# Coherent state quantization and the Heisenberg uncertainty principle in the quaternionic setting

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[1]

[1] Adler, S.L., *Quaternionic quantum mechanics and Quantum fields*, Oxford University Press, New York, 1995.

## Quaternions

The quaternion field is

$$\mathbb{H} = \{\mathfrak{q} = q_0 + q_1 i + q_2 j + q_3 k \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

where $i, j, k$ are imaginary units such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$.

The quaternionic conjugate of $\mathfrak{q}$ is

$$\overline{\mathfrak{q}} = q_0 - q_1 i - q_2 j - q_3 k.$$

The quaternion norm is

$$|\mathfrak{q}| = (\overline{\mathfrak{q}}\mathfrak{q})^{1/2} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

Also $|\mathfrak{p}\mathfrak{q}| = |\mathfrak{p}||\mathfrak{q}|$. for $\mathfrak{p}, \mathfrak{q} \in \mathbb{H}$

# Quaternions

Quaternions by $2 \times 2$ complex matrices:

$$\mathfrak{q} = q_0 \sigma_0 + i \mathbf{q} \cdot \underline{\sigma}, \tag{1}$$

with $q_0 \in \mathbb{R}, \quad \mathbf{q} = (q_1, q_2, q_3) \in \mathbb{R}^3$, $\sigma_0 = \mathbb{I}_2$, the $2 \times 2$ identity matrix, and $\underline{\sigma} = (\sigma_1, -\sigma_2, \sigma_3)$, where the $\sigma_\ell, \ \ell = 1, 2, 3$ are the usual Pauli matrices. The quaternionic imaginary units are identified as, $i = \sqrt{-1}\sigma_1, \ \ j = -\sqrt{-1}\sigma_2, \ \ k = \sqrt{-1}\sigma_3$. Thus,

$$\mathfrak{q} = \begin{pmatrix} q_0 + iq_3 & -q_2 + iq_1 \\ q_2 + iq_1 & q_0 - iq_3 \end{pmatrix} \tag{2}$$

and $\overline{\mathfrak{q}} = \mathfrak{q}^\dagger$ (matrix adjoint) .

# Quaternions

Introducing the polar coordinates:

$$\begin{aligned} q_0 &= r\cos\theta, \\ q_1 &= r\sin\theta\sin\phi\cos\psi, \\ q_2 &= r\sin\theta\sin\phi\sin\psi, \\ q_3 &= r\sin\theta\cos\phi, \end{aligned}$$

where $(r, \phi, \theta, \psi) \in [0, \infty) \times [0, \pi] \times [0, 2\pi)^2$, we may write

$$\mathfrak{q} = A(r)e^{i\theta\sigma(\widehat{n})}, \tag{3}$$

where

$$A(r) = r\sigma_0 \tag{4}$$

and

$$\sigma(\widehat{n}) = \begin{pmatrix} \cos\phi & \sin\phi e^{i\psi} \\ \sin\phi e^{-i\psi} & -\cos\phi \end{pmatrix}. \tag{5}$$

The matrices $A(r)$ and $\sigma(\widehat{n})$ satisfy the conditions,

$$A(r) = A(r)^{\dagger},\ \sigma(\widehat{n})^2 = \sigma_0,\ \sigma(\widehat{n})^{\dagger} = \sigma(\widehat{n}) \tag{6}$$

and $[A(r), \sigma(\widehat{n})] = 0$.

## Quaternion

Let

$$\mathbb{S} \quad = \quad \{I = x_1 i + x_2 j + x_3 k \mid x_1, x_2, x_3 \in \mathbb{R},\ x_1^2 + x_2^2 + x_3^2 = 1\},$$

we call it a quaternion sphere.

### Proposition

[a] For any non-real quaternion $\mathfrak{q} \in \mathbb{H} \setminus \mathbb{R}$, there exist, and are unique, $x, y \in \mathbb{R}$ with $y > 0$, and $I_{\mathfrak{q}} \in \mathbb{S}$ such that $\mathfrak{q} = x + I_{\mathfrak{q}} y$.

[a] Gentili, G.,Struppa, D.C., *A new theory of regular functions of a quaternionic variable*, Adv. Math. **216** (2007), 279-301.

# Quaternion Slice

For every quaternion $I \in \mathbb{S}$, the complex line $L_I = \mathbb{R} + I\mathbb{R}$ passing through the origin, and containing $1$ and $I$, is called a quaternion slice. It can be seen that

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} L_I \quad \text{and} \quad \bigcap_{I \in \mathbb{S}} L_I = \mathbb{R} \tag{7}$$

Further,

1. $L_I \subset \mathbb{H}$ is commutative.
2. Elements from two different quaternion slices, $L_I$ and $L_J$ (for $I, J \in \mathbb{S}$ with $I \neq J$), do not necessarily commute.

# Right quaternionic Hilbert space

## Definition

Let $V_{\mathbb{H}}^R$ be a linear vector space under right multiplication by quaternionic scalars. For $f, g, h \in V_{\mathbb{H}}^R$ and $\mathfrak{q} \in \mathbb{H}$, the inner product

$$\langle \cdot \mid \cdot \rangle : V_{\mathbb{H}}^R \times V_{\mathbb{H}}^R \longrightarrow \mathbb{H}$$

satisfies the following properties

(i) $\overline{\langle f \mid g \rangle} = \langle g \mid f \rangle$

(ii) $\|f\|^2 = \langle f \mid f \rangle > 0$ unless $f = 0$, a real norm

(iii) $\langle f \mid g + h \rangle = \langle f \mid g \rangle + \langle f \mid h \rangle$

(iv) $\langle f \mid g\mathfrak{q} \rangle = \langle f \mid g \rangle \mathfrak{q}$

(v) $\langle f\mathfrak{q} \mid g \rangle = \overline{\mathfrak{q}} \langle f \mid g \rangle$

where $\overline{\mathfrak{q}}$ stands for the quaternionic conjugate.

# Right quaternionic Hilbert spaces

We assume that the space $V_{\mathbb{H}}^{R}$

1. is complete under the norm given above.
2. together with $\langle\cdot \mid \cdot\rangle$ this defines a right quaternionic Hilbert space.
3. we shall assume it to be separable.

Quaternionic Hilbert spaces share most of the standard properties of complex Hilbert spaces. In particular, the Cauchy-Schwartz inequality holds on quaternionic Hilbert spaces as well as the Riesz representation theorem for their duals. Thus, the Dirac bra-ket notation can be adapted to quaternionic Hilbert spaces:

$$\mid f\mathfrak{q}\rangle = \mid f\rangle\mathfrak{q}, \qquad \langle f\mathfrak{q}\mid = \overline{\mathfrak{q}}\langle f\mid ,$$

for a right quaternionic Hilbert space, with $|f\rangle$ denoting the vector $f$ and $\langle f|$ its dual vector, see for more detail [2] .

[2] Thirulogasanthar, K., Twareque Ali, S., J. Math. Phys., **54** (2013), 013506.

## Right quaternionic Hilbert spaces

The field of quaternions $\mathbb{H}$ itself can be turned into a right quaternionic Hilbert space with

$$\langle \mathfrak{q} \mid \mathfrak{q}' \rangle = \mathfrak{q}^{\dagger}\mathfrak{q}' = \overline{\mathfrak{q}}\mathfrak{q}'.$$

Further note that, due to the non-commutativity of quaternions the sum $\sum_{m=0}^{\infty} \mathfrak{p}^m \mathfrak{q}^m / m!$ cannot be written as $\exp(\mathfrak{p}\mathfrak{q})$. However, in any Hilbert space the norm convergence implies the convergence of the series and

$$\sum_{m=0}^{\infty} |\mathfrak{p}^m \mathfrak{q}^m / m!| = e^{|\mathfrak{p}||\mathfrak{q}|}.$$

## Right quaternionic square integrable functions

Let $(X, \mu)$ be a measure space and $\mathbb{H}$ the field of quaternions, then

$$\left\{ f : X \to \mathbb{H} \,\middle|\, \int_X |f(x)|^2 d\mu(x) < \infty \right\}$$

is a right quaternionic Hilbert space which is denoted by $L^2_{\mathbb{H}}(X, \mu)$, with the (right) scalar product

$$\langle f \mid g \rangle = \int_X \overline{f(x)} g(x) d\mu(x), \tag{8}$$

where $\overline{f(x)}$ is the quaternionic conjugate of $f(x)$, and (right) scalar multiplication $f\mathfrak{a},\ \mathfrak{a} \in \mathbb{H}$, with $(f\mathfrak{a})(\mathfrak{q}) = f(\mathfrak{q})\mathfrak{a}$ [3]

[3]Viswanath, K., *Normal operators on quaternionic Hilbert spaces*, Trans. Am. Math. Soc. **162** (1971), 337ï¿$\frac{1}{2}$350.

## CS on $V_{\mathbb{H}}^R$

Let $\{| f_m\rangle\}_{m=0}^{\infty}$ be an orthonormal basis of $V_{\mathbb{H}}^R$. For $\mathfrak{q} \in \mathbb{H}$, the coherent states are defined as vectors in $V_{\mathbb{H}}^R$ in the form

$$| \mathfrak{q}\rangle = \mathcal{N}(| \mathfrak{q} |)^{-\frac{1}{2}} \sum_{m=0}^{\infty} | f_m\rangle \frac{\mathfrak{q}^m}{\sqrt{\rho(m)}}, \tag{9}$$

where $\mathcal{N}(| \mathfrak{q} |)$ is the normalization factor and $\{\rho(m)\}_{m=0}^{\infty}$ is a positive sequence of real numbers. [4]

[4] Thirulogasanthar, K., Honnouvo, G., Krzyzak, A., *Coherent states and Hermite polynomials on Quaternionic Hilbert spaces*, J. Phys.A: Math. Theor. **43** (2010), 385205.

## CS on $V_{\mathbb{H}}^R$

The resolution of the identity is,

$$\int_{\mathcal{D}} \mid \mathfrak{q}\rangle\langle\mathfrak{q} \mid d\varsigma(r,\theta,\phi,\psi) = \mathbb{I}_{V_{\mathbb{H}}^R}, \tag{10}$$

where $\mathbb{I}_{V_{\mathbb{H}}^R}$ is the identity operator on $V_{\mathbb{H}}^R$.
Particularly, if $\rho(m) = m!$, then the normalization factor $\mathcal{N}(\mid \mathfrak{q} \mid) = e^{|\mathfrak{q}|^2}$. The resolution of the identity is obtained with $d\varsigma(r,\theta,\phi,\psi) = \dfrac{r}{2\pi}e^{-r^2}\sin\phi\, dr\, d\theta\, d\phi\, d\psi$.
When $\rho(m) = m!$, the (CS) defined by (9) are called *right quaternionic canonical coherent states*. For the purpose of quantizing the quaternions we shall use these canonical set of CS.

# Quantization of quaternions

Since $(\mathbb{H}, d\varsigma(r,\theta,\phi,\psi))$ is a measure space, the set

$$\left\{ f : \mathbb{H} \to \mathbb{H} \mid \int_{\mathbb{H}} |f(\mathfrak{q})|^2 d\varsigma(r,\theta,\phi,\psi) < \infty \right\}$$

is the space of right quaternionic square integrable functions and is denoted by $L^2_{\mathbb{H}}(\mathbb{H}, d\varsigma(r,\theta,\phi,\psi))$. [5]

[5] Muraleetharan. B., Thirulogasanthar, *coherent state quantization of quaternions*, J. Math. Phys., **56** (2015), 083510.

## Quantization of quaternions

Define the sequence of functions $\{\phi_n\}_{n=0}^{\infty}$ such that

$$\phi_n : \mathbb{H} \longrightarrow \mathbb{H}$$

by

$$\phi_n(\mathfrak{q}) = \frac{\overline{\mathfrak{q}}^n}{\sqrt{n!}}, \quad \text{for all} \quad \mathfrak{q} \in \mathbb{H}. \tag{11}$$

Then $\phi_n \in L^2_{\mathbb{H}}(\mathbb{H}, d\varsigma(r,\theta,\phi,\psi))$, for all $n = 0, 1, 2\cdots$ and $\langle \phi_m \mid \phi_n \rangle = \delta_{mn}$ . That is,

$$\mathcal{O} = \{\phi_n \mid n = 0, 1, 2\cdots\}$$

is an orthonormal set in $L^2_{\mathbb{H}}(\mathbb{H}, d\varsigma(r,\theta,\phi,\psi))$. The right quaternionic span of $\mathcal{O}$ is the space of anti-right-regular functions [6] (the counter part of complex anti-holomorphic functions).

[6] Thirulogasanthar, K., Twareque Ali, S., J. Math. Phys., **54** (2013), 013506.

# Quantization of quaternions

Let $\mathfrak{H}$ be a separable right quaternionic Hilbert space with an orthonormal basis

$$\mathcal{E} = \{ \mid e_n\rangle \mid n = 0, 1, 2 \cdots \}$$

which is in $1-1$ correspondence with $\mathcal{O}$. Then the coherent states (9) become

$$\mid \gamma_{\mathfrak{q}}\rangle = e^{-|\mathfrak{q}|^2/2} \sum_{m=0}^{\infty} \mid e_m\rangle \overline{\phi_m}. \qquad (12)$$

Using the set of CS (12) we shall establish the coherent state quantization on $\mathfrak{H}$ by associating a function

$$\mathbb{H} \ni \mathfrak{q} \longmapsto f(\mathfrak{q}, \overline{\mathfrak{q}}).$$

# Quantization of quaternions

Now let us define the operator on $\mathfrak{H}$ by

$$f(\mathfrak{q},\overline{\mathfrak{q}}) \mapsto A_f, \tag{13}$$

where $A_f$ is given by the operator valued integral

$$A_f = \int_{\mathbb{H}} \mid \gamma_{\mathfrak{q}}\rangle f(\mathfrak{q},\overline{\mathfrak{q}})\langle \gamma_{\mathfrak{q}} \mid d\varsigma(r,\theta,\phi,\psi) = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \frac{\mid e_m\rangle J_{m,l}\langle e_l \mid}{\sqrt{m!\, l!}}; \tag{14}$$

where the integral $J_{m,l}$ is given by

$$\iiiint\limits_{[0,\infty)\times[0,\pi]\times[0,2\pi)^2} \frac{\mathfrak{q}^m f(\mathfrak{q},\overline{\mathfrak{q}})\overline{\mathfrak{q}}^l}{e^{r^2}} d\varsigma(r,\theta,\phi,\psi).$$

# Quantization of quaternions

By direct calculation we can see that if $f(\mathfrak{q}, \overline{\mathfrak{q}}) = \mathfrak{q}$, then

$$A_{\mathfrak{q}} = \sum_{m=0}^{\infty} \sqrt{(m+1)} \mid e_m\rangle\langle e_{m+1} \mid \tag{15}$$

and if $f(\mathfrak{q}, \overline{\mathfrak{q}}) = \overline{\mathfrak{q}}$, then

$$A_{\overline{\mathfrak{q}}} = \sum_{m=0}^{\infty} \sqrt{(m+1)} \mid e_{m+1}\rangle\langle e_m \mid . \tag{16}$$

Moreover if $f(\mathfrak{q}, \overline{\mathfrak{q}}) = 1$, then $A_1 = \mathbb{I}_{\mathfrak{H}}$.

# Quantization of quaternions

Since

$$\langle A_{\overline{\mathfrak{q}}} g \mid f\rangle = \langle g \mid A_{\mathfrak{q}} f\rangle; \quad \text{for all} \quad |f\rangle, |g\rangle \in \mathfrak{H},$$

$A_{\overline{\mathfrak{q}}}$ is the adjoint of $A_{\mathfrak{q}}$ and vice-versa.
Now if $\mathfrak{H} = \overline{span\mathcal{O}}$ (right linear span over $\mathbb{H}$), then it is a subspace of $L^2_{\mathbb{H}}(\mathbb{H}, d\varsigma(r,\theta,\phi,\psi))$ and

$$A_f : \mathfrak{H} \longrightarrow \mathfrak{H} \quad \text{by}$$

$$A_f(u) = A_f \mid u\rangle = \int_{\mathbb{H}} \mid \gamma_{\mathfrak{q}}\rangle f(\mathfrak{q}, \overline{\mathfrak{q}}) \langle \gamma_{\mathfrak{q}} \mid u\rangle d\varsigma(r,\theta,\phi,\psi),$$

for all $u \in \mathfrak{H}$. Moreover, for each $u \in \mathfrak{H}, \ \ A_f \mid u\rangle \in \mathfrak{H}$.

# Quantization of quaternions

For $|u\rangle, |v\rangle \in \mathfrak{H}$, it can also be considered as a function

$$A_f : \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathbb{H} \quad \text{by}$$

$$\begin{aligned} A_f(u,v) &= \langle u \mid A_f \mid v\rangle \\ &= \int_{\mathbb{H}} \langle u \mid \gamma_{\mathfrak{q}}\rangle f(\mathfrak{q},\overline{\mathfrak{q}})\langle \gamma_{\mathfrak{q}} \mid v\rangle d\varsigma(r,\theta,\phi,\psi). \end{aligned}$$

# Quantization of quaternions

Since $\mid \gamma_{\mathfrak{q}}\rangle$ is a column vector and $\langle\gamma_{\mathfrak{q}} \mid$ is a row vector, we can see that the operator $A_f$ is a matrix and the matrix elements with respect to the basis $\{\mid e_n\rangle\}$ are given by

$$(A_f)_{mn} = \langle e_m \mid A_f \mid e_n\rangle = \int_{\mathbb{H}} \langle e_m \mid \gamma_{\mathfrak{q}}\rangle f(\mathfrak{q}, \overline{\mathfrak{q}})\langle\gamma_{\mathfrak{q}} \mid e_n\rangle d\varsigma(r, \theta, \phi, \psi).$$

We have

$$\langle e_m \mid \gamma_{\mathfrak{q}}\rangle = \mathcal{N}(\mid \mathfrak{q} \mid)^{-\frac{1}{2}} \, \overline{\phi_m(\mathfrak{q})}$$

and

$$\langle\gamma_{\mathfrak{q}} \mid e_n\rangle = \overline{\langle e_n \mid \gamma_{\mathfrak{q}}\rangle} = \mathcal{N}(\mid \mathfrak{q} \mid)^{-\frac{1}{2}} \, \phi_n(\mathfrak{q}).$$

# Quantization of quaternions

Therefore

$$(A_f)_{mn} = \int_{\mathbb{H}} \mathcal{N}(|\ \mathfrak{q}\ |)^{-1}\overline{\phi_m(\mathfrak{q})} f(\mathfrak{q}, \overline{\mathfrak{q}})\phi_n(\mathfrak{q}).d\varsigma(r, \theta, \phi, \psi).$$

Hence, it can easily be seen that

$$\begin{aligned}
(A_{\mathfrak{q}})_{k,l} &= \langle e_k | A_{\mathfrak{q}} | e_l \rangle = \begin{cases} \sqrt{k+1} & \text{if} \quad l = k+1 \\ 0 & \text{if} \quad l \neq k+1, \end{cases} \\
(A_{\overline{\mathfrak{q}}})_{k,l} &= \langle e_k | A_{\overline{\mathfrak{q}}} | e_l \rangle = \begin{cases} \sqrt{k} & \text{if} \quad l = k-1 \\ 0 & \text{if} \quad l \neq k-1. \end{cases}
\end{aligned}$$

## Quantization of quaternions

Let us realize the operator $A_f$ as annihilation and creation operators. From (15) and (16) we have $A_{\mathfrak{q}} \mid e_0\rangle = 0$ ,

$$A_{\mathfrak{q}} \mid e_m\rangle = \sqrt{m} \mid e_{m-1}\rangle \,;\; m = 1, 2, \cdots$$

and

$$A_{\overline{\mathfrak{q}}} \mid e_m\rangle = \sqrt{m+1} \mid e_{m+1}\rangle \,;\; m = 0, 1, 2, \cdots$$

That is, $A_{\mathfrak{q}}, A_{\overline{\mathfrak{q}}}$ are annihilation and creation operators respectively.

# Quantization of quaternions

Moreover, one can easily see that $A_{\mathfrak{q}} \mid \gamma_{\mathfrak{q}}\rangle = \mid \gamma_{\mathfrak{q}}\rangle \mathfrak{q}$, which is in complete analogy with the action of the annihilation operator on the ordinary harmonic oscillator CS. We can also write

$$\mid e_n\rangle = \frac{(A_{\overline{\mathfrak{q}}})^n}{\sqrt{n!}} \mid e_0\rangle.$$

# Quantization of quaternions

Now a direct calculation shows that

$$A_{\mathfrak{q}} A_{\overline{\mathfrak{q}}} \;=\; \sum_{m=0}^{\infty} (m+1) \mid e_m\rangle\langle e_m \mid$$

and

$$A_{\overline{\mathfrak{q}}} A_{\mathfrak{q}} \;=\; \sum_{m=0}^{\infty} (m+1) \mid e_{m+1}\rangle\langle e_{m+1} \mid .$$

Therefore the commutator of $A_{\overline{\mathfrak{q}}}, A_{\mathfrak{q}}$ takes the form

$$\begin{aligned} [A_{\mathfrak{q}}, A_{\overline{\mathfrak{q}}}] &= A_{\mathfrak{q}} A_{\overline{\mathfrak{q}}} - A_{\overline{\mathfrak{q}}} A_{\mathfrak{q}} \\ &= \sum_{m=0}^{\infty} \mid e_m\rangle\langle e_m \mid = \mathbb{I}_{\mathfrak{H}}. \end{aligned}$$

## Quantization of quaternions

### Remark

The operator $A_f$ in (14) is formed by the vector $\mid \gamma_{\mathfrak{q}}\rangle f(\mathfrak{q}, \overline{\mathfrak{q}})$, which is the right scalar multiple of the vector $\mid \gamma_{\mathfrak{q}}\rangle$ by the scalar $f(\mathfrak{q}, \overline{\mathfrak{q}})$, and the dual vector $\langle\gamma_{\mathfrak{q}} \mid$. Instead if one takes

$$A_f = \int_{\mathbb{H}} f(\mathfrak{q}, \overline{\mathfrak{q}}) \mid \gamma_{\mathfrak{q}}\rangle\langle\gamma_{\mathfrak{q}} \mid d\varsigma(r, \theta, \phi, \psi), \tag{17}$$

then it is formed by $f(\mathfrak{q}, \overline{\mathfrak{q}}) \mid \gamma_{\mathfrak{q}}\rangle$ (a left scalar multiple of a right Hilbert space vector) and the dual vector $\langle\gamma_{\mathfrak{q}} \mid$, which is unconventional . Further, due to the non-commutativity of quaternions, an $A_f$ in the form (17) would have caused severe technical problems in the follow up computations.

# Number, position, momentum operators and Hamiltonian

Let $N = A_{\overline{\mathfrak{q}}} A_{\mathfrak{q}}$, then we have

$$\begin{aligned} N \mid e_k\rangle &= A_{\overline{\mathfrak{q}}} A_{\mathfrak{q}} \mid e_k\rangle \\ &= \sum_{m=0}^{\infty} \mid e_{m+1}\rangle\langle e_{m+1} \mid e_k\rangle(m+1) \\ &= \mid e_k\rangle k. \end{aligned}$$

Thereby $N$ acts as the number operator and the Hilbert space $\mathfrak{H}$ is the quaternionic Fock space [7]

[7] Alpay, D., Colombo, F., Sabadini, I., Salomon, G., *The Fock space in the slice hyperholomorphic setting*, Hypercomplex Analysis: New perspective and applications, Trends in Mathematics, Birkhüser, Basel (2014), 43-59.

## Number, position, momentum operators and Hamiltonian

As an analogue of the usual harmonic oscillator Hamiltonian, if we take $\mathcal{H}_h = N + \mathbb{I}_{\mathfrak{H}}$, then $\mathcal{H}_h \mid e_n\rangle =\mid e_n\rangle(n+1)$, which is a Hamiltonian in the right quaternionic Hilbert space $\mathfrak{H}$ with spectrum $(n+1)$ and eigenvector $\mid e_n\rangle$.

# Number, position, momentum operators and Hamiltonian

Following the complex formalism, for $\mathfrak{q} \in \mathbb{H}$ if we take $q = \dfrac{1}{\sqrt{2}}(\mathfrak{q} + \overline{\mathfrak{q}})$, then we can have a self-adjoint position operator as

$$Q = \frac{1}{\sqrt{2}}(A_{\mathfrak{q}} + A_{\overline{\mathfrak{q}}}).$$

## Remark

In the complex quantum mechanics, for the canonical CS $\mid z\rangle, \quad z \in \mathbb{C}$, the lower symbol or the expectation value of the number operator, $\langle z|N|z\rangle$, is precisely $|z|^2$. The position and momentum coordinates are $q = \frac{1}{\sqrt{2}}(z + \overline{z})$ and $p = \frac{-i}{\sqrt{2}}(z - \overline{z})$ and by linearity one infers that the position and momentum operators as $Q = \frac{1}{\sqrt{2}}(A_z + A_{\overline{z}})$ and $P = \frac{-i}{\sqrt{2}}(A_z - A_{\overline{z}})$. The CS quantized classical harmonic oscillator, $H_c = \frac{1}{2}(q^2 + p^2)$, is $A_{H_c} = A_{|z|^2} = N + \mathbb{I}_{\mathfrak{H}_c}$, where $\mathbb{I}_{\mathfrak{H}_c}$ is the identity operator of the complex Fock space $\mathfrak{H}_c$. The operators $Q$ and $P$ satisfy the commutation rule $[Q, P] = i\mathbb{I}_{\mathfrak{H}_c}$ and are self-adjoint. If one simply takes the canonical quantization of the classical Hamiltonian it becomes $\hat{H}_c = \frac{1}{2}(Q^2 + P^2) = N + \frac{1}{2}\mathbb{I}_{\mathfrak{H}_c}$. [a]

[a] Gazeau, J-P., *Coherent states in quantum physics*, Wiley-VCH, Berlin (2009).

# Non-self-adjointness of P

In the case of the momentum operator, the complex formalism does not transfer to quaternions. In the case of quaternions we have three imaginary units, $i, j$ and $k$, and if we try to duplicate the complex momentum coordinate with one of $i, j$ or $k$, that is, if we take

$$\mathfrak{p} = \frac{-i}{\sqrt{2}}(\mathfrak{q} - \overline{\mathfrak{q}}),$$

then the operator $P$ becomes

$$P = \frac{-i}{\sqrt{2}}(A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}})$$

and due to the non-commutativity of quaternions $P$ is not self-adjoint.

# Non-self-adjointness of P

Further, a simple calculation shows that, the analogue of the complex operator $H_c$ in remark (4) is $H_h = \frac{1}{2}(\mathfrak{q}^2 + \mathfrak{p}^2) \neq |\mathfrak{q}|^2$. However, the lower symbol of $N$ is $\langle \gamma_{\mathfrak{q}} \mid N \mid \gamma_{\mathfrak{q}} \rangle = |\mathfrak{q}|^2$ and through a rather lengthy calculation we can see that $A_{|\mathfrak{q}|^2} = N + \mathbb{I}_{\mathfrak{H}}$.

# Non-self-adjointness of P

For a complex scalar $\alpha \in \mathbb{C}$ and an operator $T$ in a complex Hilbert space, the adjoint of the scalar multiple, $\alpha T$, is taken as

$$(\alpha T)^\dagger = \overline{\alpha} T^\dagger.$$

However, in general, this is not true for a non-real quaternionic scalar multiple of an operator on a quaternionic Hilbert space.

## Non-self-adjointness of P

The quaternion field $\mathbb{H}$ with the inner product $\langle \mathfrak{p}|\mathfrak{q}\rangle = \overline{\mathfrak{p}}\mathfrak{q}; \;\; \mathfrak{p},\mathfrak{q}\in\mathbb{H}$, is a right quaternionic Hilbert space. The identity operator, $\mathfrak{I}$, on $\mathbb{H}$ is self-adjoint. For a fixed $\alpha\in\mathbb{H}\setminus\mathbb{R}$, if $(\alpha\mathfrak{I})^{\dagger} = \overline{\alpha}\,\mathfrak{I}^{\dagger}$, then for $\mathfrak{p},\mathfrak{q}\in\mathbb{H}$, we have

$$\langle \mathfrak{p}|(\alpha\mathfrak{I})(\mathfrak{q})\rangle = \langle \mathfrak{p}|\mathfrak{I}(\mathfrak{q})\overline{\alpha}\rangle = \langle \mathfrak{p}|\mathfrak{q}\overline{\alpha}\rangle = \overline{\mathfrak{p}}\mathfrak{q}\overline{\alpha}$$

and

$$\begin{aligned}\langle (\alpha\mathfrak{I})^{\dagger}(\mathfrak{p})|\mathfrak{q}\rangle &= \langle (\overline{\alpha}\mathfrak{I}^{\dagger})(\mathfrak{p})|\mathfrak{q}\rangle = \langle (\overline{\alpha}\mathfrak{I})(\mathfrak{p})|\mathfrak{q}\rangle = \langle \mathfrak{I}(\mathfrak{p})\alpha|\mathfrak{q}\rangle = \langle \mathfrak{p}\alpha|\mathfrak{q}\rangle \\ &= \overline{\mathfrak{p}\alpha}\mathfrak{q} = \overline{\alpha}\,\overline{\mathfrak{p}}\mathfrak{q}.\end{aligned}$$

# Non-self-adjointness of P

## Example

For example, if $\alpha = i + 2j,\ \mathfrak{q} = j,\ \mathfrak{p} = k$, then we get

$$\langle \mathfrak{p} | (\alpha \mathfrak{I})(\mathfrak{q}) \rangle = 1 - 2k \quad \text{and} \quad \langle (\alpha \mathfrak{I})^{\dagger}(\mathfrak{p}) | \mathfrak{q} \rangle = 1 + 2k.$$

Therefore

$$(\alpha \mathfrak{I})^{\dagger} \neq \overline{\alpha} \mathfrak{I}^{\dagger}.$$

## Solution with quaternion slice

However, if we restrict ourselves to a quaternion slice, then we can have self-adjoint position and momentum operators with all the expected properties of their complex counterparts.

## Solution with quaternion slice

In order to exhibit this, let us see the structure of CS on a slice.

- Since elements in a quaternion slice commute, a quaternion slice is isomorphic to the complex plane. That is, for each $I \in \mathbb{S}$, $L_I$ is isomorphic to $\mathbb{C}$.
- While we are on a slice, $L_I$, the set of CS is formed with elements from the slice $L_I$ and the CS belongs to the right quaternionic Hilbert space over the field $L_I$ and we denote this Hilbert space by $\mathfrak{H}_{L_I}$.
- Let $\mathfrak{q}_I \in L_I$, $\mathfrak{q}_I = re^{I\theta}; r > 0, 0 \leq \theta < 2\pi$, then the normalization factor of the CS, over the slice $L_I$, is given by $\mathcal{N}(\mathfrak{q}_I) = e^{|\mathfrak{q}_I|^2}$ and a resolution of the identity is obtained with the measure $d\mu_I(r,\theta) = \frac{1}{2\pi} re^{-r^2} drd\theta$.

# Solution with quaternion slice

Even though a quaternion slice is isomorphic to $\mathbb{C}$, Hilbert space over a slice is not similar to a complex Hilbert space. In particular, the inner product of two elements from a slice-Hilbert space does not commute with the elements of the slice. For $\mathfrak{q}_I \in L_I$, let us define the position and momentum coordinates by

$$\mathfrak{q}_I = \frac{1}{\sqrt{2}}(\mathfrak{q}_I + \overline{\mathfrak{q}}_I) \quad \text{and} \quad \mathfrak{p}_I = \frac{-I}{\sqrt{2}}(\mathfrak{q}_I - \overline{\mathfrak{q}}_I),$$

then, since commutativity holds among $I, \mathfrak{q}$ and $\overline{\mathfrak{q}}$, the Hamiltonian can be calculated as

$$H_I = \frac{1}{2}\left(\mathfrak{q}_I^2 + \mathfrak{p}_I^2\right) = |\mathfrak{q}_I|^2.$$

## Solution with quaternion slice

Recall that on a right quaternionic Hilbert space operators are multiplied on the left by quaternion scalars. From the position and momentum coordinates, using linearity, we get the position operator, $Q_I$, and the momentum operator, $P_I$, as

$$Q_I = \frac{1}{\sqrt{2}}\left(A_{\mathfrak{q}_I} + A_{\overline{\mathfrak{q}}_I}\right) \quad \text{and} \quad P_I = \frac{-I}{\sqrt{2}}\left(A_{\mathfrak{q}_I} - A_{\overline{\mathfrak{q}}_I}\right).$$

Since $(A_{\overline{\mathfrak{q}}_I})^\dagger = A_{\mathfrak{q}_I}$ and $(-I)^\dagger = I$, the operators $P_I$ and $Q_I$ are self-adjoint. Using the fact $(\mathfrak{q}O_R)|f\rangle = (O_R|f\rangle)\overline{\mathfrak{q}}$ we can see that $A_{\overline{\mathfrak{q}}_I}(IA_{\mathfrak{q}_I}) = IA_{\overline{\mathfrak{q}}_I}A_{\mathfrak{q}_I}$.

## Solution with quaternion slice

With the aid of this we get

$$Q_I P_I \quad = \quad = -\frac{1}{2} I \, [{A_{\mathfrak{q}_I}}^2 + A_{\overline{\mathfrak{q}}_I} A_{\mathfrak{q}_I} - A_{\mathfrak{q}} A_{\overline{\mathfrak{q}}_I} - {A_{\overline{\mathfrak{q}}_I}}^2]$$

and

$$P_I Q_I \quad = \quad = -\frac{1}{2} I \, [{A_{\mathfrak{q}_I}}^2 - A_{\overline{\mathfrak{q}}_I} A_{\mathfrak{q}_I} + A_{\mathfrak{q}_I} A_{\overline{\mathfrak{q}}_I} - {A_{\overline{\mathfrak{q}}_I}}^2].$$

# Solution with quaternion slice

Thereby we have the commutator

$$[Q_I, P_I] = Q_I P_I - P_I Q_I = I\,[A_{\mathfrak{q}_I}, A_{\overline{\mathfrak{q}}_I}] = I\mathbb{I}_{\mathfrak{H}_{L_I}}.$$

We also have

$$\begin{aligned} Q_I^2 &= \frac{1}{2}\left[{A_{\mathfrak{q}_I}}^2 + A_{\overline{\mathfrak{q}}_I}A_{\mathfrak{q}_I} + A_{\mathfrak{q}_I}A_{\overline{\mathfrak{q}}_I} + {A_{\overline{\mathfrak{q}}_I}}^2\right] \quad \text{and} \\ P_I^2 &= -\frac{1}{2}\left[{A_{\mathfrak{q}_I}}^2 - A_{\overline{\mathfrak{q}}_I}A_{\mathfrak{q}_I} - A_{\mathfrak{q}_I}A_{\overline{\mathfrak{q}}_I} + {A_{\overline{\mathfrak{q}}_I}}^2\right] \end{aligned}$$

Hence

$$\begin{aligned}\hat{H}_I &= \frac{Q_I^2 + P_I^2}{2} = \frac{1}{2}[A_{\overline{\mathfrak{q}}_I} A_{\mathfrak{q}_I} + A_{\mathfrak{q}_I} A_{\overline{\mathfrak{q}}_I}] \\ &= A_{\overline{\mathfrak{q}}_I} A_{\mathfrak{q}_I} + \frac{1}{2}[A_{\mathfrak{q}_I} A_{\overline{\mathfrak{q}}_I} - A_{\overline{\mathfrak{q}}_I} A_{\mathfrak{q}_I}] = N_I + \frac{1}{2}\mathbb{I}_{\mathfrak{H}_{L_I}},\end{aligned}$$

which is in complete analogy with the complex case in the sense of *canonical quantization*, which simply replaces the classical coordinates by quantum observables (corresponding self-adjoint operators).

# Heisenberg Uncertainty on slices

In the following we shall show that the RQCS saturate the Heisenberg uncertainty relation and thereby they form a set minimum uncertainty states.
For Notational simplicity we use the same symbols for the operators and vectors as for $\mathbb{H}$. However they are now restricted to a slice-Hilbert space. For example:

$$A_{\mathfrak{q}} = A_{\mathfrak{q}}|_{V_{L_I}^R}.$$

## Heisenberg Uncertainty on slices

In order to compute the expectation values of the involved operators recall that $A_{\mathfrak{q}}|e_0\rangle = 0$,

$$\begin{aligned} A_{\mathfrak{q}}|e_m\rangle &= \sqrt{m}|e_{m-1}\rangle; \quad m = 1, 2, \cdots \\ A_{\bar{\mathfrak{q}}}|e_m\rangle &= \sqrt{m+1}|e_{m+1}\rangle; \quad m = 0, 1, \cdots \end{aligned}$$

and

$$A_{\mathfrak{q}}|\gamma_{\mathfrak{q}}\rangle = |\gamma_{\mathfrak{q}}\rangle\mathfrak{q}. \tag{18}$$

# Heisenberg Uncertainty on slices

Using (18) we can easily see that

$$A_{\mathfrak{q}}^2|\gamma_{\mathfrak{q}}\rangle = A_{\mathfrak{q}}|\gamma_{\mathfrak{q}}\rangle\mathfrak{q} = |\gamma_{\mathfrak{q}}\rangle\mathfrak{q}^2.$$

Hence, as $\langle\gamma_{\mathfrak{q}}|\gamma_{\mathfrak{q}}\rangle = 1$, we get

$$\langle\gamma_{\mathfrak{q}}|A_{\mathfrak{q}}|\gamma_{\mathfrak{q}}\rangle = \mathfrak{q} \quad \text{and} \quad \langle\gamma_{\mathfrak{q}}|A_{\mathfrak{q}}^2|\gamma_{\mathfrak{q}}\rangle = \mathfrak{q}^2.$$

Let $a_m = \sqrt{m+1}$ and $b_m = \sqrt{(m+1)(m+2)}$. The action of the operators, $A_{\bar{\mathfrak{q}}}$, $A_{\bar{\mathfrak{q}}}^2$, $A_{\bar{\mathfrak{q}}}A_{\mathfrak{q}}$ and $A_{\mathfrak{q}}A_{\bar{\mathfrak{q}}}$ on the RQCS takes the form

$$\begin{aligned} A_{\bar{\mathfrak{q}}}|\gamma_{\mathfrak{q}}\rangle &= e^{-|\mathfrak{q}|^2/2}\sum_{m=0}^{\infty} A_{\bar{\mathfrak{q}}}|e_m\rangle\frac{\mathfrak{q}^m}{\sqrt{m!}} \\ &= e^{-|\mathfrak{q}|^2/2}\sum_{m=0}^{\infty} |e_{m+1}\rangle a_m\frac{\mathfrak{q}^m}{\sqrt{m!}}, \end{aligned}$$

## Heisenberg Uncertainty on slices

and similarly,

$$A_{\bar{\mathfrak{q}}}^2|\gamma_{\mathfrak{q}}\rangle = e^{-|\mathfrak{q}|^2/2}\sum_{m=0}^{\infty}|e_{m+2}\rangle b_m \frac{\mathfrak{q}^m}{\sqrt{m!}},$$

$$A_{\bar{\mathfrak{q}}}A_{\mathfrak{q}}|\gamma_{\mathfrak{q}}\rangle = e^{-|\mathfrak{q}|^2/2}\sum_{m=0}^{\infty}|e_{m+1}\rangle a_m \frac{\mathfrak{q}^{m+1}}{\sqrt{m!}}$$

and

$$A_{\mathfrak{q}}A_{\bar{\mathfrak{q}}}|\gamma_{\mathfrak{q}}\rangle = e^{-|\mathfrak{q}|^2/2}\sum_{m=0}^{\infty}|e_m\rangle a_m^2 \frac{\mathfrak{q}^m}{\sqrt{m!}}.$$

## Heisenberg Uncertainty on slices

The dual of the CS is

$$\langle \gamma_{\mathfrak{q}}| = e^{-|\mathfrak{q}|^2/2} \sum_{m=0}^{\infty} \frac{\bar{\mathfrak{q}}^m}{\sqrt{m!}} \langle e_m|.$$

Thereby we get the expectation values

$$\begin{aligned}
&\langle \gamma_{\mathfrak{q}}|A_{\bar{\mathfrak{q}}}|\gamma_{\mathfrak{q}}\rangle \\
= \;& e^{-|\mathfrak{q}|^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\bar{\mathfrak{q}}^m}{\sqrt{m!}} \langle e_m|e_{n+1}\rangle a_n \frac{\mathfrak{q}^n}{\sqrt{n!}} \\
= \;& e^{-|\mathfrak{q}|^2} \sum_{m=0}^{\infty} \frac{\bar{\mathfrak{q}}^{m+1}\mathfrak{q}^m}{m!} \\
= \;& e^{-|\mathfrak{q}|^2} \bar{\mathfrak{q}} \sum_{m=0}^{\infty} \frac{|\mathfrak{q}|^{2m}}{m!} \\
= \;& \bar{\mathfrak{q}},
\end{aligned}$$

# Heisenberg Uncertainty on slices

and similarly,

$$\begin{aligned}
\langle \gamma_{\mathfrak{q}} | A_{\bar{\mathfrak{q}}}^2 | \gamma_{\mathfrak{q}} \rangle &= \bar{\mathfrak{q}}^2, \\
\langle \gamma_{\mathfrak{q}} | A_{\bar{\mathfrak{q}}} A_{\mathfrak{q}} | \gamma_{\mathfrak{q}} \rangle &= \bar{\mathfrak{q}}\mathfrak{q} = |\mathfrak{q}|^2, \\
\langle \gamma_{\mathfrak{q}} | A_{\mathfrak{q}} A_{\bar{\mathfrak{q}}} | \gamma_{\mathfrak{q}} \rangle &= 1 + |\mathfrak{q}|^2.
\end{aligned}$$

# Heisenberg Uncertainty on slices

Using the above expectation values we can get the expectation values of $Q$ and $Q^2$ as follows.

$$\begin{aligned}
\langle \gamma_{\mathfrak{q}} | Q | \gamma_{\mathfrak{q}} \rangle &= \frac{1}{\sqrt{2}} \langle \gamma_{\mathfrak{q}} | A_{\mathfrak{q}} + A_{\bar{\mathfrak{q}}} | \gamma_{\mathfrak{q}} \rangle \\
&= \frac{1}{\sqrt{2}} [\langle \gamma_{\mathfrak{q}} | A_{\mathfrak{q}} | \gamma_{\mathfrak{q}} \rangle + \langle \gamma_{\mathfrak{q}} | A_{\bar{\mathfrak{q}}} | \gamma_{\mathfrak{q}} \rangle] \\
&= \frac{1}{\sqrt{2}} (\mathfrak{q} + \bar{\mathfrak{q}}),
\end{aligned}$$

and hence

$$\langle \gamma_{\mathfrak{q}} | Q | \gamma_{\mathfrak{q}} \rangle^2 = \frac{1}{2} (\mathfrak{q}^2 + 2|\mathfrak{q}|^2 + \bar{\mathfrak{q}}^2).$$

# Heisenberg Uncertainty on slices

Now for $Q^2$

$$
\begin{aligned}
& \langle\gamma_{\mathfrak{q}}|Q^2|\gamma_{\mathfrak{q}}\rangle \\
= & \frac{1}{2}\langle\gamma_{\mathfrak{q}}|A_{\mathfrak{q}}^2 + A_{\mathfrak{q}}A_{\bar{\mathfrak{q}}} + A_{\bar{\mathfrak{q}}}A_{\mathfrak{q}}A_{\bar{\mathfrak{q}}}^2|\gamma_{\mathfrak{q}}\rangle \\
= & \frac{1}{2}[\mathfrak{q}^2 + 1 + |\mathfrak{q}|^2 + |\mathfrak{q}|^2 + \bar{\mathfrak{q}}^2] \\
= & \frac{1}{2}[\mathfrak{q}^2 + 1 + 2|\mathfrak{q}|^2 + \bar{\mathfrak{q}}^2].
\end{aligned}
$$

Therefore the variance of $Q$ becomes

$$
\begin{aligned}
\langle\Delta Q\rangle^2 &= \langle\gamma_{\mathfrak{q}}|Q^2|\gamma_{\mathfrak{q}}\rangle - \langle\gamma_{\mathfrak{q}}|Q|\gamma_{\mathfrak{q}}\rangle^2 \\
&= 1/2.
\end{aligned}
$$

That is,

$$\langle\Delta Q\rangle = \frac{1}{\sqrt{2}}.$$

## Heisenberg Uncertainty on slices

For the momentum operator $P$, we have

$$\begin{aligned}P|\gamma_{\mathfrak{q}}\rangle &= \left(\frac{-I}{\sqrt{2}}[A_{\mathfrak{q}} - A_{\bar{\mathfrak{q}}}]\right)|\gamma_{\mathfrak{q}}\rangle \\ &= ([A_{\mathfrak{q}} - A_{\bar{\mathfrak{q}}}]|\gamma_{\mathfrak{q}}\rangle)\overline{\left(\frac{-I}{\sqrt{2}}\right)} \\ &= ([A_{\mathfrak{q}} - A_{\bar{\mathfrak{q}}}]|\gamma_{\mathfrak{q}}\rangle)\left(\frac{I}{\sqrt{2}}\right).\end{aligned}$$

Thereby we get

$$\begin{aligned}\langle\gamma_{\mathfrak{q}}|P|\gamma_{\mathfrak{q}}\rangle &= \langle\gamma_{\mathfrak{q}}|A_{\mathfrak{q}} - A_{\bar{\mathfrak{q}}}|\gamma_{\mathfrak{q}}\rangle\frac{I}{\sqrt{2}} \\ &= [\langle\gamma_{\mathfrak{q}}|A_{\mathfrak{q}}|\gamma_{\mathfrak{q}}\rangle - \langle\gamma_{\mathfrak{q}}|A_{\bar{\mathfrak{q}}}|\gamma_{\mathfrak{q}}\rangle]\frac{I}{\sqrt{2}} \\ &= (\mathfrak{q} - \bar{\mathfrak{q}})\frac{I}{\sqrt{2}},\end{aligned}$$

# Heisenberg Uncertainty on slices

hence, as $I^2 = -1$, we obtain

$$\langle \gamma_{\mathfrak{q}} | P | \gamma_{\mathfrak{q}} \rangle^2 = \frac{1}{2}(-\mathfrak{q}^2 + 2|\mathfrak{q}|^2 - \bar{\mathfrak{q}}^2).$$

Now for $P^2$

$$\begin{aligned} & \langle \gamma_{\mathfrak{q}} | P^2 | \gamma_{\mathfrak{q}} \rangle \\ = & -\frac{1}{2} \langle \gamma_{\mathfrak{q}} | A_{\mathfrak{q}}^2 - A_{\mathfrak{q}} A_{\bar{\mathfrak{q}}} - A_{\bar{\mathfrak{q}}} A_{\mathfrak{q}} + A_{\bar{\mathfrak{q}}}^2 | \gamma_{\mathfrak{q}} \rangle \\ = & -\frac{1}{2} [\mathfrak{q}^2 - 1 - |\mathfrak{q}|^2 - |\mathfrak{q}|^2 + \bar{\mathfrak{q}}^2] \\ = & -\frac{1}{2} [\mathfrak{q}^2 - 1 - 2|\mathfrak{q}|^2 + \bar{\mathfrak{q}}^2]. \end{aligned}$$

Therefore the variance of $P$ becomes

$$\begin{aligned} \langle \Delta P \rangle^2 &= \langle \gamma_{\mathfrak{q}} | P^2 | \gamma_{\mathfrak{q}} \rangle - \langle \gamma_{\mathfrak{q}} | P | \gamma_{\mathfrak{q}} \rangle^2 \\ &= 1/2. \end{aligned}$$

# Heisenberg Uncertainty on slices

That is,

$$\langle \Delta P \rangle = \frac{1}{\sqrt{2}}.$$

As the conclusion of the above, we have

$$\langle \Delta Q \rangle \langle \Delta P \rangle = \frac{1}{2}.$$

Further, since $[Q, P] = I\mathbb{I}_{\mathfrak{H}}$, we have

$$\begin{aligned} [Q, P]|\gamma_{\mathfrak{q}}\rangle &= (I\mathbb{I}_{\mathfrak{H}})|\gamma_{\mathfrak{q}}\rangle = (\mathbb{I}_{\mathfrak{H}}|\gamma_{\mathfrak{q}}\rangle)\bar{I} \\ &= |\gamma_{\mathfrak{q}}\rangle(-I). \end{aligned}$$

# Heisenberg Uncertainty on slices

Therefore

$$\langle \gamma_{\mathfrak{q}} | [Q, P] | \gamma_{\mathfrak{q}} \rangle = \langle \gamma_{\mathfrak{q}} | \gamma_{\mathfrak{q}} \rangle (-I) = -I.$$

Hence

$$\frac{1}{2} |\langle [Q, P] \rangle| = \frac{1}{2} | - I| = \frac{1}{2}.$$

The above can be recapitulated in one line as

$$\langle \Delta Q \rangle \langle \Delta P \rangle = \frac{1}{2} |\langle [Q, P] \rangle| = \frac{1}{2}.$$

That is, the RQCS $|\gamma_{\mathfrak{q}}\rangle$ saturate the Heisenberg uncertainty, which is in complete analogy with the canonical CS of CQM.

## Left Scalar Multiplications on $V_{\mathbb{H}}^{R}$

### Solution with left scalar multiplication:

The left scalar multiple of vectors on a right quaternionic Hilbert space is an extremely non-canonical operation associated with a choice of preferred Hilbert basis.
Since the Hilbert space $V_{\mathbb{H}}^{R}$ is separable it has a Hilbert basis

$$\mathcal{O} = \{\varphi_k \mid k \in N\}, \tag{19}$$

where $N$ is a countable index set.

# Left Scalar Multiplications on $V_{\mathbb{H}}^R$

The left scalar multiplication '$\cdot$' on $V_{\mathbb{H}}^R$ induced by $\mathcal{O}$ is defined as the map $\mathbb{H} \times V_{\mathbb{H}}^R \ni (\mathfrak{q}, \phi) \longmapsto \mathfrak{q} \cdot \phi \in V_{\mathbb{H}}^R$ given by

$$\mathfrak{q} \cdot \phi := \sum_{k \in N} \varphi_k \mathfrak{q} \langle \varphi_k \mid \phi \rangle, \tag{20}$$

for all $(\mathfrak{q}, \phi) \in \mathbb{H} \times V_{\mathbb{H}}^R$. Since all left multiplications are made with respect to some basis, assume that the basis $\mathcal{O}$ given by (19) is fixed in the rest of this presentation.

## Left Scalar Multiplications on $V_{\mathbb{H}}^R$

### Proposition

[17] The left product defined in (20) satisfies the following properties. For every $\phi, \psi \in V_{\mathbb{H}}^R$ and $\mathfrak{p}, \mathfrak{q} \in \mathbb{H}$,

(a) $\mathfrak{q} \cdot (\phi + \psi) = \mathfrak{q} \cdot \phi + \mathfrak{q} \cdot \psi$ and $\mathfrak{q} \cdot (\phi \mathfrak{p}) = (\mathfrak{q} \cdot \phi)\mathfrak{p}$.

(b) $\|\mathfrak{q} \cdot \phi\| = |\mathfrak{q}|\|\phi\|$.

(c) $\mathfrak{q} \cdot (\mathfrak{p} \cdot \phi) = (\mathfrak{q}\mathfrak{p} \cdot \phi)$.

(d) $\langle \overline{\mathfrak{q}} \cdot \phi \mid \psi \rangle = \langle \phi \mid \mathfrak{q} \cdot \psi \rangle$.

(e) $r \cdot \phi = \phi r$, for all $r \in \mathbb{R}$.

(f) $\mathfrak{q} \cdot \varphi_k = \varphi_k \mathfrak{q}$, for all $k \in N$.

# Left Scalar Multiplications on $V_{\mathbb{H}}^{R}$

**Remark**

One can trivially see that $(\mathfrak{p}+\mathfrak{q})\cdot\phi=\mathfrak{p}\cdot\phi+\mathfrak{q}\cdot\phi$, for all $\mathfrak{p},\mathfrak{q}\in\mathbb{H}$ and $\phi\in V_{\mathbb{H}}^{R}$. Moreover, with the aid of (b) in above Proposition (6), we can have, if $\{\phi_n\}$ in $V_{\mathbb{H}}^{R}$ such that $\phi_n\longrightarrow\phi$, then $\mathfrak{q}\cdot\phi_n\longrightarrow\mathfrak{q}\cdot\phi$. Also if $\sum_n\phi_n$ is a convergent series in $V_{\mathbb{H}}^{R}$, then $\mathfrak{q}\cdot(\sum_n\phi_n)=\sum_n\mathfrak{q}\cdot\phi_n$.

## Left Scalar Multiplications on $V_{\mathbb{H}}^R$

For any fixed $\mathfrak{q} \in \mathbb{H}$ and a given right $\mathbb{H}$-linear operator $A : \mathfrak{D}(A) \longrightarrow V_{\mathbb{H}}^R$, the left scalar multiplication '$\cdot$' of $A$ is defined as a map $\mathfrak{q} \cdot A : \mathfrak{D}(A) \longrightarrow V_{\mathbb{H}}^R$ by the setting

$$(\mathfrak{q} \cdot A)\phi := \mathfrak{q} \cdot (A\phi) = \sum_{k \in N} \varphi_k \mathfrak{q} \langle \varphi_k \mid A\phi \rangle, \tag{21}$$

for all $\phi \in D(A)$. It is straightforward that $\mathfrak{q} \cdot A$ is a right $\mathbb{H}$-linear operator.

# Left Scalar Multiplications on $V_{\mathbb{H}}^R$

If $\mathfrak{q} \cdot \phi \in \mathfrak{D}(A)$, for all $\phi \in \mathfrak{D}(A)$, one can define right scalar multiplication '$\cdot$' of the right $\mathbb{H}$-linear operator $A : \mathfrak{D}(A) \longrightarrow V_{\mathbb{H}}^R$ as a map $A \cdot \mathfrak{q} : \mathfrak{D}(A) \longrightarrow V_{\mathbb{H}}^R$ by the setting

$$(A \cdot \mathfrak{q})\phi := A(\mathfrak{q} \cdot \phi), \tag{22}$$

for all $\phi \in D(A)$. It is also right $\mathbb{H}$-linear operator. One can easily obtain that, if $\mathfrak{q} \cdot \phi \in \mathfrak{D}(A)$, for all $\phi \in \mathfrak{D}(A)$ and $\mathfrak{D}(A)$ is dense in $V_{\mathbb{H}}^R$, then

$$(\mathfrak{q} \cdot A)^\dagger = A^\dagger \cdot \overline{\mathfrak{q}} \;\text{ and }\; (A \cdot \mathfrak{q})^\dagger = \overline{\mathfrak{q}} \cdot A^\dagger. \tag{23}$$

# Action with left multiplication

Further, real numbers commute with quaternions. Therefore according to (21), for example, we have

$$\begin{aligned}
(\mathfrak{q}\cdot A_{\overline{\mathfrak{q}}})^2 \mid e_0\rangle &= (\mathfrak{q}\cdot A_{\overline{\mathfrak{q}}})(\mathfrak{q}\cdot A_{\overline{\mathfrak{q}}})|e_0\rangle \\
&= (\mathfrak{q}\cdot A_{\overline{\mathfrak{q}}})(\mathfrak{q}\cdot A_{\overline{\mathfrak{q}}}|e_0\rangle) \\
&= (\mathfrak{q}\cdot A_{\overline{\mathfrak{q}}})(\mathfrak{q}\cdot |e_1\rangle)\sqrt{1} \\
&= (\mathfrak{q}\cdot A_{\overline{\mathfrak{q}}})|e_1\rangle\mathfrak{q} \\
&= \mathfrak{q}\cdot (A_{\overline{\mathfrak{q}}}|e_1\rangle)\mathfrak{q} \\
&= \mathfrak{q}\cdot (|e_2\rangle\sqrt{2})\mathfrak{q} \\
&= |e_2\rangle\mathfrak{q}^2\sqrt{2!}.
\end{aligned}$$

That is, $\dfrac{(\mathfrak{q}\cdot A_{\overline{\mathfrak{q}}})^2 \mid e_0\rangle}{\sqrt{2!}} = |e_2\rangle\mathfrak{q}^2$. By induction, for each $n = 0, 1, 2, \cdots$, we have $\dfrac{(\mathfrak{q}\cdot A_{\overline{\mathfrak{q}}})^n \mid e_0\rangle}{\sqrt{n!}} = |e_n\rangle\mathfrak{q}^n$.

# Action with left multiplication

Using this one can see that

$$\begin{aligned}
(e^{-|\mathfrak{q}|^2/2} \cdot e^{\mathfrak{q}\cdot A_{\overline{\mathfrak{q}}}})|e_0\rangle &= e^{-|\mathfrak{q}|^2/2} \cdot (e^{\mathfrak{q}\cdot A_{\overline{\mathfrak{q}}}}|e_0\rangle) \\
&= e^{-|\mathfrak{q}|^2/2} \cdot \left[\sum_{n=0}^{\infty} \frac{(\mathfrak{q}\cdot A_{\overline{\mathfrak{q}}})^n \mid e_0\rangle}{n!}\right] \\
&= e^{-|\mathfrak{q}|^2/2} \cdot \left[\sum_{n=0}^{\infty} |e_n\rangle \frac{\mathfrak{q}^n}{\sqrt{n!}}\right] \\
&= \mid \gamma_{\mathfrak{q}}\rangle.
\end{aligned}$$

That is, $\mid \gamma_{\mathfrak{q}}\rangle = (e^{-|\mathfrak{q}|^2/2} \cdot e^{\mathfrak{q}\cdot A_{\overline{\mathfrak{q}}}})|e_0\rangle$.

# Action with left multiplication

The following Proposition gives the commutativity between quaternions and the operator $A_{\mathfrak{q}}$ under the operations of left (21) and right (22) scalar multiplication of right linear operators. This result plays an important role in having momentum operator.

**Proposition**

For each $\mathfrak{r} \in \mathbb{H}$, we have $\mathfrak{r} \cdot A_{\mathfrak{q}} = A_{\mathfrak{q}} \cdot \mathfrak{r}$.

**Proof.**

For an arbitrary $\mathfrak{r} \in \mathbb{H}$, calculating $\mathfrak{r} \cdot A_{\mathfrak{q}}$ and $A_{\mathfrak{q}} \cdot \mathfrak{r}$ manually, the equality can be obtained. $\square$

# Proof in detail.

Let $\mathfrak{x} \in \mathbb{H}$, and $\phi \in \mathfrak{H}$, now

$$\begin{aligned}
(\mathfrak{x} \cdot A_{\mathfrak{q}})\phi &= \sum_{n=0}^{\infty} |e_n\rangle \mathfrak{x} \langle e_n | A_{\mathfrak{q}} \phi \rangle \\
&= \sum_{n=0}^{\infty} |e_n\rangle \mathfrak{x} \left( \sum_{m=0}^{\infty} \sqrt{m+1} \langle e_n | e_m \rangle \langle e_{m+1} | \phi \rangle \right) \\
&= \sum_{n=0}^{\infty} \sqrt{n+1}\, |e_n\rangle \mathfrak{x} \langle e_{n+1} | \phi \rangle
\end{aligned}$$

# Proof in detail.....

and

$$
\begin{aligned}
(A_{\mathfrak{q}} \cdot \mathfrak{x})\phi &= A_{\mathfrak{q}}(\mathfrak{x} \cdot \phi) \\
&= \sum_{n=0}^{\infty} \sqrt{n+1} |e_n\rangle\langle e_{n+1}|\mathfrak{x} \cdot \phi\rangle \\
&= \sum_{n=0}^{\infty} \sqrt{n+1} |e_n\rangle \left( \sum_{m=0}^{\infty} \langle e_{n+1}|e_m\rangle \mathfrak{x} \langle e_m|\phi\rangle \right) \\
&= \sum_{n=0}^{\infty} \sqrt{n+1}\, |e_n\rangle \mathfrak{x} \langle e_{n+1}|\phi\rangle.
\end{aligned}
$$

That is, $(\mathfrak{x} \cdot A_{\mathfrak{q}})\phi = (A_{\mathfrak{q}} \cdot \mathfrak{x})\phi$. Since $\phi \in \mathfrak{H}$ is arbitrary, we have $\mathfrak{x} \cdot A_{\mathfrak{q}} = A_{\mathfrak{q}} \cdot \mathfrak{x}$. Similarly $\mathfrak{x} \cdot A_{\bar{\mathfrak{q}}} = A_{\bar{\mathfrak{q}}} \cdot \mathfrak{x}$ can be obtained. Hence the result follows.

## Momentum Operator with left multiplication

In the case of the momentum operator, the complex formalism does not transfer to quaternions. In the case of quaternions we have three imaginary units, $i, j$ and $k$, and if we try to duplicate the complex momentum coordinate with $i, j$ or $k$, that is, if we take

$$q = \frac{1}{\sqrt{2}}(\mathfrak{q} + \overline{\mathfrak{q}}) \quad \text{and}$$

$$p_i = \frac{-i}{\sqrt{2}}(\mathfrak{q} - \overline{\mathfrak{q}}),$$

$$p_j = \frac{-j}{\sqrt{2}}(\mathfrak{q} - \overline{\mathfrak{q}})$$

and

$$p_k = \frac{-k}{\sqrt{2}}(\mathfrak{q} - \overline{\mathfrak{q}})$$

# Momentum Operator with left multiplication

then the momentum operators with respect to the above coordinates becomes

$$P_i = \frac{-i}{\sqrt{2}} \cdot (A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}}),$$

$$P_j = \frac{-j}{\sqrt{2}} \cdot (A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}})$$

and

$$P_k = \frac{-k}{\sqrt{2}} \cdot (A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}})$$

respectively.

# Momentum operator with left multiplication

Now for each $\tau \in \{i, j, k\}$, the operators $Q$ and $P_\tau$ are self-adjoint. For, It is trivial to say that the position operator $Q$ is self-adjoint. Since $A_{\overline{\mathfrak{q}}}$ is the adjoint of $A_{\mathfrak{q}}$ and vice-versa, we have, for any $\tau \in \{i, j, k\}$,

$$\begin{aligned}
P_\tau^\dagger &= \left[\frac{-\tau}{\sqrt{2}} \cdot (A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}})\right]^\dagger \\
&= (A_{\mathfrak{q}}^\dagger - A_{\overline{\mathfrak{q}}}^\dagger) \cdot \frac{\tau}{\sqrt{2}} \qquad \text{by (23)} \\
&= (A_{\overline{\mathfrak{q}}} - A_{\mathfrak{q}}) \cdot \frac{\tau}{\sqrt{2}} \\
&= \frac{-\tau}{\sqrt{2}} \cdot (A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}}) \qquad \text{by Proposition 8} \\
&= P_\tau.
\end{aligned}$$

Thus for each $\tau \in \{i, j, k\}$, the operators $P_\tau$ is self-adjoint.

# Number, position, momentum operators and Hamiltonian

Thus for each $\tau \in \{i, j, k\}$, the operators $P_\tau$ is self-adjoint. We can have the generalized Hamiltonian with respect to $\tau \in \{i, j, k\}$ that $H_\tau = \frac{1}{2}\left(\mid q \mid^2 + \mid p_\tau \mid^2\right) = |\mathfrak{q}|^2$. Moreover, there is another Hamiltonian which we can have as a combined one in terms of all of above three coordinates, as follows

$$H_c = \frac{1}{2}\left(q^2 - p_i^2 - p_j^2 - p_k^2\right) = |\mathfrak{q}|^2.$$

The lower symbol of $N$ is $\langle \gamma_{\mathfrak{q}} \mid N \mid \gamma_{\mathfrak{q}} \rangle = |\mathfrak{q}|^2$ and through a rather lengthy calculation we can see that $A_{|\mathfrak{q}|^2} = N + \mathbb{I}_{\mathfrak{H}}$.

# Number, position, momentum operators and Hamiltonian

Now for each $\tau \in \{i, j, k\}$, it can be obtained that

$$
\begin{aligned}
QP_\tau\phi &= \left[\frac{(A_{\mathfrak{q}} + A_{\overline{\mathfrak{q}}})}{\sqrt{2}}\right]\left[(-\tau)\cdot\frac{(A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}})}{\sqrt{2}}\right]\phi \\
&= \left[\frac{(A_{\mathfrak{q}} + A_{\overline{\mathfrak{q}}})}{\sqrt{2}}\right]\left[(-\tau)\cdot\left(\frac{(A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}})}{\sqrt{2}}\phi\right)\right] \\
&= \left[\frac{(A_{\mathfrak{q}} + A_{\overline{\mathfrak{q}}})}{\sqrt{2}}\cdot(-\tau)\right]\left[\left(\frac{(A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}})}{\sqrt{2}}\phi\right)\right] \quad \text{by (22)} \\
&= \left[(-\tau)\cdot\left(\frac{(A_{\mathfrak{q}} + A_{\overline{\mathfrak{q}}})}{\sqrt{2}}\right)\right]\left[\left(\frac{(A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}})}{\sqrt{2}}\phi\right)\right] \quad \text{by Proposition 8} \\
&= -\frac{1}{2}\tau\cdot[{A_{\mathfrak{q}}}^2 + A_{\overline{\mathfrak{q}}}A_{\mathfrak{q}} - A_{\mathfrak{q}}A_{\overline{\mathfrak{q}}} - {A_{\overline{\mathfrak{q}}}}^2]\phi
\end{aligned}
$$

# Number, position, momentum operators and Hamiltonian

and

$$\begin{aligned} P_\tau Q\phi &= \left[-\tau \cdot \frac{(A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}})}{\sqrt{2}}\right]\left[\frac{(A_{\mathfrak{q}} + A_{\overline{\mathfrak{q}}})}{\sqrt{2}}\right]\phi \\ &= -\frac{1}{2}\tau \cdot [A_{\mathfrak{q}}{}^2 - A_{\overline{\mathfrak{q}}}A_{\mathfrak{q}} + A_{\mathfrak{q}}A_{\overline{\mathfrak{q}}} - A_{\overline{\mathfrak{q}}}{}^2]\phi, \end{aligned}$$

for all $\phi \in V_{\mathbb{H}}^R$. Thereby for each $\tau \in \{i, j, k\}$, we have the commutator that

$$[Q, P_\tau] = QP_\tau - P_\tau Q = \tau \cdot [A_{\mathfrak{q}}, A_{\overline{\mathfrak{q}}}] = \tau \cdot \mathbb{I}_{\mathfrak{H}}.$$

We can also obtain, in a similar fashion, for each $\tau \in \{i, j, k\}$,

$$\begin{aligned} Q^2 &= \frac{1}{2}[A_{\mathfrak{q}}{}^2 + A_{\overline{\mathfrak{q}}}A_{\mathfrak{q}} + A_{\mathfrak{q}}A_{\overline{\mathfrak{q}}} + A_{\overline{\mathfrak{q}}}{}^2] \quad \text{and} \\ P_\tau^2 &= -\frac{1}{2}[A_{\mathfrak{q}}{}^2 - A_{\overline{\mathfrak{q}}}A_{\mathfrak{q}} - A_{\mathfrak{q}}A_{\overline{\mathfrak{q}}} + A_{\overline{\mathfrak{q}}}{}^2] \end{aligned}$$

# Number, position, momentum operators and Hamiltonian

Hence for each $\tau \in \{i, j, k\}$,

$$\begin{aligned} \hat{H}_\tau &= \frac{Q^2 + P_\tau^2}{2} = \frac{1}{2}[A_{\overline{\mathfrak{q}}} A_{\mathfrak{q}} + A_{\mathfrak{q}} A_{\overline{\mathfrak{q}}}] \\ &= A_{\overline{\mathfrak{q}}} A_{\mathfrak{q}} + \frac{1}{2}[A_{\mathfrak{q}} A_{\overline{\mathfrak{q}}} - A_{\overline{\mathfrak{q}}} A_{\mathfrak{q}}] \\ &= N + \frac{1}{2}\mathbb{I}_{\mathfrak{H}}, \end{aligned}$$

which does not depend on the choice of $\tau \in \{i, j, k\}$, and is in complete analogy with the complex case in the sense of *canonical quantization*, which simply replaces the classical coordinates by quantum observables (corresponding self-adjoint operators).

## Number, position, momentum operators and Hamiltonian

Let us try with the momentum coordinate

$$p^* = -\frac{(i+j+k)}{\sqrt{3}} \cdot \frac{(\mathfrak{q} - \overline{\mathfrak{q}})}{\sqrt{2}}$$

to define another momentum operator $P$ as

$$P^* = -\frac{(i+j+k)}{\sqrt{3}} \cdot \frac{(A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}})}{\sqrt{2}}.$$

One can realize that $P^*$ is self-adjoint, and the Hamiltonian $H$ becomes

$$H^* = \frac{1}{2}\left(\mid q \mid^2 + \mid p^* \mid^2\right) = |\mathfrak{q}|^2.$$

# Number, position, momentum operators and Hamiltonian

Furthermore, we have

$$[Q, P^*] = \frac{(i+j+k)}{\sqrt{3}} \cdot \mathbb{I}_{\mathfrak{H}}$$

and

$$\hat{H}^* = \frac{Q^2 + P^{*2}}{2} = N + \frac{1}{2}\mathbb{I}_{\mathfrak{H}}.$$

In more general, we can define the momentum coordinate for each $I \in \mathbb{S}$, such that

$$p_I = \frac{-I}{\sqrt{2}}(\mathfrak{q} - \overline{\mathfrak{q}}), \quad \mathfrak{q} \in \mathbb{H}$$

and the momentum operator

$$P_I = \frac{-I}{\sqrt{2}} \cdot (A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}}).$$

## Number, position, momentum operators and Hamiltonian

Then the Hamiltonian $H_I = \frac{1}{2}\left(\mid q \mid^2 + \mid p_I \mid^2\right) = |\mathfrak{q}|^2$. Also we can have

$$[Q, P] = I \cdot \mathbb{I}_{\mathfrak{H}}$$

and

$$\hat{H}_I = \frac{Q^2 + P_I^2}{2} = N + \frac{1}{2}\mathbb{I}_{\mathfrak{H}}.$$

In quaternion case, we have a set of self-adjoint momentum operators as

$$\mathfrak{P} = \left\{ P_I = \frac{-I}{\sqrt{2}} \cdot (A_{\mathfrak{q}} - A_{\overline{\mathfrak{q}}}) \ \mid \ I \in \mathbb{S} \right\}.$$

## Heisenberg Uncertainty

As before, we have

$$\begin{aligned}
\langle \gamma_{\mathfrak{q}} | A_{\mathfrak{q}} | \gamma_{\mathfrak{q}} \rangle &= \mathfrak{q} \\
\langle \gamma_{\mathfrak{q}} | A_{\mathfrak{q}}^2 | \gamma_{\mathfrak{q}} \rangle &= \mathfrak{q}^2 \\
\langle \gamma_{\mathfrak{q}} | A_{\bar{\mathfrak{q}}}^2 | \gamma_{\mathfrak{q}} \rangle &= \bar{\mathfrak{q}}^2, \\
\langle \gamma_{\mathfrak{q}} | A_{\bar{\mathfrak{q}}} A_{\mathfrak{q}} | \gamma_{\mathfrak{q}} \rangle &= \bar{\mathfrak{q}}\mathfrak{q} = |\mathfrak{q}|^2, \\
\langle \gamma_{\mathfrak{q}} | A_{\mathfrak{q}} A_{\bar{\mathfrak{q}}} | \gamma_{\mathfrak{q}} \rangle &= 1 + |\mathfrak{q}|^2.
\end{aligned}$$

and

$$\begin{aligned}
\langle \Delta Q \rangle^2 &= \langle \gamma_{\mathfrak{q}} | Q^2 | \gamma_{\mathfrak{q}} \rangle - \langle \gamma_{\mathfrak{q}} | Q | \gamma_{\mathfrak{q}} \rangle^2 \\
&= 1/2.
\end{aligned}$$

# Heisenberg Uncertainty

Let $I \in \mathbb{S}$, then for the momentum operator $P_I$, we have

$$\begin{aligned} P_I|\gamma_{\mathfrak{q}}\rangle &= \left(\frac{-I}{\sqrt{2}} \cdot [A_{\mathfrak{q}} - A_{\bar{\mathfrak{q}}}]\right)|\gamma_{\mathfrak{q}}\rangle \\ &= \frac{-I}{\sqrt{2}} \cdot ([A_{\mathfrak{q}} - A_{\bar{\mathfrak{q}}}]|\gamma_{\mathfrak{q}}\rangle) \\ &= \frac{-I}{\sqrt{2}} \cdot (A_{\mathfrak{q}}|\gamma_{\mathfrak{q}}\rangle - A_{\bar{\mathfrak{q}}}|\gamma_{\mathfrak{q}}\rangle)\,. \end{aligned}$$

# Heisenberg Uncertainty

Now one can see that

$$I \cdot A_{\mathfrak{q}} = \sum_{m=0}^{\infty} \sqrt{(m+1)} \mid e_m \rangle I \langle e_{m+1} \mid .$$

Thus

$$\begin{aligned}(I \cdot A_{\mathfrak{q}}) \mid \gamma_{\mathfrak{q}} \rangle &= e^{-|\mathfrak{q}|^2/2} \sum_{m=0}^{\infty} \sqrt{(m+1)} \mid e_m \rangle I \sum_{n=0}^{\infty} \langle e_{m+1} | e_n \rangle \frac{\mathfrak{q}^n}{\sqrt{n!}} \\ &= e^{-|\mathfrak{q}|^2/2} \sum_{m=0}^{\infty} \sqrt{(m+1)} \mid e_m \rangle I \frac{\mathfrak{q}^{m+1}}{\sqrt{m+1!}}\end{aligned}$$

# Heisenberg Uncertainty

and

$$\begin{aligned}
\langle \gamma_{\mathfrak{q}} \mid (I \cdot A_{\mathfrak{q}}) \mid \gamma_{\mathfrak{q}} \rangle &= e^{-|\mathfrak{q}|^2/2} \sum_{m=0}^{\infty} \sqrt{(m+1)} \langle \gamma_{\mathfrak{q}} \mid e_m \rangle I \frac{\mathfrak{q}^{m+1}}{\sqrt{m+1!}} \\
&= e^{-|\mathfrak{q}|^2} \sum_{m=0}^{\infty} \sqrt{(m+1)} \sum_{n=0}^{\infty} \frac{\bar{\mathfrak{q}}^n}{\sqrt{n!}} \langle e_n | e_m \rangle I \frac{\mathfrak{q}^{m+1}}{\sqrt{m+1!}} \\
&= e^{-|\mathfrak{q}|^2} \sum_{m=0}^{\infty} \sqrt{(m+1)} \frac{\bar{\mathfrak{q}}^m}{\sqrt{m!}} I \frac{\mathfrak{q}^{m+1}}{\sqrt{m+1!}} \\
&= \left( e^{-|\mathfrak{q}|^2} \sum_{m=0}^{\infty} \frac{\bar{\mathfrak{q}}^m I \mathfrak{q}^m}{m!} \right) \mathfrak{q} = \mathfrak{C}_I \mathfrak{q};
\end{aligned}$$

# Heisenberg Uncertainty

where $\mathfrak{C}_I = e^{-|\mathfrak{q}|^2} \sum_{m=0}^{\infty} \frac{\bar{\mathfrak{q}}^m I \mathfrak{q}^m}{m!}$ and this series absolutely converges to 1, i.e. $|\mathfrak{C}_I| \leq 1$. It is nice to note that, $\bar{\mathfrak{C}_I} = -\mathfrak{C}_I$ and $|\mathfrak{C}_I|^2 = -\mathfrak{C}_I^2$. From this, one can say that, there exist $\mathfrak{I} \in \mathbb{S}$ and $r \in [0, 1]$ such that $\mathfrak{C}_I = r\mathfrak{I}$.

# Heisenberg Uncertainty

Also we can find that

$$(I \cdot A_{\bar{\mathfrak{q}}}) \mid \gamma_{\mathfrak{q}}\rangle = e^{-|\mathfrak{q}|^2/2} \sum_{m=0}^{\infty} \sqrt{(m+1)} \mid e_{m+1}\rangle I \frac{\mathfrak{q}^m}{\sqrt{m!}}$$

and

$$\langle \gamma_{\mathfrak{q}} \mid (I \cdot A_{\bar{\mathfrak{q}}}) \mid \gamma_{\mathfrak{q}}\rangle = \bar{\mathfrak{q}} \left( e^{-|\mathfrak{q}|^2} \sum_{m=0}^{\infty} \frac{\bar{\mathfrak{q}}^m I \mathfrak{q}^m}{m!} \right) = \bar{\mathfrak{q}} \mathfrak{C}_I.$$

# Heisenberg Uncertainty

So,

$$\begin{aligned}\langle\gamma_{\mathfrak{q}}|P_I|\gamma_{\mathfrak{q}}\rangle &= \frac{1}{\sqrt{2}}[\langle\gamma_{\mathfrak{q}}|(I\cdot A_{\mathfrak{q}})|\gamma_{\mathfrak{q}}\rangle - \langle\gamma_{\mathfrak{q}}|(I\cdot A_{\bar{\mathfrak{q}}})|\gamma_{\mathfrak{q}}\rangle] \\ &= \frac{1}{\sqrt{2}}(\mathfrak{C}_I\mathfrak{q} - \bar{\mathfrak{q}}\mathfrak{C}_I).\end{aligned}$$

# Heisenberg Uncertainty

We obtain

$$\begin{aligned}
\langle \gamma_{\mathfrak{q}} | P_I | \gamma_{\mathfrak{q}} \rangle^2 &= \frac{1}{2}(\mathfrak{C}_I \mathfrak{q} - \bar{\mathfrak{q}} \mathfrak{C}_I)^2 \\
&= \frac{1}{2}(\mathfrak{C}_I \mathfrak{q} + \overline{\mathfrak{C}_I \mathfrak{q}})^2 \\
&= \frac{1}{2}[(\mathfrak{C}_I \mathfrak{q})^2 + 2|\mathfrak{C}_I \mathfrak{q}|^2 + (\overline{\mathfrak{C}_I \mathfrak{q}})^2].
\end{aligned}$$

## Heisenberg Uncertainty

Since $I^2 = -1$, we have

$$\begin{aligned} & \langle \gamma_{\mathfrak{q}} | P^2 | \gamma_{\mathfrak{q}} \rangle \\ = & -\frac{1}{2} \langle \gamma_{\mathfrak{q}} | A_{\mathfrak{q}}^2 - A_{\mathfrak{q}} A_{\bar{\mathfrak{q}}} - A_{\bar{\mathfrak{q}}} A_{\mathfrak{q}} + A_{\bar{\mathfrak{q}}}^2 | \gamma_{\mathfrak{q}} \rangle \\ = & -\frac{1}{2} [\mathfrak{q}^2 - 1 - |\mathfrak{q}|^2 - |\mathfrak{q}|^2 + \bar{\mathfrak{q}}^2] \\ = & -\frac{1}{2} [\mathfrak{q}^2 - 1 - 2|\mathfrak{q}|^2 + \bar{\mathfrak{q}}^2]. \end{aligned}$$

# Heisenberg Uncertainty

Therefore the variance of $P_I$ becomes

$$\begin{aligned}\langle \Delta P_I \rangle^2 &= \langle \gamma_{\mathfrak{q}} | P_I^2 | \gamma_{\mathfrak{q}} \rangle - \langle \gamma_{\mathfrak{q}} | P_I | \gamma_{\mathfrak{q}} \rangle^2 \\ &= -\frac{1}{2}[\mathfrak{q}^2 - 1 - 2|\mathfrak{q}|^2 + \bar{\mathfrak{q}}^2] - \frac{1}{2}[(\mathfrak{C}_I \mathfrak{q})^2 + 2|\mathfrak{C}_I \mathfrak{q}|^2 + (\overline{\mathfrak{C}_I \mathfrak{q}})^2].\end{aligned}$$

## Heisenberg Uncertainty

Since $\langle \Delta Q \rangle, \langle \Delta P_I \rangle \in \mathbb{R}$, we have

$$\begin{aligned}
& \langle \Delta Q \rangle^2 \langle \Delta P_I \rangle^2 \\
= & -\frac{1}{4}[(\mathfrak{q}^2 - 1 - 2|\mathfrak{q}|^2 + \bar{\mathfrak{q}}^2) + ((\mathfrak{C}_I \mathfrak{q})^2 + 2|\mathfrak{C}_I \mathfrak{q}|^2 + (\overline{\mathfrak{C}_I \mathfrak{q}})^2)] \\
\geq & -\frac{1}{4}[(\mathfrak{q}^2 - 1 - 2|\mathfrak{q}|^2 + \bar{\mathfrak{q}}^2) + ((\mathfrak{C}_I \mathfrak{q})^2 + 2|\mathfrak{q}|^2 + (\overline{\mathfrak{C}_I \mathfrak{q}})^2)] \quad \text{as } |\mathfrak{C}_I| \leq 1 \\
= & \frac{1}{4} - \frac{1}{4}[(\mathfrak{q}^2 + \bar{\mathfrak{q}}^2) + ((\mathfrak{C}_I \mathfrak{q})^2 + (\overline{\mathfrak{C}_I \mathfrak{q}})^2)]
\end{aligned}$$

# Heisenberg Uncertainty

From this,

$$\begin{aligned}
|\langle \Delta Q\rangle^2 \langle \Delta P_I\rangle^2| &\geq \frac{1}{4} - \frac{1}{4}(|\mathfrak{q}|^2(1+|\mathfrak{C}_I|^2) + |\bar{\mathfrak{q}}|^2(1+|\mathfrak{C}_I|^2|)) \\
&\geq \frac{1}{4} - \frac{1}{2}|\mathfrak{q}^2|\,(1+\mathfrak{C}_I|^2) \\
&\geq \frac{1}{4} - |\mathfrak{q}^2| \quad \text{as} \quad |\mathfrak{C}_I| \leq 1.
\end{aligned}$$

Thus $|\langle \Delta Q\rangle^2 \langle \Delta P_I\rangle^2| \geq \frac{1}{4} - |\mathfrak{q}|^2$. Likewise, it is not difficult to see that $|\langle \Delta Q\rangle^2 \langle \Delta P_I\rangle^2| \leq \frac{1}{4} + |\mathfrak{q}|^2$.

# Heisenberg Uncertainty

As a summary, we have

$$||\langle \Delta Q\rangle^2 \langle \Delta P_I\rangle^2| - \frac{1}{4}| \leq |\mathfrak{q}|^2.$$

From this, one can say that

$$\lim_{|\mathfrak{q}|\longrightarrow 0} |\langle \Delta Q\rangle \langle \Delta P_I\rangle| = \frac{1}{2}.$$

Further, since $[Q, P_I] = I \cdot \mathbb{I}_{\mathfrak{H}}$, we have

$$\begin{aligned} [Q, P_I]|\gamma_{\mathfrak{q}}\rangle &= (I \cdot \mathbb{I}_{\mathfrak{H}})|\gamma_{\mathfrak{q}}\rangle = I \cdot (\mathbb{I}_{\mathfrak{H}}|\gamma_{\mathfrak{q}}\rangle) \\ &= |I \cdot \gamma_{\mathfrak{q}}\rangle. \end{aligned}$$

## Heisenberg Uncertainty

Therefore

$$\langle \gamma_{\mathfrak{q}} | [Q, P_I] | \gamma_{\mathfrak{q}} \rangle = \langle \gamma_{\mathfrak{q}} | I \cdot \gamma_{\mathfrak{q}} \rangle = e^{-|\mathfrak{q}|^2} \sum_{m=0}^{\infty} \frac{\bar{\mathfrak{q}}^m I \mathfrak{q}^m}{m!} = \mathfrak{C}_I = r\mathfrak{I}.$$

Hence

$$\frac{1}{2} |\langle [Q, P_I] \rangle| = \frac{1}{2} |r\mathfrak{I}| = \frac{1}{2} r \leq \frac{1}{2},$$

as $r \leq 1$. As a conclusion we can say that

$$\lim_{|\mathfrak{q}| \longrightarrow 0} |\langle \Delta Q \rangle \langle \Delta P_I \rangle| \geq \frac{1}{2} |\langle [Q, P_I] \rangle|.$$

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