Franks-Misiurewicz conjecture for extensions of irrational rotations

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1. if I has rational slope, then I contains rational points.
2. if I has irrational slope, then one end-point of I is rational.

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LeCalvez and Tal have shown that whenever a rotation set is an interval of irrational slope containing a rational point, this rational point must be an endpoint.

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- *are non-compactly generated,*
- *have points realizing both extremal rotation vectors.*

Theorem (Koropecki- _ -Sambarino)

Let $f \in Homeo_0(\mathbb{T}^2)$ *be an extension of an irrational rotation. Then* $\rho(F)$ *is a singleton.*

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Theorem

Let $f \in Homeo_0(\mathbb{T}^2)$ be minimal. Then the rotation set is either

1. *a singleton or*
2. *an irrational slope interval containing no rational points,*

where both situation can be realized.

Dragging Lemma

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Lemma (Dragging Lemma)

Assume the existence of an essential loop $\gamma \subset \mathbb{A}$ and a pair of points $z, w \in \mathbb{A}$ so that for some closed topological disks $V_z, V_w, V_{f(z)}, V_{f(w)}$ of diameter bounded above by ε with $z \in V_z, w \in V_w, f(z) \in V_{f(z)}, f(w) \in V_{f(w)}$, we have that $\gamma \cap V_z,\ \gamma \cap V_w,\ f(\gamma) \cap V_{f(z)},\ f(\gamma) \cap V_{f(z)}$ are all singletons. Then, for any two arcs $I \subset \gamma$ joining V_z with V_w and $J \subset f(\gamma)$ joining $V_{f(z)}$ with $V_{f(z)}$, we have

$$\omega(J) \geq \omega(f(I)) - 5.$$

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For any extension of an irrational rotation f consider a lift $\tilde{f} \in \mathrm{Homeo}_0(\mathbb{A})$. Let ε given as in the Dragging lemma. We obtain as corollary:

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Corollary (Relaxed version)

Assume f is an e.i.r. having a non-trivial interval as rotation set $[\rho^-, \rho^+] \times \{\alpha\}$, with $\rho^- < -20$ and $\rho^+ > 20$. Let $x, y \in \mathbb{A}$ be the projection of two points realizing (ρ^-, α) and (ρ^+, α) respectively, and an essential curve $\gamma \subset \mathbb{A}$ so that for every $n \in \mathbb{N}$

$$d(\tilde{f}^n(x), \tilde{f}^n(\gamma)) < \varepsilon \text{ and } d(\tilde{f}^n(y), \tilde{f}^n(\gamma)) < \varepsilon.$$

Then, there exists a sequence of arcs $(I_n)_{n\in\mathbb{N}}$ with $I_n \subset \tilde{f}^n(\gamma)$ so that

$$\omega(I_n) > 5n.$$

In particular $\omega(I_n) \to \infty$.