Shearlets: Theory, Applications, and Generalizations

Felix Voigtlaender (*on behalf of Gitta Kutyniok*)

Coherent States and their Applications: A Contemporary Panorama
November 15, 2016

Outline

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1. Multiscale Systems inspired by Coherent States
   - The Quasi-regular Representation
   - Wavelets and their Limitations
   - Continuous Shearlet Systems
   - Discrete Shearlet Systems

2. Shearlets and their Applications
   - Compactly supported Shearlets
   - Optimal sparse Approximation
   - Applications with Compressed Sensing

3. Shearlets, Coherent States & Decomposition spaces

The quasi-regular representation of $\mathbb{R}^d \rtimes \mathrm{GL}\left(\mathbb{R}^d\right)$

The quasi-regular representation

$$\pi : \mathbb{R}^d \rtimes \mathrm{GL}\left(\mathbb{R}^d\right) \to \mathcal{U}\left(L^2\left(\mathbb{R}^d\right)\right), (x,h) \mapsto \mathsf{T}_x \mathsf{D}_h$$

with

$$\left(\mathsf{T}_x f\right)(y) = f(y-x) \qquad \text{and} \qquad \mathsf{D}_h f = \left|\det h\right|^{-1/2} \cdot f \circ h^{-1}$$

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Theorem (Führ)

$\pi|_G$ is irreducible and square-integrable if and only if

1. *There is $\xi_0 \in \mathbb{R}^d$ such that the dual orbit $\mathcal{O} := H^T \xi_0 \subset \mathbb{R}^d$ is open and of full measure.*
2. *The isotropy group $H_{\xi_0} := \left\{ h \in H \,\middle|\, h^T \xi_0 = \xi_0 \right\} \leq H$ is compact.*

Wavelets

Definition: Let $\psi \in L^2(\mathbb{R})$ be a wavelet, i.e., $\int_{\mathbb{R}} |\widehat{\psi}(\omega)|^2 \frac{d\omega}{|\omega|} < \infty$. The associated **homogeneous continuous** wavelet system is

$$\left(|a|^{-1/2} \cdot \psi\left(a^{-1}(\bullet - b)\right)\right)_{a\in\mathbb{R}^*, b\in\mathbb{R}} = (\pi(b,a)\,\psi)_{(b,a)\in\mathbb{R}\rtimes\mathbb{R}^*}.$$

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Definition: Let $\phi, \psi \in L^2(\mathbb{R})$ be a scaling function and a wavelet. Then the **inhomogeneous discrete** wavelet system generated by ϕ, ψ is

$$\left(\phi(\bullet - m)\right)_{m\in\mathbb{Z}} \cup \left(2^{j/2} \cdot \psi\left(2^j \bullet - m\right)\right)_{j\in\mathbb{N}_0, m\in\mathbb{Z}}.$$

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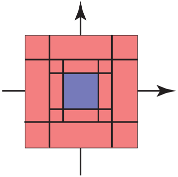
Definition: Let ϕ, ψ as above. A 2D inhomogeneous discrete wavelet system is defined by

$$\left(\phi^{(1)}(\bullet - m)\right)_{m\in\mathbb{Z}^2} \cup \left(2^j \cdot \psi^{(\ell)}\left(2^j \bullet - m\right)\right)_{j\in\mathbb{N}_0, m\in\mathbb{Z}^2, \ell\in\{1,2,3\}}$$



where

$$\phi^{(1)} := \phi \otimes \phi, \quad \psi^{(1)} := \phi \otimes \psi, \quad \psi^{(2)} := \psi \otimes \phi, \quad \psi^{(3)} := \psi \otimes \psi.$$

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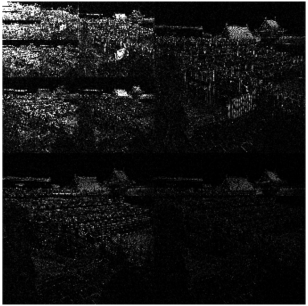

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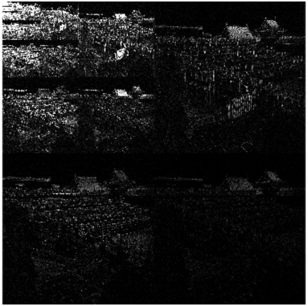
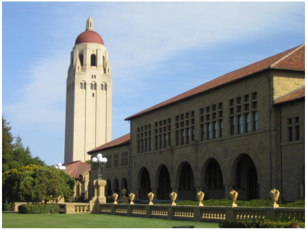
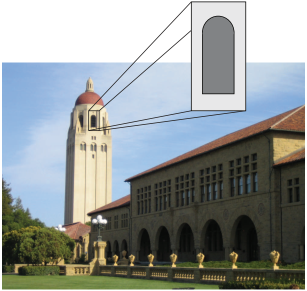


Theorem: Discrete wavelets provide optimal approximation rates for those $f \in L^2(\mathbb{R}^2)$ which are C^2 apart from **point singularities**:

$$\|f - f_N\|_{L^2} \lesssim N^{-1/2} \qquad (N \to \infty).$$



Living on the ~~point~~ edge



Natural images are governed by curved singularities, not point singularities!

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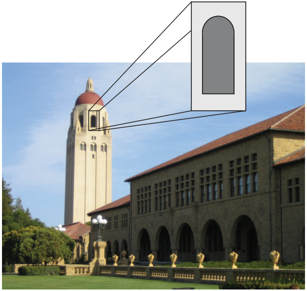

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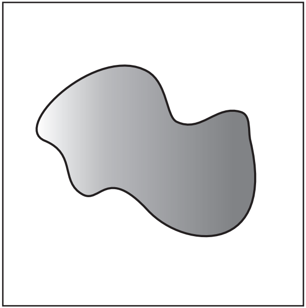

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Definition (Donoho; 2001)

With $Q := (0,1)^2$, the set of cartoon-like functions is defined as

$$\mathcal{E}^2(\mathbb{R}^2) = \left\{ f_0 + f_1 \cdot \mathbb{1}_B \,\middle|\, \partial B \subset Q \text{ closed } C^2 \text{ curve and } f_0, f_1 \in C_c^2(Q) \right\}.$$

Approximation of cartoon-like functions

Theorem: For $f \in \mathcal{E}^2(\mathbb{R}^2)$, the best N-term approximation f_N

- using Fourier basis: $\|f - f_N\|_{L^2} \lesssim N^{-1/4}$,
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$$\|f - f_N\|_{L^2} \lesssim N^{-\theta} \qquad \forall N \in \mathbb{N} \qquad \forall f \in \mathcal{E}^2(\mathbb{R}^2),$$

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Intuitive explanation: This is caused by the scalar dilations:

Design Goals for a new representation system

Design a representation system $\Psi = (\psi_\lambda)_\lambda \subset L^2\left(\mathbb{R}^2\right)$ such that:

- Ψ is a frame:

$$f \in L^2 \mapsto (\langle f, \psi_\lambda \rangle)_\lambda \in \ell^2 \mapsto \sum_\lambda \langle f, \psi_\lambda \rangle \widetilde{\psi_\lambda} = f.$$

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- Fast algorithms which treat the analog and digital world uniformly.

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Non-exhaustive list of approaches:

- Ridgelets (Candès and Donoho; 1999)
- Curvelets (Candès and Donoho; 2002)
- Contourlets (Do and Vetterli; 2002)
- Bandlets (LePennec and Mallat; 2003)
- Shearlets (Kutyniok and Labate; 2006)

Continuous shearlet systems

Instead of the dilation group $H = \mathbb{R}^* \cdot \mathrm{SO}\left(\mathbb{R}^d\right)$ leading to wavelets, we consider the dilation group

$$H = \left\{ \varepsilon \cdot \underbrace{\begin{pmatrix} a & 0 \\ 0 & a^{1/2} \end{pmatrix}}_{=:A_a} \underbrace{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}}_{=:S_s} \,\middle|\, \varepsilon \in \{\pm 1\}, a \in (0,\infty), s \in \mathbb{R} \right\}.$$

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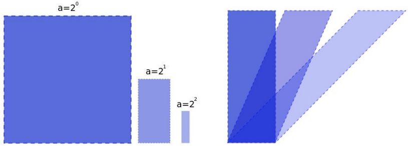
Main properties:

- Parabolic scaling
- Different orientations via shearing.



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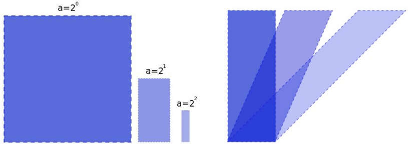

Advantages of shearing:

- The shearing matrices $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ leave the digital grid $\mathbb{Z}^2$ invariant.
- Uniform theory for the analog and digital situation.

The set

$$\left\{ (S_k A_{2^j})^{-1} = \left[\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix} \right]^{-1} : j,k \in \mathbb{Z} \right\}$$

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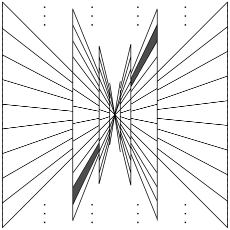
Hence, coorbit theory (Feichtinger & Gröchenig) motivates the following definition:

Definition (Kutyniok, Labate; 2006): For $\psi \in L^2(\mathbb{R}^2)$, the associated discrete shearlet system is

$$\left(2^{\frac{3}{4}j} \cdot \psi(S_k A_{2^j} \bullet - m) \right)_{j,k \in \mathbb{Z}, m \in \mathbb{Z}^2} = \left(\mathbf{D}_{(S_k A_{2^j})^{-1}} \mathbf{T}_m \psi \right)_{j,k \in \mathbb{Z}, m \in \mathbb{Z}^2}.$$

The induced frequency tiling



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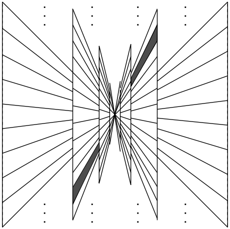

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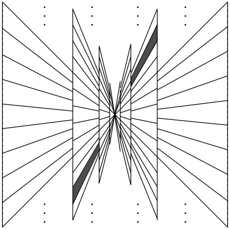

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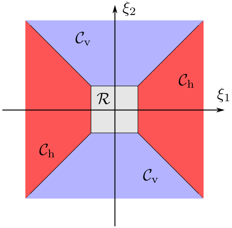
Solution: Use cone-adapted shearlets:



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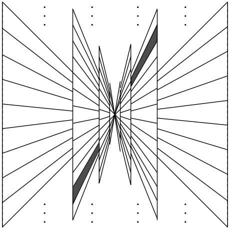

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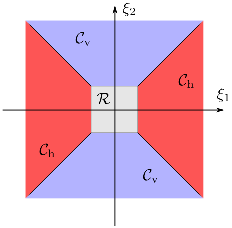

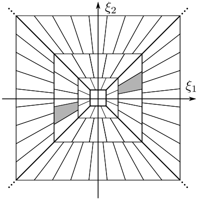

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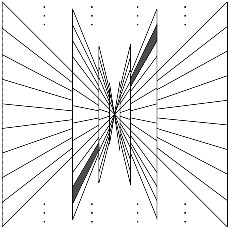

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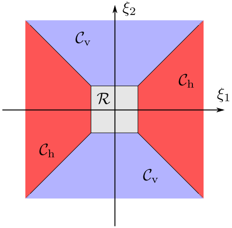
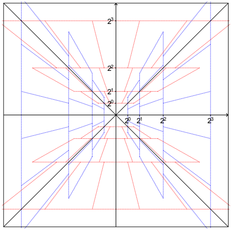




Cone-adapted Discrete Shearlet Systems

Definition (Kutyniok, Labate; 2006):

The cone-adapted discrete shearlet system $\mathcal{SH}(\phi, \psi, \tilde{\psi}; c)$ with sampling density $c > 0$ generated by $\phi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2)$ is the union of

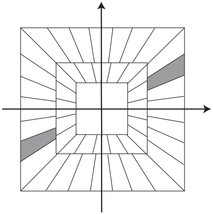
$$\left\{\phi(\bullet - c \cdot m) \,\middle|\, m \in \mathbb{Z}^2\right\},$$

$$\left\{\psi_{j,k,m,h} := \psi\left(S_k A_{2^j} \bullet - c \cdot m\right) \,\middle|\, j \in \mathbb{N}_0, |k| \leq \lceil 2^{j/2} \rceil, m \in \mathbb{Z}^2\right\},$$

$$\left\{\psi_{j,k,m,v} := \tilde{\psi}\left(S_k^T \tilde{A}_{2^j} \bullet - c \cdot m\right) \,\middle|\, j \in \mathbb{N}_0, |k| \leq \lceil 2^{j/2} \rceil, m \in \mathbb{Z}^2\right\},$$

where

$$S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \qquad A_{2^j} = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}, \qquad \tilde{A}_{2^j} = \begin{pmatrix} 2^{j/2} & 0 \\ 0 & 2^j \end{pmatrix}.$$



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Recall: Design goals for shearlet system Ψ:

- Ψ is a frame.
- Ψ is an affine system, motivated by coherent states. ✓
- Ψ is a multiscale representation system, with an associated tiling of the Fourier domain. ✓
- It should be possibly to choose a compactly supported generator for Ψ.
- Ψ should provide optimally sparse approximations of cartoons.
- Fast algorithms treating the analog and digital world uniformly. (✓)

Conclusion of first part

We have seen:

- Wavelets are not optimal for approximation of functions with curvilinear singularities.
- Shearlets are an alternative multiscale system which uses
  - parabolic dilations
  - shearings

  instead of scalar dilations.

Recall: Design goals for shearlet system Ψ:

- Ψ is a frame.
- Ψ is an affine system, motivated by coherent states. ✓
- Ψ is a multiscale representation system, with an associated tiling of the Fourier domain. ✓
- It should be possibly to choose a compactly supported generator for Ψ.
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Outline

Theorem (Kittipoom, Kutyniok, Lim; 2012):
Let $\phi, \psi, \tilde{\psi} \in L^2\left(\mathbb{R}^2\right)$ be compactly supported and assume that

- $\widehat{\psi}, \widehat{\tilde{\psi}}$ satisfy certain decay conditions,
- we have

$$|\widehat{\phi}(\xi)|^2 + \sum_{j,k} |\widehat{\psi}_{j,k}(\xi)|^2 + |\widehat{\tilde{\psi}_{j,k}}(\xi)|^2 \geq C > 0 \quad \text{a.e.} \qquad (\dagger)$$

Then there is some $c > 0$ such that $\mathcal{SH}(\phi, \psi, \tilde{\psi}; c)$ is a frame for $L^2\left(\mathbb{R}^2\right)$.

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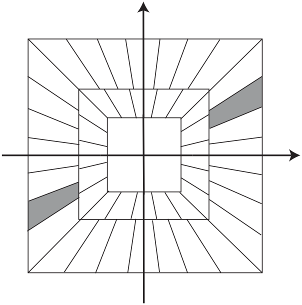

Compactly supported shearlets

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Remarks

- For $(\dagger)$, it suffices to have $\tilde{\psi}((\xi_1, \xi_2)) = \psi((\xi_2, \xi_1))$ as well as

$$\begin{aligned} \left|\widehat{\phi}(\xi)\right| &\gtrsim 1 \qquad \text{for } \xi \in [-1,1]^2, \\ |\widehat{\psi}(\xi)| &\gtrsim 1 \qquad \text{for } \xi_1 \in [1/3,\, 3] \text{ and } |\xi_2| \leq |\xi_1|, \end{aligned}$$

- There are special examples with frame bounds $B/A \approx 4$.

Theorem (Kutyniok, Lim; 2011):
Let $\phi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2)$ be compactly supported and such that $\mathcal{SH}(\phi, \psi, \tilde{\psi}; c)$ is a frame for $L^2(\mathbb{R}^2)$. Further, assume that $\widehat{\psi}, \widehat{\tilde{\psi}}$ satisfy certain decay conditions. Then $\mathcal{SH}(\phi, \psi, \tilde{\psi}; c)$ provides an optimally sparse approximation of all $f \in \mathcal{E}^2(\mathbb{R}^2)$, i.e.,

$$\|f - f_N\|_{L^2} \lesssim N^{-1} \cdot (\log N)^{3/2} \qquad \forall N \in \mathbb{N} \qquad \forall f \in \mathcal{E}^2(\mathbb{R}^2).$$

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Here, f_N is a linear combination of (at most) N elements of the dual frame of $\mathcal{SH}(\phi, \psi, \tilde{\psi}; c)$.

Optimal Sparse Approximation of Cartoon-like functions

Theorem (Kutyniok, Lim; 2011):
Let $\phi, \psi, \tilde{\psi} \in L^2\left(\mathbb{R}^2\right)$ be compactly supported and such that $\mathcal{SH}(\phi, \psi, \tilde{\psi}; c)$ is a frame for $L^2\left(\mathbb{R}^2\right)$. Further, assume that $\widehat{\psi}, \widehat{\tilde{\psi}}$ satisfy certain decay conditions. Then $\mathcal{SH}(\phi, \psi, \tilde{\psi}; c)$ provides an optimally sparse approximation of all $f \in \mathcal{E}^2\left(\mathbb{R}^2\right)$, i.e.,

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Remark

The proof shows

$$\sum_{n>N} |\theta(f)|_n^2 \lesssim N^{-2} \cdot (\log N)^3 \qquad \forall N \in \mathbb{N} \qquad \forall f \in \mathcal{E}^2(\mathbb{R}^2),$$

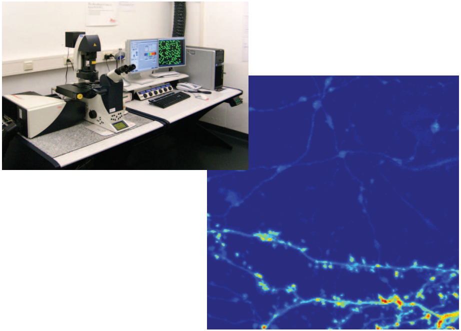
where $|\theta(f)|_n$ denotes the n-th largest shearlet coefficient. Hence, the shearlet coefficients are in ℓ^p for all $p > \frac{2}{3}$.

Applications

Feature separation in Images

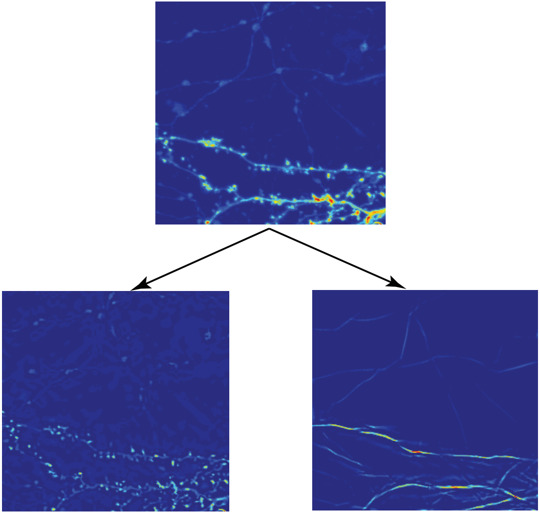
Problem from Neurobiology: Alzheimer Research

- Detection of characteristics of Alzheimer
- Separation of spines (point-like) and dendrites (curvilinear)



(Confocal-Laser Scanning-Microscopy)

Numerical results of feature separation



(Source: Brandt, Kutyniok, Lim, Sündermann; 2010)

Separation using Sparsity

General approach:

- Let $x^0 = x_1^0 + x_2^0$ be a signal.
- Let Φ_1, Φ_2 be frames such that $x_i^0 = \Phi_i c_i^0$ with c_i^0 sparse, $i = 1, 2$.
- This yields

$$x^0 = \left[\ \Phi_1 \ \middle|\ \Phi_2\ \right] \begin{bmatrix} c_1^0 \\ c_2^0 \end{bmatrix}.$$

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Idea for determining c_1^0, c_2^0 and hence x_1^0, x_2^0: **ℓ^1 minimization (Elad, Starck, Querre, Donoho; 2005)**:

$$(\hat{c}_1, \hat{c}_2) = \operatorname{argmin} \|c_1\|_{\ell^1} + \|c_2\|_{\ell^1} \text{ subject to } x^0 = \Phi_1 c_1 + \Phi_2 c_2.$$

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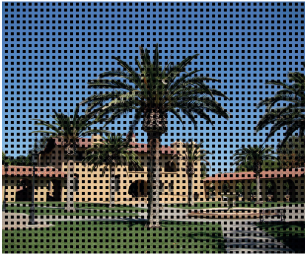
Theorems (Donoho, Kutyniok; 2013), (Kutyniok; 2014):

- Wavelets: Optimally sparse approximations of points.
- Shearlets: Optimally sparse approximations of curves.

$\rightsquigarrow$ **Provable asymptotic separation of points and curves!**

Inpainting using Shearlets



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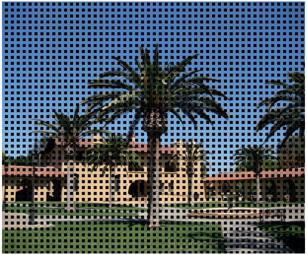

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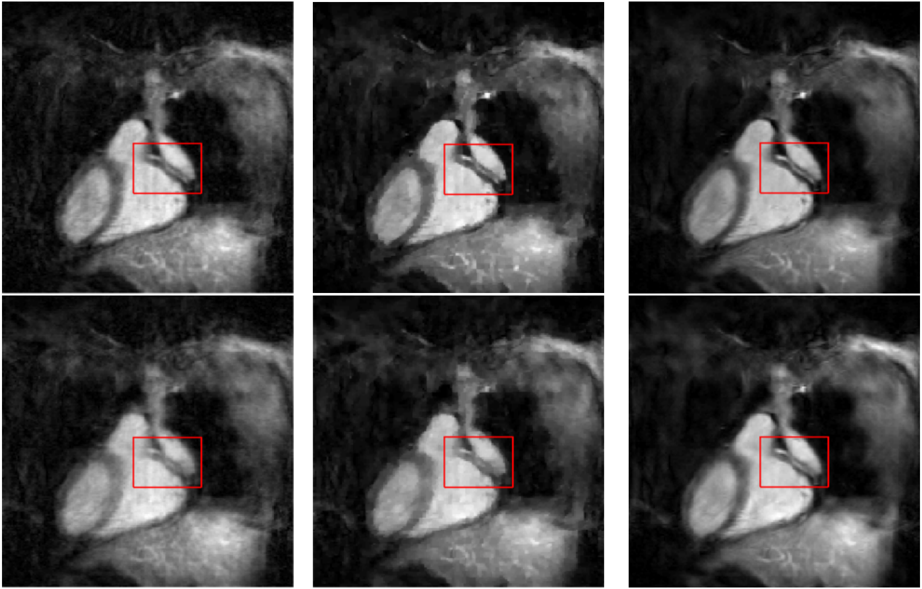
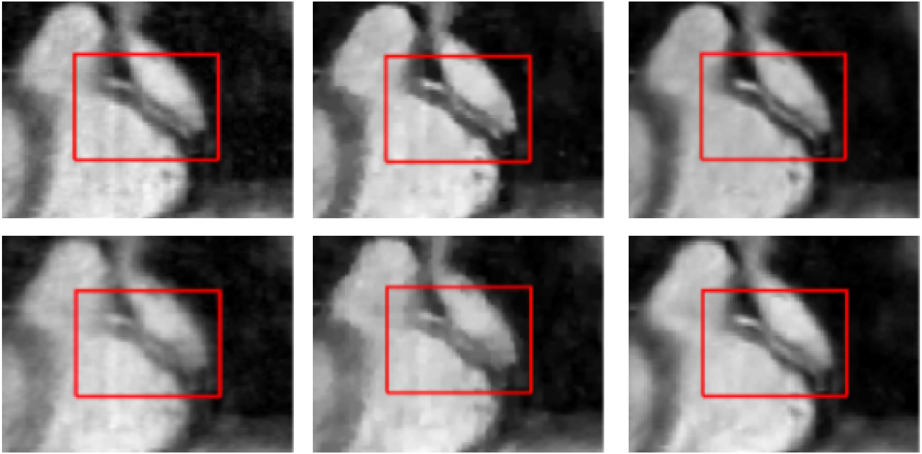
(Source: Lim; 2014)

MRI reconstruction



SENSE TV Shearlets

MRI reconstruction



SENSE TV Shearlets

Outline

1. Multiscale Systems inspired by Coherent States
   - The Quasi-regular Representation
   - Wavelets and their Limitations
   - Continuous Shearlet Systems
   - Discrete Shearlet Systems
2. Shearlets and their Applications
   - Compactly supported Shearlets
   - Optimal sparse Approximation
   - Applications with Compressed Sensing
3. Shearlets, Coherent States & Decomposition spaces

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$$\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q) := \left\{ g \in \;\; ??? \;\; \middle| \left(\left\| \mathcal{F}^{-1} (\varphi_i \cdot \widehat{g}) \right\|_{L^p} \right)_{i \in I} \in \ell_w^q (I) \right\}.$$

Note: $\ell_w^q (I) = \left\{ (c_i)_{i \in I} \in \mathbb{C}^I \,\middle|\, (w_i \cdot c_i)_{i \in I} \in \ell^q (I) \right\}.$

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In most (but not all cases) where $\mathcal{O} = \mathbb{R}^d$, we can use $\mathcal{S}'(\mathbb{R}^d)$ as the reservoir, i.e.,

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Why are decomposition spaces useful?

Many important smoothness spaces are **decomposition spaces**:

- Besov spaces (homogeneous *and* inhomogeneous),
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$\rightsquigarrow$ We can study coorbit spaces using decomposition space theory.

Examples of coverings I: Besov spaces

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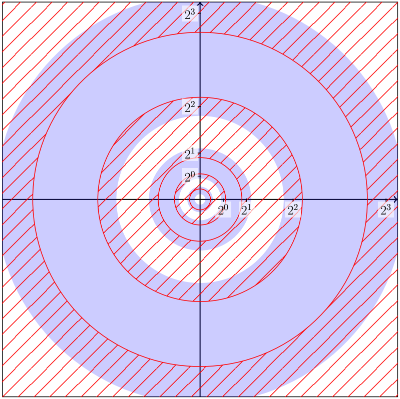
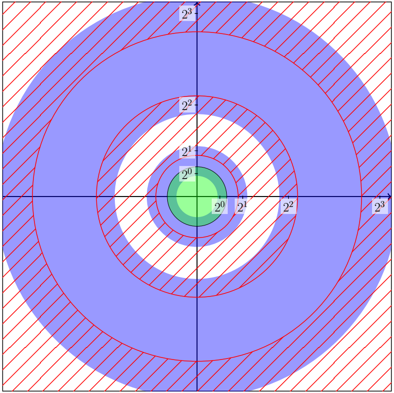
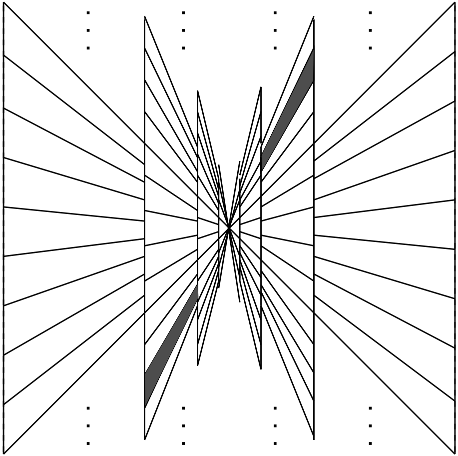
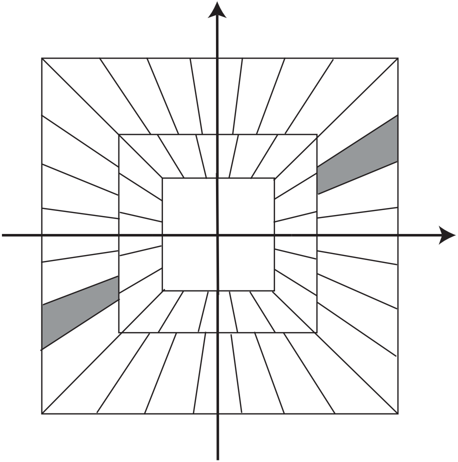
homogeneous	inhomogeneous
	

Examples of coverings II: Shearlet coorbit spaces



Examples of coverings III: Shearlet smoothness spaces



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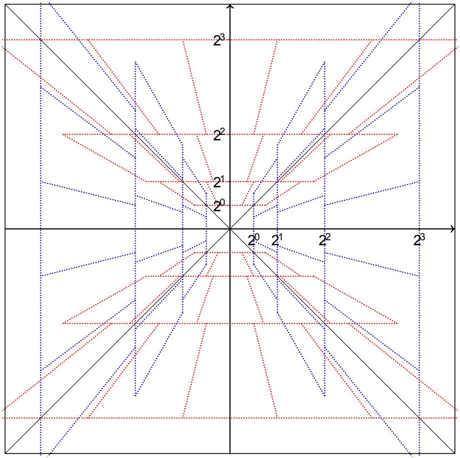

Structured Banach frames for decomposition spaces I

Assume that $\mathcal{Q} = (Q_i)_{i \in I}$ is a structured admissible covering, i.e.,

- $Q_i = T_i Q + b_i$ for $Q \subset \mathbb{R}^d$ fixed, $T_i \in \mathrm{GL}(\mathbb{R}^d)$ and $b_i \in \mathbb{R}^d$,

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Idea: For fixed prototype $\gamma \in L^1(\mathbb{R}^d)$, consider the structured family

$$\Psi^{(\delta)} := \left(\mathbf{T}_{\delta \cdot T_i^{-T} k} \, \gamma^{[i]} \right)_{i \in I, k \in \mathbb{Z}^d} \quad \text{with} \quad \gamma^{[i]} := |\det T_i|^{1/2} \cdot \mathbf{M}_{b_i} \left[\gamma \circ T_i^T \right].$$

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Note:

- $\widehat{\gamma^{[i]}} = |\det T_i|^{-1/2} \cdot \mathbf{T}_{b_i} \left[\widehat{\gamma} \circ T_i^{-1}\right]$.
- In particular, **if $\widehat{\gamma}$ is essentially supported in Q, then $\widehat{\gamma^{[i]}}$ is essentially supported in Q_i**.

Structured Banach frames for decomposition spaces II

Theorem (Voigtlaender; 2016): Let $\mathcal{Q} = (T_i Q + b_i)_{i \in I}$ and let $w = (w_i)_{i \in I}$ be a suitable weight. For $p, q \in (0, \infty]$, there is an (explicitly given) coefficient space $\mathscr{C}_{p,q,w} \leq \mathbb{C}^{I \times \mathbb{Z}^d}$ such that the following holds:

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If γ satisfies certain technical conditions, then there is $\delta_0 > 0$ such that

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Precisely:

- The analysis operator

  $$A^{(\delta)} : \mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right) \to \mathscr{C}_{p,q,w}, f \mapsto \left[\left(f * \gamma^{[i]} \right) \left(\delta \cdot T_i^{-T} k \right) \right]_{i \in I, k \in \mathbb{Z}^d}$$

  is well-defined and bounded.

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$$A^{(\delta)} : \mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right) \to \mathscr{C}_{p,q,w}, f \mapsto \left[\left(f * \gamma^{[i]} \right) \left(\delta \cdot T_i^{-T} k \right) \right]_{i \in I, k \in \mathbb{Z}^d}$$
is well-defined and bounded.
- There is a bounded reconstruction operator
$$R^{(\delta)} : \mathscr{C}_{p,q,w} \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$$
satisfying $R^{(\delta)} \circ A^{(\delta)} = \mathrm{id}_{\mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)}$.

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If γ satisfies certain technical conditions, then there is $\delta_0 > 0$ such that $\Psi^{(\delta)}$ forms an atomic decomposition for $\mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$ for all $0 < \delta \leq \delta_0$.

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- The synthesis operator

$$S^{(\delta)} : \mathscr{C}_{p,qw} \to \mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$$

$$(c_k^{(i)})_{i \in I, k \in \mathbb{Z}^d} \mapsto \sum_{i \in I} \sum_{k \in \mathbb{Z}^d} c_k^{(i)} \cdot \mathsf{T}_{\delta \cdot T_i^{-T} k} \, \gamma^{[i]}$$

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Structured Banach frames for decomposition spaces III

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Each $f \in \mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$ has a (more or less) sparse expansion w.r.t $\Psi^{(\delta)}$.

Structured Banach frames & Shearlets

Lemma: If $\mathcal{Q}$ is the covering associated to Shearlet smoothness spaces $\mathscr{S}_s^{p,q}(\mathbb{R}^2)$ (introduced by Labate et al.), then $\left(\gamma^{[i]}\right)_{i \in I}$ is a cone-adapted shearlet system.

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Corollary (Voigtlaender; 2016): The N-term approximation of cartoon-like functions w.r.t. the **primal** shearlet frame satisfies

$$\|f - f_N\|_{L^2} \lesssim_\varepsilon N^{-(1-\varepsilon)} \qquad \forall N \in \mathbb{N} \qquad \forall \varepsilon \in (0,1).$$

Let’s conclude!

What to take Home?

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- Shearlets are a multiscale system based on the quasi-regular representation of the shearlet group.
- They employ parabolic scaling and shearing and provide optimally sparse approximations for curvilinear features.
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- Wavelet-type coorbit spaces **are** decomposition spaces.
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Final remark: Existence of embeddings between decomposition spaces can be decided by comparing the geometry of the coverings.

Consequence: Sparsity in one system $\rightsquigarrow$ sparsity in another system.

Thank you!

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Questions, comments, counterexamples?

☺

Embeddings of decomposition spaces – General question

General assumptions

- $\mathcal{Q} = (Q_i)_{i \in I} = (T_i Q + b_i)_{i \in I}$ and $\mathcal{P} = (P_j)_{j \in J} = (S_j P + c_j)_{j \in J}$ are coverings of $\mathcal{O}, \mathcal{O}' \subset \mathbb{R}^d$.
- We are given weights $w = (w_i)_{i \in I}$ and $v = (v_j)_{j \in J}$.
- We have $p_1, p_2, q_1, q_2 \in (0, \infty]$.

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$$\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}\right) \hookrightarrow \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right)?$$

Strong additional assumption: $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$, i.e.,

$$\exists N \in \mathbb{N} \, \forall i \in I \, \exists j_i \in J : \qquad Q_i \subset P_{j_i}^{N*}.$$

Roughly: $\mathcal{Q}$ is finer than $\mathcal{P}$.

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Theorem (Voigtlaender; 2015)

If • $\mathcal{Q}$ *is almost subordinate to* $\mathcal{P}$,

- $p_1 \leq p_2$,
- $(\blacklozenge_{p_2^{\triangledown}}) < \infty$, *for* $p_2^{\triangledown} := \min \{ p_2, p_2' \}$

then

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Necessary criteria

Recall: With

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Conversely, if $\mathcal{Q}$ is almost subordinate to $\mathcal{P}$ and if

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Theorem (Voigtlaender; 2015)

If $\mathcal{O} = \mathcal{O}'$ and if

- *$\mathcal{Q}$ is almost subordinate to $\mathcal{P}$,*
- *$\mathcal{Q}$ and w are relatively $\mathcal{P}$-moderate,*

then

$$\mathcal{D}\left(\mathcal{Q}, L^{p_1}, \ell_w^{q_1}\right) \hookrightarrow \mathcal{D}\left(\mathcal{P}, L^{p_2}, \ell_v^{q_2}\right) \qquad \Longleftrightarrow \qquad p_1 \leq p_2 \text{ and } \left(\blacklozenge_{p_2^\triangledown}\right) < \infty.$$

A sample application

Theorem (Voigtlaender; 2015)

We have $\mathscr{S}_s^{p_1,q_1}\left(\mathbb{R}^2\right) \hookrightarrow \mathcal{B}_r^{p_2,q_2}\left(\mathbb{R}^2\right)$ *if and only if* $p_1 \leq p_2$ *and*

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with $\frac{1}{p^{\pm\triangle}} = \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\}$.