(1,0)$
$(0,-1)$

Corollary

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-contracting and orientation preserving homeomorphism such that $\{0\}$ is a local attractor. Denote by U the basin of attraction of $\{0\}$. Let C be a positively invariant compact cone with vertex 0. Then, $C \subset U$ if and only if $Fix(h) \cap C = \{0\}$.

This is because there are topological discs B, neighborhoods of 0, such that $h(B) \subset int(B)$ and $h(B \setminus C) \cap (B \setminus C) \neq \emptyset$. $\square$

Example. There are area contracting non-dissipative orientation preserving homeomorphisms $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $Fix(\phi) = Per(\phi) = \{0\}$, 0 is an asymptotically stable fixed point and ϕ^* has not periodic points in the circle of prime ends of $\partial U \cup \infty$. Then, $\phi^* : U^* \to U^*$ is an orientation preserving homeomorphisms such that 0 is an asymptotically stable fixed point, ϕ^* contracts area in U. No cones as in the last corollary exist.

Theorem

Let $W \subset \mathbb{C}$ be an open set and $h : W \to h(W) \subset \mathbb{C}$ be an analytic diffeomorphism such that $\{0\} \subset W$ is a local attractor. Denote by $U \subset W$ the basin of attraction of $\{0\}$. Let $C \subset W$ be a positively invariant non-separating continuum such that $0 \in C$ and $|Dh(z)| < 1$ for every $z \in C$. Then, $C \subset U$ if and only if $Per(h) \cap C = \{0\}$.

Let us consider the invariant continuum $K = \bigcap_{i \in \mathbb{N}} h^i(C)$. If K is nontrivial, let $\rho(h)$ be the rotation number of $h^* : \mathbb{P} \to \mathbb{P}$, the homeomorphism induced by h in the circle of prime ends of $S^2 \setminus K$. Using Schwarz's reflection principle we have that h^* is analytic in $\mathbb{P}$. Then h^* can not be a Denjoy homeomorphism. This implies that $\rho(h) \in \mathbb{Q}$. Then, $C \subset U$ if and only if $Per(h) \cap C = \{0\}$.

Theorem

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ *be an area-contracting and orientation reversing homeomorphism such that* $\{0\}$ *is a local attractor. Denote by* U *the basin of attraction of* $\{0\}$*. Let* C *be a positively invariant continuum such that* $0 \in C$*, Then,* $C \subset U$ *if and only if* $Per^2(h) \cap C = \{0\}$*.*

Example. Now we show how orientation-reversing maps appear naturally in the context of certain periodic differential system with symmetries.
Let S denote the symmetry $S((x_1, x_2)) = (x_1, -x_2)$ and consider the differential system

$$\dot{x} = F(t, x), \quad x \in \mathbb{R}^2 \tag{5}$$

where $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$F(t + \pi, S(x)) = S(F(t, x)), \ \text{for each } (t, x) \in \mathbb{R} \times \mathbb{R}^2.$$

We shall assume in addition that: F is C^1, $F(t,0) = 0$ for all t, the trivial solution $x = 0$ is asymptotically stable and that there is global existence and uniqueness for the initial value problem.
The solution satisfying the initial condition $x(t_0) = x_0$ will be denoted by $x(t; t_0, x_0)$ and it is well defined for all $t \in \mathbb{R}$.

Now we consider the maps

$$P_1(x_0) = x(\pi; 0, x_0) \quad \text{and} \quad P_2(x_0) = x(2\pi; \pi, x_0).$$

One has that

$$S \circ P_1 = P_2 \circ S.$$

In fact, if $x(t)$ is a solution then $S(x(t+\pi))$ is also a solution. Then

$$x(t; 0, S(x_0)) = S(x(t+\pi; \pi, x_0)).$$

This implies that $S \circ P_2 = P_1 \circ S$.

From the symmetry of the vector field we have that the system is 2π-periodic with respect to time and the Poincaré map $P(x_0) = x(2\pi; 0, x_0)$ satisfies

$$P = P_2 \circ P_1 = P_2 \circ S \circ S \circ P_1 = (S \circ P_1)^2.$$

It is well known that the maps P_1, P_2 and P preserve the orientation. Then, P can be decompose as $P = r \circ r$ where $r = S \circ P_1$ reverses the orientation.

We are interested in the case when P is area-contracting. This happens, for instance, when the divergence of F with respect to the x variable is negative everywhere: $\sum \frac{\partial F_i}{\partial x_i}(x,t) < 0$ for every x and every t.

Under the above conditions, any P-positively invariant continuum C with $0 \in C$ is contained in U if and only if $(Fix(P) \setminus \{0\}) \cap C = \emptyset$.

To illustrate the previous result consider the system

$$\dot{x}_1 = -x_1 + \psi(x_2), \quad \dot{x}_2 = -x_2 + \lambda(\sin t)x_1,$$

where $\psi \in C^1(\mathbb{R})$ is even and bounded, $\psi(0) = \psi'(0) = 0$ and $\lambda \in \mathbb{R}$ is a parameter. The general conditions imposed to system, including the symmetry, are satisfied in this case.

To check the dissipativity one can employ the Lyapunov function

$$V(x_1, x_2) = \alpha x_1^2 + \beta x_2^2$$

with α and β positive numbers satisfying $\alpha > \frac{\lambda^2}{4}\beta$. It satisfies

$$\dot{V}(x) \leq -\gamma V(x) \quad \text{whenever } ||x|| \geq R.$$

Here γ and R are positive numbers that can be determined. This is sufficient to guarantee dissipativity. The divergence of the system is constant and equal to -2 so that the second condition holds.

It remains to study the stability of the 2π-periodic linear system

$$\dot{y}_1 = -y_1, \quad \dot{y}_2 = -y_2 + \lambda(\sin t)y_1.$$

It has the Floquet solution $y(t) = \mathrm{col}(0, e^{-t})$ and so the corresponding multiplier is $\mu_1 = e^{-2\pi}$. The Jacobi-Liouville formula implies that the product of the multipliers satisfies $\mu_1\mu_2 = e^{-4\pi}$ and so also $\mu_2 = e^{-2\pi}$. This proves the asymptotic stability of the linearized system.

Theorem

Let $h : \mathbb{D} \to \mathbb{D}$ be an orientation preserving C^1-diffeomorphism such that 0 is a fixed point with $\sigma(Dh(0)) = \{\lambda, \bar{\lambda} \in \mathbb{C} \setminus \mathbb{R}, 0 < |\lambda| < 1\}$ and $Per(h) = \{0\}$. Suppose that A is a non-separating and invariant continuum such that $0 \in \partial A$. If there exists a cone with vertex 0, H, such that $(H \setminus \{0\}) \cap A = \emptyset$, then $\{0\} = A$.

Take the double of the disk $\mathbb{D}$, $d(\mathbb{D})$, obtained by pasting two copies of $\mathbb{D}$ along the boundary. It is clear that $d(\mathbb{D})$ is a 2-sphere. Let $d(h) : d(\mathbb{D}) \to d(\mathbb{D})$ the obvious orientation preserving homeomorphism induced by h. Our map $d(h)$ has exactly two attracting periodic points (fixed). Denote them by 0_+ and 0_-.

Assume that A is nontrivial. Then A gives rise to a couple of $d(h)$-invariant and non-separating continua A_- and A_+ of $d(\mathbb{D})$. Let $U = d(\mathbb{D}) \setminus A_-$. One has that U is a simply connected open set containing 0_+ and $i(d(h), 0_+) = i(d(h), U) = 1$. Using previous arguments we have that $\rho(d(h), U) \in \mathbb{Q}$. Now using $d(h)^{-1}$, we obtain periodic orbits of $d(h)$ different from 0_- and 0_+.

Remark.. The map $\phi^* : U^* \to U^*$ of Example 2 admits nontrivial invariant continua as in the last corollary showing that the assumption about $D(h(0))$ can not be removed.

Theorem

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving and dissipative C^1-diffeomorphism having a continuum and non-separating invariant set A such that $Per(h) \subset A$.
Assume that $0 \in A$ is a fixed point such that $\sigma(Dh(0)) = \{\lambda, \bar{\lambda} \in \mathbb{C} \setminus \mathbb{R}, 0 < |\lambda| < 1\}$. If there exists a cone with vertex 0, H, such that $(H \setminus \{0\}) \cap A = \emptyset$, then $\{0\} = A$ if and only if $Per(h) = \{0\}$.

Assume that A is nontrivial. One can easily define an euclidean structure in the plane such that h preserves angles near 0. Let $\gamma(H) \in [0, 2\pi)$ the angle determined by H. Then $\gamma(h^k(H)) = \gamma(H)$ for every $k \in \mathbb{Z}$. If $m \in \mathbb{N}$ is such that $m\gamma(H) > 2\pi$ we have that $h^m(H) \cap h^j(H) \neq \emptyset$ for a $j \leq m$.

Let $\rho(h)$ be the rotation number of $h^* : \mathbb{P} \to \mathbb{P}$, the homeomorphism induced by h in the circle of prime ends of $S^2 \setminus A$.

We have that $\rho(h) \in \mathbb{Q}$ if A is nontrivial. Since A is cellular it has a base of neighborhoods formed by topological discs. Take a topological disc D containing A such that $h^k(D) \subset D$ for a big enough $k \in \mathbb{N}$.

We have that $i(h^k, int(D)) = 1$. Since $Per(h) \subset A$ we have that $i(h^k, \infty) = 1$. Now we can find periodic orbits different from 0. Then $Per(h) \setminus \{0\} \neq \emptyset$.

Remark. Note that in the last theorem we do not assume h to contracts area nor H to be positively/negatively invariant. Note also that H can be arbitrarily small. Example 2 provides a construction in $\mathbb{D}$, even if h^* contracts area in $int(\mathbb{D})$, showing that in the above theorem differentiability and $D(h(0))$ play an essential role.

Such an attractor A appears naturally when h is dissipative. Next corollary is a direct consequence.

Corollary

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving dissipative diffeomorphism. Let W be the basin of attraction of $\{\infty\}$ for h^{-1} and consider the global attractor $A = \mathbb{R}^2 \setminus W$. Assume that $0 \in A$ is a fixed point such that $\sigma(Dh(0)) = \{\lambda, \bar{\lambda} \in \mathbb{C} \setminus \mathbb{R}, 0 < |\lambda| < 1\}$. If there exists a cone with vertex 0, H, such that $(H \setminus \{0\}) \subset W$, then $\{0\} = A$ if and only if $Per(h) = \{0\}$.

Remark. The above results give sufficient conditions under which a continuum containing the closure of the periodic orbits of a planar diffeomorphism can not be isolated as an attractor.

THANK YOU VERY MUCH