Spin-coherent, Basis–coherent, and Anti-coherent States

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Quantum Uncertainty Relations (HUR)

Heisenberg uncertainty relation (1927)

Formulation of **Kennard** (1927) for the product of variances of position and momentum ($\hbar = 1$)

$$\Delta^2 x \, \Delta^2 p \ \geq \ \frac{1}{4} \, .$$

A more general (but **state dependent** !)

formulation of Robertson (1929)

for arbitrary operators A an B. Let $\Delta^2 A = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$ be the variance of an operator A. Then for any state $|\psi\rangle$

$$\Delta^2 A \, \Delta^2 B \ \geq \ \frac{1}{4} \big|\langle\psi|AB - BA|\psi\rangle\big|^2$$

As $[x, p] = xp - px = i$ the latter form implies the former bound.


beeline

Otton Nikodym & Stefan Banach,

talking at a bench in Planty Garden, **Cracow, summer 1916**

Infinite Hilbert space $\mathcal{H}_\infty$

Harmonic Oscillator Coherent States (CS)

Vacuum state, $|0\rangle$ and **commutation relation**, $[a, a^\dagger] = 1$,
with $a = (\hat{x} + i\hat{p})/\sqrt{2}$ and with $z = (x + ip)/\sqrt{2}$ yield "standard"

Displacement operator **coherent states**: $|z\rangle := \exp(za^\dagger - z^* a)|0\rangle$

satisfying identity resolution: $\frac{1}{2\pi}\int d^2 z |z\rangle\langle z| = \mathbb{1}$.

Equivalent conditions:

Anihilation operator CS: $a|z\rangle = z|z\rangle$,

Minimum uncertainty CS: $\Delta x \Delta p = 1/2$ (saturation of HUR)

Husimi function & Wehrl entropy

Q–representation: $Q_\rho(z) := \mathrm{Tr}\rho|z\rangle\langle z| = \langle z|\rho|z\rangle$.

Wehrl entropy: $S_W(\rho) := -\frac{1}{2\pi}\int d^2 z\, Q_\rho(z) \log Q_\rho(z)$.

Wehrl conjecture (1978) $\longrightarrow$ **Lieb theorem** (1978):

Minimum of S_W is achieved for **coherent** states, $S_W(\rho) \geq 1$.

Finite dimensional Hilbert space $\mathcal{H}_N$

SU(2) [Bloch] Coherent States

Let $N = 2j + 1$ where j is the total spin.
For **vacuum state** set the eigenstate $|j,j\rangle$ of momentum operator J_z and **commutation relation**, $[J_i, J_k] = 2\mathrm{i} J_l e_{ikl}$ [group $SU(2)$] with $z = \tan(\theta/2)e^{\mathrm{i}\phi}$ yield **Bloch CS**

$$|z\rangle = |\theta,\phi\rangle := \frac{1}{(1+|z|^2)^j} \exp[z(J_x - \mathrm{i}J_y)]\, |j,j\rangle$$

satisfying identity resolution: $\frac{N}{4\pi}\int_\Omega d\Omega\, |z\rangle\langle z| = \mathbb{1}$.

Husimi function & Wehrl entropy

Q–representation: $Q_\rho(z) := \mathrm{Tr}\rho|z\rangle\langle z| = \langle z|\rho|z\rangle$.

Wehrl entropy: $S_W(\rho) := -\frac{1}{2\pi}\int_\Omega d\Omega\; Q_\rho(z)\log Q_\rho(z)$.

Lieb conjecture (1978) $\rightarrow$ **Lieb-Solovej theorem** (2014):
Minimum of S_W is achieved for **coherent** states, $S_W(\rho) \geq 1 - 1/N$.

Wawel castle in Cracow

Stellar Representation & Anti-coherent states

Stellar representation of a pure state $|\psi\rangle \in \mathcal{H}_N$

Husim function of a pure state $Q_\psi(z) := |\langle z|\psi\rangle|^2$
forms a polynomial $f(z)$ or order $n = N - 1 = 2j$.

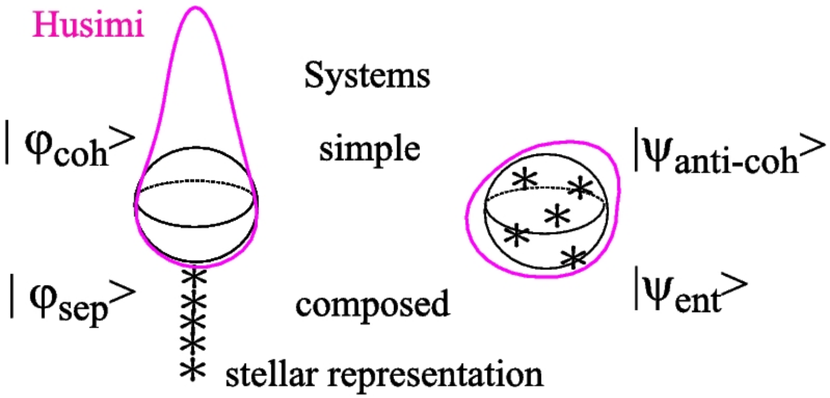
Thus it has n zeros (possibly degenerated!) on the complex plane
or on the sphere – stereographic projection $z = \tan(\theta/2)e^{i\phi}$.

Hence any state $|\psi\rangle \in \mathcal{H}_N$ can be **uniquely** defined
by a collection of n points on the sphere, called **stars**.

For coherent state all stars sit in the antipodal point
One defines **anti-coherent states** as these which:

a) **maximize** the Wehrl entropy (among pure states)
b) are **most distant** from the set of coherent states
(e.g. with respect to the geodesic, Fubini–Study distance)

Thus **anti-coherent states** correspond to
'uniform' distribution of **stars** on the sphere
(observation: random states are close to anti-coherent!)



Stellar representation and **Husimi function**
for **coherent** and **anti-coherent** states

Vector Coherent States & Separable States

Higher vector coherent states – group $SU(K)$ CS

- Take generators S_k of the group $SU(K)$, a **highest weight** state $|\mu\rangle$,
a vector $z = (z_1, \dots, z_m)$, and obtain
a **vector coherent state** $|z\rangle = C_z \prod_k \exp(z_k S_k)|\mu\rangle$

Lieb-Solovej theorem (2016):

Coherent states miminize the (generalized) Wehrl entropy.

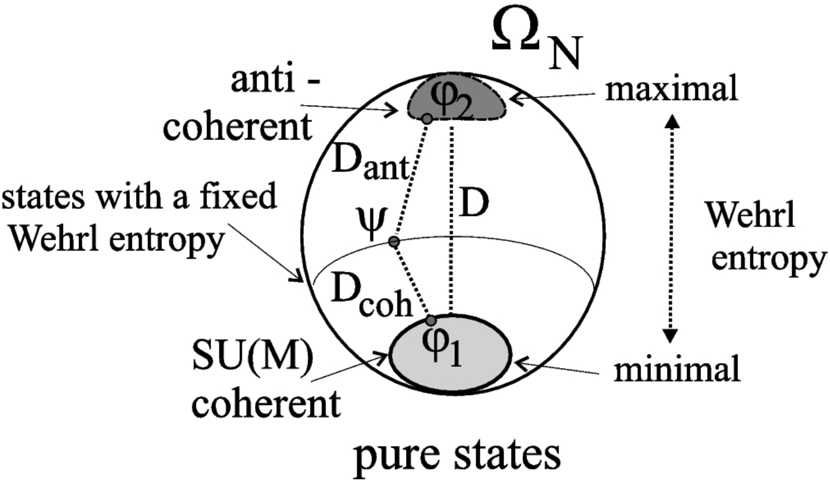
Stellar representation: now 'stars' live in $\mathbb{C}P^{K-1}$.

Texas effect: for $N = K$ every state is $SU(K)$ coherent!

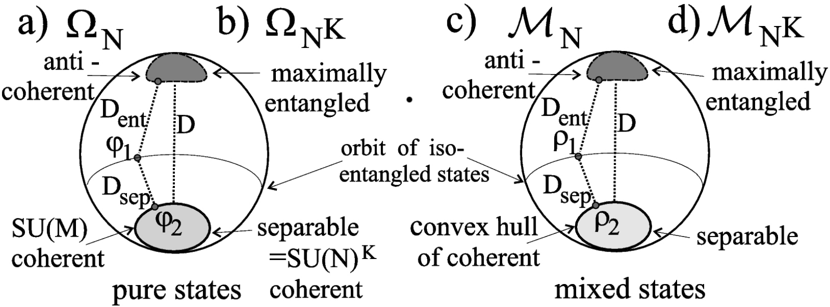
Separable & Entangled States

Consider a composed Hilbert space $\mathcal{H}_{\mathcal{KM}} = \mathcal{H}_{\mathcal{K}} \otimes \mathcal{H}_{\mathcal{M}} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$.
Definition: a product state $|\phi_{\mathrm{sep}}\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$ is called **separable**,
while any other state is called **entangled**.

A separable state is **coherent** with respect to the group $SU(K) \times SU(M)$,
a maximally **entangled** state is **anti-coherent**
with respect to a certain measure of non-coherence.



Stratification of the manifold $\Omega_N = \mathbb{C}P^{N-1}$ of pure states of a **simple** system into **strata** of states with the same **degree of coherence** (the Wehrl entropy or the distance to the set of **coherent** states).



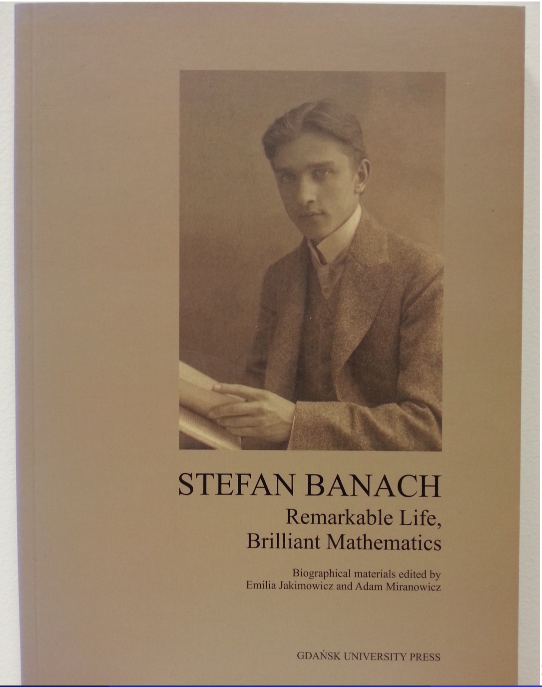
Stratification of set Ω of **pure states** of
a) **simple** system with N levels, b) **composed** system $N^{\times K}$;
set $\mathcal{M}$ of **mixed states** for
c) **simple** system with N levels, d) **composed** system $N^{\times K}$.

Wawel castle in Cracow

Ciesielski theorem

Ciesielski theorem: With probability $1 - \epsilon$ the bench **Banach** talked to **Nikodym** in 1916 was localized in η-neighbourhood of the **red arrow**.

Bench commemorating discussion between
Stefan Banach and **Otton Nikodym** (**Kraków, summer 1916**)


STEFAN BANACH
Remarkable Life,
Brilliant Mathematics
Biographical materials edited by
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Entropic Uncertainty Relations (EUR)

Continuous case

Define continuous **(Boltzmann–Gibbs)** **entropies**:

$$S(x) = -\int dx|\psi(x)|^2 \ \ln|\psi(x)|^2$$

and

$$S(p) = -\int dp|\psi(p)|^2 \ \ln|\psi(p)|^2 .$$

Then

$$S(x) + S(p) \geq \ln(e\pi) \ .$$

Białynicki-Birula, **Mycielski (1975)** and **Beckner, (1975)**

generalizations for **Rényi** α–entropies,

$$S_\alpha(x) := \frac{1}{1-\alpha} \ln\left(\int dx|\psi(x)|^{2\alpha}\right)$$

Białynicki-Birula, **(2006)**

Entropic Uncertainty Relations - N dimensional case

State $|\psi\rangle = \sum_i^N a_i|i\rangle = \sum_j b_j|\beta_j\rangle$ is expanded in the eigenbases of operators A and B, related by a unitary matrix $U_{ij} = \langle i|\beta_j\rangle$.

Let Shannon entropies in both expansion be
$S^A(\psi) = -\sum_{i=1}^N p_i \ln p_i = S(p)$ with $p_i = |a_i|^2$, $\sum_i p_i = 1$ and
$S^B(\psi) = -\sum_{j=1}^N q_j \ln q_j = S(q)$ with $q_j = |b_j|^2$, $\sum_j q_j = 1$.

Let $c_1(A,B) = \max_{ij} |U_{ij}|^2$. Then for any state $|\psi\rangle \in \mathcal{H}_N$ we have

$$S^A(\psi) + S^B(\psi) \geq -2\ln[(1+\sqrt{c_1})/2] =: B_D$$

Deutsch, (1983), later improved

$$S^A(\psi) + S^B(\psi) \geq -\ln c_1 =: B_{MU}$$

by **Maassen, Uffink, (1988)**,

Entropic Uncertainty Relations - N dimensional case

Example: the Fourier matrix F_N

Unitary matrix which defines the second (**unbiased !**) basis

$$U_{jk} = (F_N)_{jk} := \frac{1}{\sqrt{N}} \exp(i\, 2\pi jk/N) \quad \text{with} \quad j,k = 0,1,\ldots,n-1.$$

then $c_1 = \max_{jk} |U_{jk}|^2 = 1/N$.
The bound of **Maassen–Uffink** gives

$$S(p) + S(q) \;\geq\; -\ln c_1 = \ln N$$

If $|\psi\rangle = (1,0,\ldots,0)$ then $S_A = 0$ and $S_B = \ln N$ so bound is saturated...

The same bound holds for any unitary **complex Hadamard matrix** H, for which $|H_{ij}|^2 = 1/N$ for all $i,j = 1,\ldots N$.

In a general case the bounds of **Maassen and Uffink** are not optimal.

How to improve them ??

An alternative approach: Key ingredients used

A) An algebraic tool: **Majorization**

Consider two probability vectors of length N ordered decreasingly, $x = (x_1 \geq x_2 \geq \dots x_N \geq 0)$ and $y = (y_1 \geq y_2 \geq \dots y_N \geq 0)$.

The vector x is called to be **majorized** by y, written $x \prec y$, if

$$\sum_{i=1}^{m} x_i \ \leq \ \sum_{i=1}^{m} y_i, \quad \text{for} \quad m = 1, \dots N-1$$

Majorization $x \prec y$ implies inequalities for **Renyi α–entropies**

$$\frac{1}{1-\alpha} \ln\Big(\sum_{i=1}^{N} x_i^{\alpha}\Big) =: S_\alpha(x) \ \geq \ S_\alpha(y) := \frac{1}{1-\alpha} \ln\Big(\sum_{i=1}^{N} y_i^{\alpha}\Big)$$

(and other **Schur–concave** functions)

B) **Bi-entropy and product probability vectors**

Let $p \otimes q = (p_1 q_1, p_1 q_2, \dots, p_1 q_N, \ \dots p_N q_N)$ denotes a product probability vector of size N^2.

Then the sum of bientropies reads $S_\alpha(p) + S_\alpha(q) \ = \ S_\alpha(p \otimes q)$.

To arrive at an **entropic uncertainty relation** we need to find a vector Q majorizing the **product** $p \otimes q$.

1) Product Majorization EUR (PRŻ. 2013)

Let $k = 1, \dots N-1$: **spectral norms** of all submatrices of unitary U

Let $A_{m,n}$ denote the **maximal** $m \times n$ submatrix of U.

Define $s_k \;:=\; \max\{||A_{1,k}||,\; ||A_{2,k-1}||, \dots, ||A_{k-1,2}||,\; ||A_{k,1}||\}$.

We have $s_k \;\geq\; s_{k-1}$ and $R_k \;:=\; \left(\frac{1+s_k}{2}\right)^2 \geq R_{k-1}$.

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Theorem: For any unitary U of order N

the following **tensor–product majorization** relation holds:

$$(p \otimes q) \;\prec\; (R_1\;, R_2 - R_1\;, \dots, R_{N-1} - R_{N-2},\; 1 - R_{N-1}) =: Q.$$

This implies an explicit **'product' majorization entropic uncertainty relation**, valid for any pure state $|\psi\rangle$ and any **Renyi** entropy S_α

$$S_\alpha(p) + S_\alpha(q) \;\geq\; S_\alpha(Q) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{N^2} Q_i^\alpha.$$

Similar results: **Friedland, Gheorghiu, Gour** (2013)

Example: matrix of size $N = 4$, the second bound ($k = 2$)

$k = 2$: norms of 2-subvectors of unitary U

We look for a majorization relation of the type

$$(p \otimes q) \prec Q = (R_1\ , R_2 - R_1\ , 1 - R_2,\ 0, \ldots 0)\,. \qquad (1)$$

Consider the **longest** 2–sub–vector of unitary U and denote its norm by $s_2 = \max\left\{ \max_{i,j_1,j_2} \sqrt{|U_{ij_1}|^2 + |U_{ij_2}|^2},\ \max_{i_1,i_2,j} \sqrt{|U_{i_1 j}|^2 + |U_{i_2 j}|^2} \right\}$

Theorem 1 implies that the above **majorization relation**

with $R_2 = \left(\frac{1+s_2}{2}\right)^2$ **holds !**

Example: On orthogonal matrix $U \in U(4)$ with entries truncated to two decimal digits

$$\begin{bmatrix} 0.19 & 0.50 & \boxed{-0.64} & \boxed{0.55} \\ -0.62 & 0.54 & -0.21 & -0.52 \\ 0.52 & -0.21 & -0.54 & -0.62 \\ -0.55 & \mathbf{-0.64} & -0.50 & 0.19 \end{bmatrix}$$

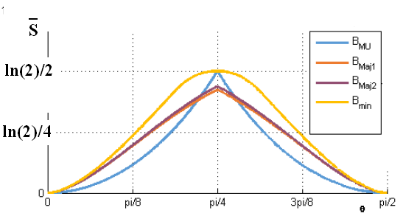
2) Strong Majorization EUR (2014)

Direct-sum majorization relation = improved lower bound

$$p \oplus q \prec \{1\} \oplus W,$$

where the majorizing vector $W = (s_1, s_2 - s_1, \dots s_N - s_{N-1}, 0, \dots, 0)$ is constructed out of the same largest norms s_k of submatrices of U.
This implies an explicit **strong majorization entropic uncertainty relation**

$$S_\alpha(p) + S_\alpha(q) \; \geq \; S_\alpha(W) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{N^2} W_i^\alpha.$$



Rudnicki, Puchała, K. Ż, PRA (2014).

Related bounds: **Coles, Piani** (2014)

← Bounds for an orthogonal rotation matrix $O(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Upper bound for the sum of entropy

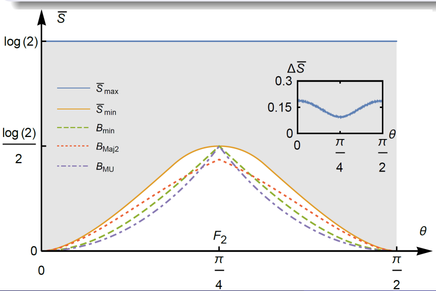
Two orthogonal measurements in $L = 2$ bases

Proposition: for any $U \in U(N)$ there exist a state $|\psi\rangle \in \mathcal{H}_N$ **mutually unbiased** with respect to a basis B and $B' = UB$, so that $|\langle i|\psi\rangle|^2 = |\langle i|U|\psi\rangle|^2 = 1/N$,

Korzekwa, Lostaglio, Jennings and Rudolph (2014).

It implies a 'trivial' *Entropic Certainty Relation*: (saturation)

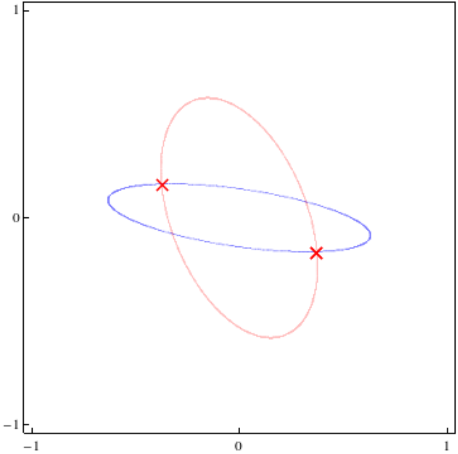
$$\bar{S} = \frac{1}{2}\Big(S(p) + S(q)\Big) \leq \log N$$



A state $|\psi\rangle \in \mathcal{H}_N$ such that $|\langle i|\psi\rangle|^2 = 1/N$ is called **coherent** with respect to basis $\{|i\rangle\}$, as the sum of **coherences** (absolute values of off-diagonal elements) is maximal.

$\leftarrow$ Upper and lower bounds for $\bar{S}$ for orthogonal matrices $O(\theta)$ of size 2.

What known theorem this figure illustrates?



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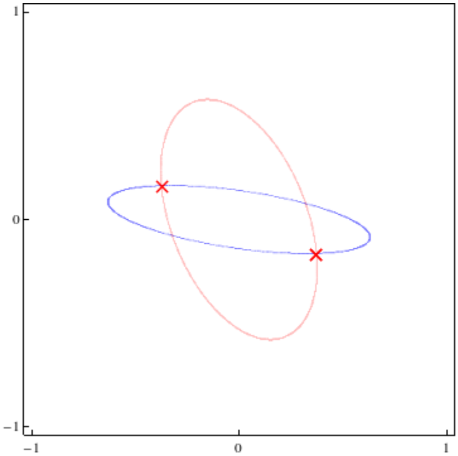


Two great circles at the sphere do cross !

$\Leftrightarrow$ **Equator** is **non-displacable** in S^2.

Non-displacable tori in CP^{N-1}

Observation. The set $\mathcal{C}$ of all N–dimensional states **mutually unbiased** with respect to a basis $\{|i\rangle\}$ forms an (N-1)– **great torus** T^{N-1}, as $|\psi\rangle = \frac{1}{\sqrt{N}}\big(1, \exp(i\phi_1), \exp(i\phi_2), \ldots, \exp(i\phi_{N-1})\big)$.

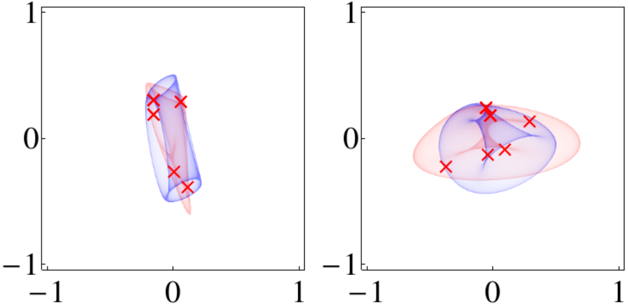
Do two **great two–tori** T_2 embedded in $\mathbb{C}P^2$ intersect?

Non-displacable tori in CP^{N-1}

Observation. The set $\mathcal{C}$ of all N–dimensional states **mutually unbiased** with respect to a basis $\{|i\rangle\}$ forms an (N-1)– **great torus** T^{N-1}, as $|\psi\rangle = \frac{1}{\sqrt{N}}(1, \exp(i\phi_1), \exp(i\phi_2), \ldots, \exp(i\phi_{N-1}))$.

Do two **great two–tori** T_2 embedded in $\mathbb{C}P^2$ intersect?

Yes, a **great K-torus** T_K is **non–displacable** in $\mathbb{C}P^K$, **Cho** (2004).



Crossing points marked **X** represent **mutually unbiased states** (mutually coherent states).

Projections of two 2–tori embedded in $\mathbb{C}P^3$: 6 crossing points marked **X**

Entropic uncertainty relations for L measurements

in basis given by L unitary matrices, $U^{(1)}, \dots, U^{(L)}$:
Define coefficients $\mathcal{S}_k$: $\{U^{(j)}\}_{j=1}^{L}$,

$$\mathcal{S}_k = \max\{\sigma_1^2(|u_{i_1}^{(j_1)}\rangle, |u_{i_2}^{(j_2)}\rangle, \dots, |u_{i_{k+1}}^{(j_{k+1})}\rangle)\},$$

being maximal squares of norms of rectangular matrices of size $N \times (k+1)$ formed by $k+1$ columns taken from the **concatenation** of all L unitary matrices.

The following **majorization** relation holds,

$$\{p_i^{(j)}\}_{i,j=1}^{N,L} \prec \{1, \mathcal{S}_1 - 1, \mathcal{S}_2 - \mathcal{S}_1, \dots\}.$$

and it implies the **poli-measurement** **entropic uncertainty relation**

$$\sum_{i=1}^{L} S\big(p^{(i)}\big) \geq -\sum_{i=1}^{NL} (\mathcal{S}_i - \mathcal{S}_{i-1}) \ln(\mathcal{S}_i - \mathcal{S}_{i-1})$$

Mutually Unbiased Bases

- Two orthogonal bases consisting of N vectors each in $\mathcal{H}_N$ are called **mutually unbiased** (MUB) if
$$|\langle\phi_i|\psi_j\rangle|^2 = \frac{1}{N}\ , \quad \text{for} \quad i,j = 1,\ldots,N\ .$$
- Full sets of $(N+1)$ MUB's are known if dimension is a **power of prime**, $N = p^k$. For $N = 6 = 2 \times 3$ only $3 < 7$ MUB's are known!
- A transition matrix $H_{ij} = \langle\phi_i|\psi_j\rangle$ from one **unbiased** basis to another forms a **complex Hadamard** matrix, which is
  a) **unitary**, $H^\dagger = H^{-1}$,
  b) has "**unimodular**" entries, $|H_{ij}|^2 = 1/N$, $i,j = 1,\ldots,N$.
- Classification of all **complex Hadamard matrices** is complete for $N = 2,3,4,5$ only. (**Haagerup** 1996)
  see Catalog of **complex Hadamard matrices**, at
  http://chaos.if.uj.edu.pl/∼karol/hadamard

3 measurements in $\mathcal{H}_2$ and Mutually Unbiased Bases.

Nontrivial upper bound $(*)$ $=$ **Certainty Relations** (**Sanchez** 1993)

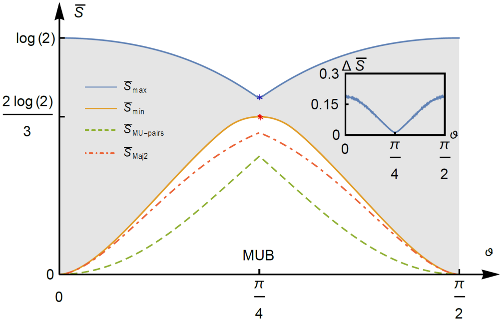
$L = 3$ measurements in 3 bases:

for one qubit: $N = 2$

$U^{(1)} = \mathbb{I}_2$

$$U^{(2)} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$U^{(3)} = \begin{pmatrix} \cos\theta & \sin\theta \\ i\sin\theta & -i\cos\theta \end{pmatrix}.$$



For $\theta = 0$ all three measurements coincide so $\bar{S}_{min} = 0$,
For $\theta = \pi/4$ these three bases become **maximally unbiased** (MUB)
so the **lower bound** $(*)$ for the sum of the entropies is **the largest**,
while the **upper bound** $(*)$ is the smallest!

The root mean square deviation of the mean entropy averaged over all pure states, $\Delta(\bar{S}) = (\langle \bar{S}^2\rangle_\psi - \langle \bar{S}\rangle_\psi^2)^{1/2}$, is **the smallest** for MUB.

3 measurements in $\mathcal{H}_2$ and Mutually Unbiased Bases.

New nontrivial **upper bounds** $B_{\max}$ = **Certainty Relations**
and **lower bounds** $B_{\min}$ = **Uncertainty Relations**, **2015**

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for one qubit: $N = 2$

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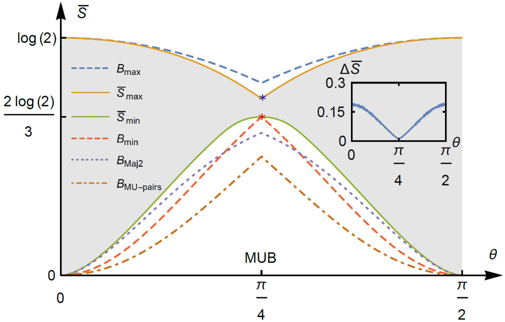

For $\theta = 0$ all three measurements coincide so $\bar{S}_{min} = 0$,
For $\theta = \pi/4$ these three bases become **maximally unbiased** (MUB)
and the **lower bound** $B_{\min}$ - - - for the average entropy $\bar{S}$ is **the largest**
- it coincides with the bound of Sanchez and becomes tight,
while the **upper bound** $B_{\max}$ - - - is the smallest!

Stefan Banach sitting at a bench close to the Wawel Castle

Sculpture: Stefan Dousa Fot. Andrzej Kobos

Concluding remarks

- **Spin coherent** states in H_N mimimize the Wehrl Entropy
- Pure states for which Wehrl Entropy is maximal are **anti-coherent**
- Composed $K \times K$ systems: separable states are **coherent** with respect to group $SU(K) \times SU(K)$; anti-coherent states are **maximally entangled**
- **Three Majorization Entropic Uncertainty Relations** (lower bounds $B \le S_{\min} \le \bar{S}$) derived for any unitary $U \in U(N)$:
  The **2014 bound** B_{Maj2} based on **simple sum** dominates
  the **2013 bound** B_{Maj1} based on *tensor product* majorization.
  The **2015 bound** $B_{\rm min}$ based on purity of the POVM works better
  in vicinity of the **Fourier** matrix (and MUBs).
- Upper bounds for mean entropy, $\bar{S} \le S_{max} \le B_{\rm max}$
  form universal **Entropic Certainty Relations**.
- Great torus T_{N-1} is **non-displacable** in $\mathbb{C}P^{N-1}$. Thus for any two bases in $\mathcal{H}_N$ there exists a **mutually basis coherent state**, for which certainty relation is saturated $\bar{S} = \log N$.
- Generalization for L orthogonal measurements.

Bench commemorating discussion between
Stefan Banach and **Otton Nikodym** (**Kraków, summer 1916**)

Sculpture: Stefan Dousa Fot. Andrzej Kobos

Opened in Planty Garden, Cracow, Oct. 14, 2016