GEOMETRIC ASPECTS OF COHERENT STATES

Mauro SPERA

Dipartimento di Matematica e Fisica
"Niccolò Tartaglia"
Università Cattolica del Sacro Cuore
via dei Musei 41, 25121 Brescia, Italia

(review of personal & joint work with G. Valli†, J. Wurzbacher, E. Previato, A. Galasso)

outline

- Kählerian coherent states in G.Q. (general)
- Application to the SSW Grassmannian
- Geometric construction of CCR representations
- Landau levels revisited

← coherent state aspects stressed

GEOMETRIC QUANTIZATION
& COHERENT STATES

$\boxed{\text{The basic idea}}$

"quantization"

$$(X, \omega) \xrightarrow{Q} \mathcal{H}_X \qquad \text{actually } P(\mathcal{H}_X)$$

symplectic manifold — Hilbert space — projective space

$\omega \in \Lambda^2(X)$ $d\omega = 0$

ω non degenerate

phase space of a classical dynamical system $\rightsquigarrow$ pure states of the classical system

(pure) states of the corresponding quantum system

If G is a Lie group acting symplectically on (X, ω) (i.e. $g^*\omega = \omega$ $\forall g \in G$) then Q should give a unitary representation of G on $\mathcal{H}_X$

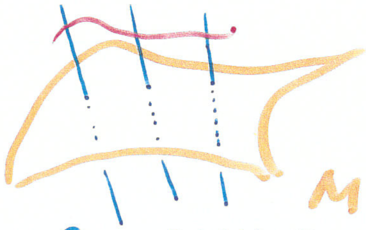
G.Q. : construct $\mathcal{H}_X$ exploiting the geometry of X
roughly : $\mathcal{H}_X$ manufactured from the space of sections of a complex line bundle $L \to X$

The celebrated Borel-Weil theorem is encompassed by G.Q.
BW yields the irreducible representations of compact, simple Lie groups as spaces of holomorphic sections of hol. line bundles over homogeneous Kähler manifolds

The coherent state map plays a special role

GENERAL ASPECTS OF KÄHLERIAN COHERENT STATES (–, 2000)

see Kirwin 2006 for extensions
to almost kähler & Spin^c frameworks
see Odjiewicz '92 as well



(M, ω) compact prequantizable Kähler manifold By Weil - Kostant, get

connection metric
$(L, \nabla, (\,,\,))$ hermitian holomorphic prequantum bundle (unique up to equivalence if M is simply connected)
Chern - Bott connection
$c_1(L) = [\omega]$

∇ : unique connection compatible with $(\,,\,)$, with curvature $= -2\pi i \omega$:

$$\boxed{\nabla^{0,1} = \bar{\partial}}$$

L^2 : space of (all) L^2 - sections of $L \to M$ (w.r. to $(\,,\,)$), integration carried out w.r. to the Liouville measure dm

H : quantum Hilbert space $\equiv H^0(L)$ holomorphic sections of $L \to M$
(finite dimensional in view of compactness)

$$H = \underbrace{\mathrm{Ker}\,\Delta}_{\text{elliptic operator}} \qquad \Delta := \nabla^{0,1\,*} \nabla^{0,1}$$

$\neq \{0\}$ if L is "sufficiently positive"

Moreover under suitable conditions

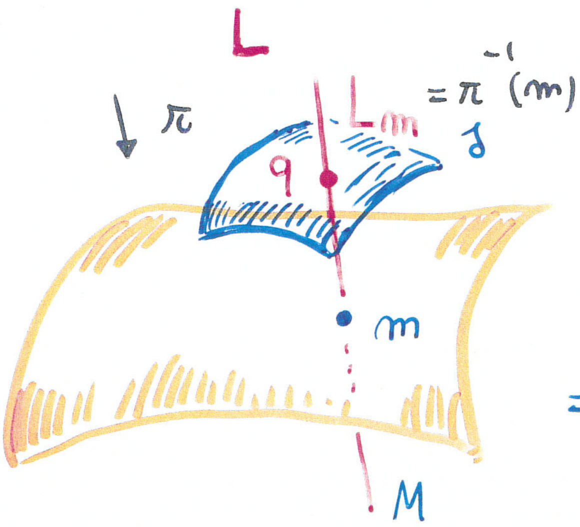
$h^0(L) := \dim H$ is a topological invariant
(R-R theorem)
+
Kodaira vanishing

under further conditions (more details below)

Normalization : $\underbrace{\mathrm{vol}(M)}_{\text{phase space volume of the classical system}} := \int_M dm = \underbrace{h^0(L)}_{\text{dim. of the quantum Hilbert space}}$

Semiclassical meaning
(Bohr's Correspondence Principle)

◇ COHERENT STATES (Rawnsley, '77)



$ev_m : H \to L_m$

$ev_m(s) := s(m)$

the evaluation map is continuous

$\Rightarrow s(m) = \langle \underbrace{e_q}_{\in H}, s \rangle \cdot q$ coherent state vectors

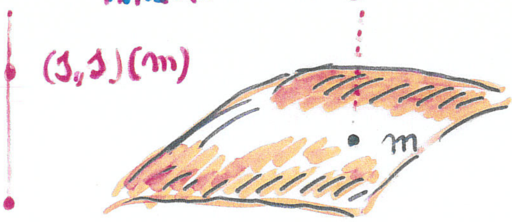
$e_{cq} = \bar{c}^{-1} e_q \quad , \quad c \in \mathbb{C}^*$

Recall: $\langle\ ,\ \rangle = \int_M (\ ,\ )\, dm$

◇ ALTERNATIVE (but equivalent) DEFINITION
M.S. (2000)

s_m , $m \in M$ coherent state vector

maximises $s \longmapsto (s,s)(m)$



$(\langle s, s\rangle = 1)$ $\qquad \overset{\shortparallel}{\langle s, A_m s\rangle}$

(s_m is clearly determined up to a phase factor)

$$(s,s)(m) = \big(\langle e_q, s\rangle \cdot q \ , \ \langle e_q, s\rangle \cdot q\big)$$

$$= (q,q)\, |\langle e_q, s\rangle|^2 \leq (q,q)\, \|e_q\|^2$$

$$\Rightarrow \quad s_m = c\, e_q$$

if $\|e_q\| = 1$, $\quad (s_m, s_m)(m) = (q,q)$

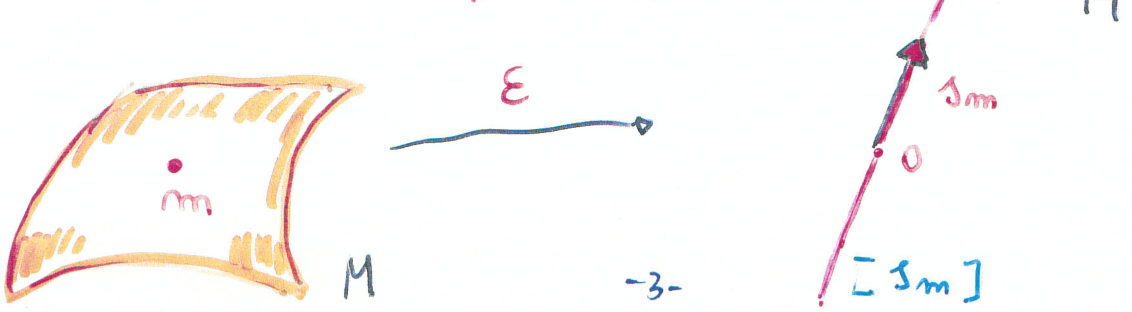
$P(H)$: projective space pertaining to H

$[\ \ ] : H \setminus \{0\} \longrightarrow P(H)$ canonical map

We wish to define

$\varepsilon : M \to P(H)$ via

$$\varepsilon(m) = [s_m]$$



Assumptions

A0 : Kodaira vanishing $\quad (\Rightarrow h^0(L) = \int_M \underbrace{\Lambda(m)}_{(s_m s_m)(m)}\, dm$

A1 $\quad \mathcal{E}$ well defined
(absence of base points)

A2 $\quad \mathcal{E}$ injective

[A1 $\not\Rightarrow$ A2 in general : consider, e.g. the canonical bundle K on a hyperelliptic Riemann surface (2 – 1 branched covering of the Riemann sphere)]

A1 , A2 $\Rightarrow$ $V_{\lambda_m} \subset H$ 1-dimensional
& $m \mapsto \Lambda(m)$ positive

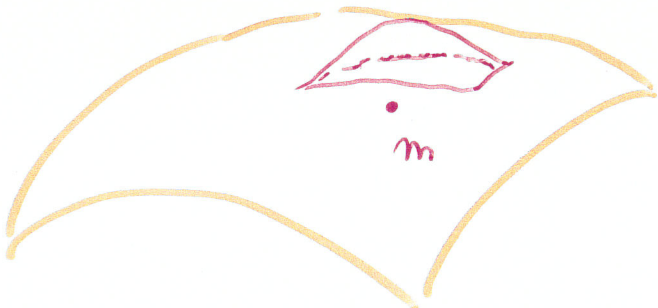
λ_m : max eig of A_m

A3 $m \mapsto \Lambda(m)$ constant (and $= 1$)
$\rightsquigarrow$ Rawnsley's function Λ constant $\quad (\Rightarrow h^0(L) = \int_M dm = \mathrm{vol}(M))$

Consequences of AO - 3

- $(s_{m'}, s_{m'})(m) < (s_m, s_m)(m) = 1$

$\star$ density of c.s. wave function $m' \mapsto (s_m, s_m)(m')$



- $\underset{\mathcal{H}}{\langle s_m, s \rangle} = (s_m, \underset{\mathcal{H}}{s})(m)$ $\qquad \lambda_m = 1$

reproducing property

- $\langle s_1, s_2 \rangle = \int_M \langle s_1, s_m \rangle \langle s_m, s_2 \rangle \, dm$

i.e

$$\int_M | s_m \rangle \langle s_m | \, dm = I$$

generalized resolution of the identity

- $\overbrace{(s_{m'}, s_{m'})(m) = (s_m, s_m)(m')}^{\text{reciprocity law}}$

$$= | \langle s_m, s_{m'} \rangle |^2$$

coherent state transition probability

- $(\nabla s_m)(m) = 0$

(in a local unitary frame $s = \tilde{s}\, s_0$

$$d\tilde{s}_m(m) = -\underbrace{\vartheta^0(m)}_{\text{connection form}}\ \underbrace{\tilde{s}_m(m)}_{\text{can be chosen to be } = 1}$$)

Moreover

Theorem A0 - 3 $\Rightarrow$ $\mathcal{E}$ symplectic embedding

$\mathcal{E}$, $\mathcal{E}_*$ injective and $\boxed{\mathcal{E}^*(\Omega) = \omega}$

Ω: Fubini - Study on $\mathbb{P}(H)$

and moreover

$$\mathcal{E}^*(\underbrace{\mathcal{O}(1)}_{\text{hyperplane bundle}}, \underbrace{\nabla_{can}}_{\text{can. connection}}) = (L, \nabla)$$

$$\vartheta_{can} = -\frac{\langle v\, dv \rangle}{\|v\|^2}$$

◇ CALABI'S DIASTASIS FUNCTION

$$D_m(m') = f(m,m) + f(m',m') - f(m',m) - f(m,m')$$

f <u>any</u> Kähler potential locally defined $\boxed{\frac{i}{2}\partial\bar\partial f = \omega}$

(extended sesquiholomorphically)

local form

$$\boxed{D(m,m') = \sum_{i=1}^{n} |z_i(m')|^2 + \underset{\text{in } z,\bar z}{\text{h.o. terms}}}$$

$z_i(m) = 0$

Calabi - Bochner

Remark:

$$(s_m, s_m)(m') = e^{-D(m,m')}$$

$$\| $$

$$|\langle s_m, s_{m'} \rangle|^2$$

Gaussian shape (Calabi - Bochner)

- "classical" evolution
- uncertainty relations minimised (e.g. —, '93)

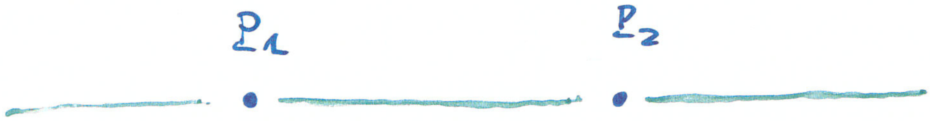
THE KLEIN QUADRIC

Prologue

$Gr(4,2)$ planes ($\mathbb{C}^2$) in $\mathbb{C}^4$

$\equiv$ lines ($\mathbb{P}^1$) in $\mathbb{P}^3$

traditional notation $Gr(2,4)$



p_1 : (x_0, x_1, x_2, x_3) $\quad (x_0,x_1,x_2,x_3) \neq (0,0,0,0)$

p_2 : (y_0, y_1, y_2, y_3) $\quad x_i \mapsto \underset{\neq 0}{\rho}\, x_i$

$$P_{ij} := \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$$

Plücker coordinates
homogeneous coordinates in $\mathbb{P}^5$

$$\begin{vmatrix} x_0 & \cdots & x_3 \\ y_0 & \cdots & y_3 \\ x_0 & \cdots & x_3 \\ y_0 & \cdots & y_3 \end{vmatrix} = 0 \quad \Rightarrow$$

$$P_{01}P_{23} - P_{02}P_{13} + P_{03}P_{12} = 0$$

$\Rightarrow \quad Gr(4,2) \cong Q \hookrightarrow \mathbb{P}^5$

★ Q : Klein quadric

The embedding

$$Gr(4,2) \cong Q \xhookrightarrow[\text{Plücker}]{} \mathbb{P}^5$$

is realized à la Kodaira via Det^*
(dual of det)

$$Gr(4,2) \ni W \longmapsto \underbrace{\mathbb{C}\, w_1 \wedge w_2}_{\text{orthonormal basis}} \equiv \underbrace{\mathrm{Det}_W}_{\text{determinant line}}$$

$$\begin{tikzcd}
\mathrm{Det} = \mathrm{Pl}^* \mathcal{O}(-1) \arrow[d] & \mathcal{O}(-1) \arrow[d] \\
Gr(4,2) \arrow[r, hook, "\mathrm{Pl}"'] & \mathbb{P}^5
\end{tikzcd}$$

tautological bundle

$[v] \mapsto \mathbb{C} v$

$v \neq 0$

Also:

$$Gr(4,2) \cong U(4) \big/ U(2) \times U(2)$$

$\Rightarrow$ one realizes an irreducible unitary representation of $U(4)$ on the space of holomorphic sections of Det^* $(= \mathrm{Pl}^* \mathcal{O}(1))$

hyperplane bundle
(dual to $\mathcal{O}(-1)$)

(Borel - Weil)

$$\boxed{Gr(4,2) = U(4)\text{-orbit}}$$

Sato – Segal – Wilson Grassmannian

• $Gr_{res}(H, H_+)$

restricted

$H = L^2(S^1, \mathbb{C})$

H_+ : non negative Fourier modes

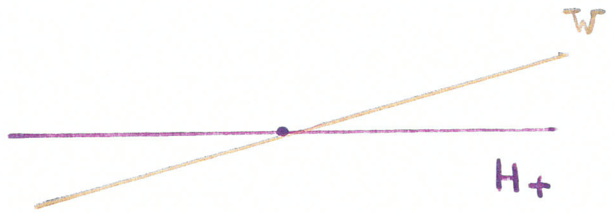
$$H = \underset{\infty}{H_+} \oplus \underset{\infty}{H_-}$$

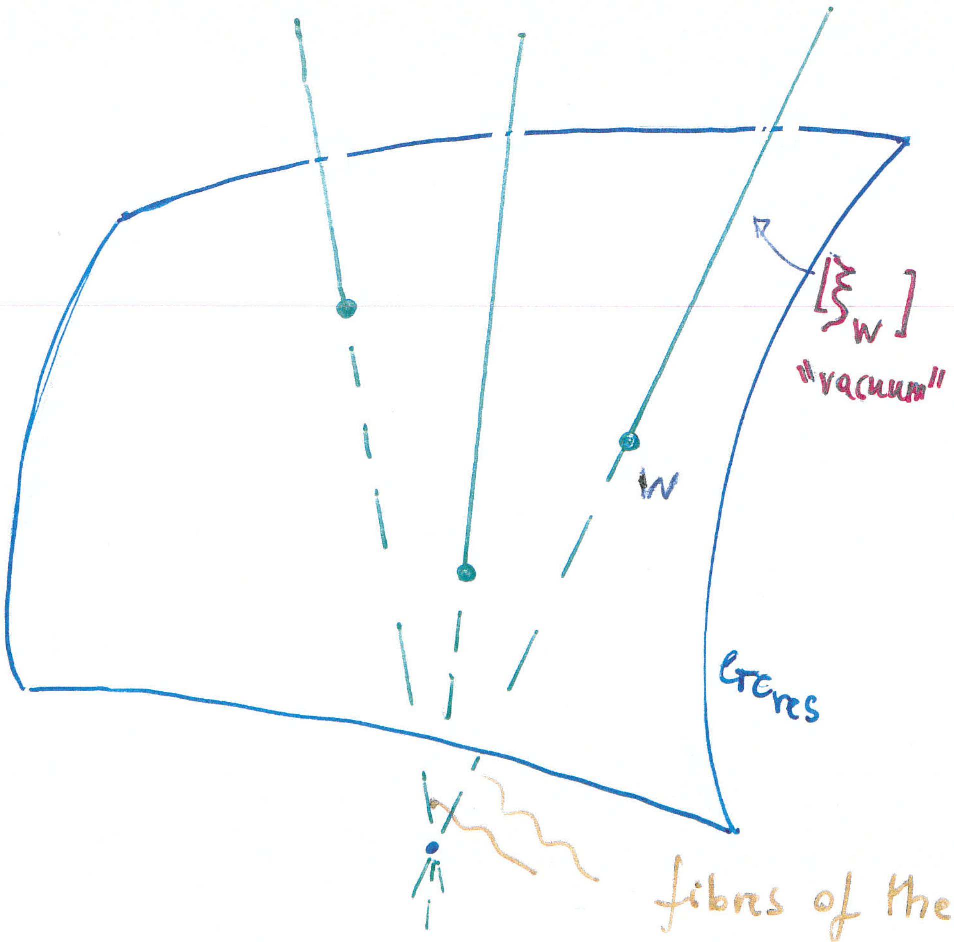
$W \in Gr_{res} \iff P_W - P_+ \in HS$ Hilbert-Schmidt

projections

" W "close" to H_+ "

(— cf Valli '94)





$\mathcal{H}_{P_+}$

" CAR-algebra representation space pertaining to a (pure) quasi-free state "

$[\xi_W]$

$(\pi_W, \mathcal{H}_W, \xi_W)$

determinant line bundle Det

$\omega(a) = \langle \xi_W, \pi_W(a) \xi_W \rangle$

" $W \sim w_1 \wedge w_2 \wedge \ldots$ "

orthonormal basis

GNS construction

$CAR(H)$: $[a^*(f), a(g)]_+ = \langle f | g \rangle_H I$

(complex Clifford algebra) $[a(f), a(g)]_+ = 0$

a^* creation
a annihilation
operators

(fermionic systems)

$$\begin{array}{ccc} \mathrm{Det}^* & & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ Gr_{res} & \hookrightarrow & \mathbb{P}(\mathcal{H}_+) \end{array}$$

← hyperplane bundle

Plücker Embedding

cf. Klein quadric

$\lambda \in \Lambda \in Gr_{res}$ $\left\{ a^*(\lambda)\Lambda = 0 \right.$ $\equiv$ Pauli Exclusion Principle

Plücker equations ~

(—, Valli (1994))

$$\boxed{Gr_{res}(H, H_+) = \frac{U_{res}(H)}{U(H_+) \times U(H_-)}}$$

U_{res}-orbit of H_+

homogeneous Kähler manifold

$U_{res}(H) = \{ u \in U(H) \mid [u, J] \in HS \}$

$J: P_+ - P_-$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

polarization operator

restricted unitary group

[Gr_{res} : hermitian symmetric space]

(— & T. Wurzbacher 2000)

Kähler form $\sim$ s Schwinger cocycle
anomaly

$$s(A,B)\mathbb{1} = [Q(A), Q(B)] - Q[A,B]$$

Q: fermionic 2^{nd} quantization (wick ordering)

$\equiv$ curvature of Det^* (with respect to the Chern-Bott connection)

(geometric quantization on Gr_{res})

$\rightsquigarrow$ $\boxed{\text{Borel-Weil for } \tilde{\bar{U}}_{res}(H, H_+)}$

$\ast$ central extension in valued

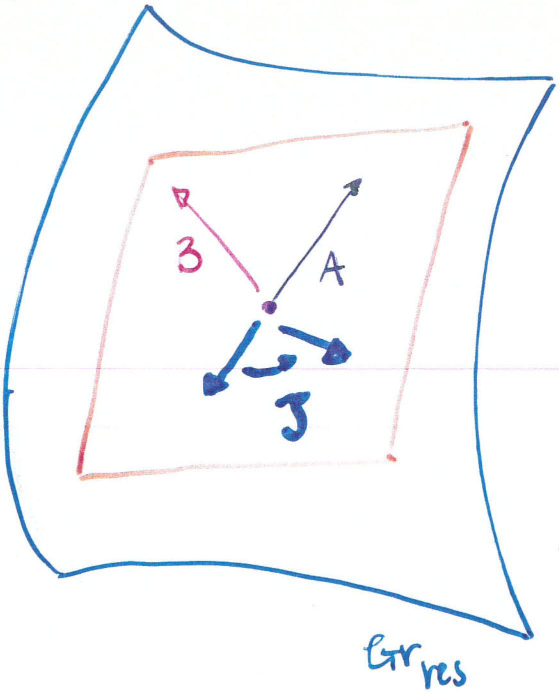
tangent space (at $F_+ \equiv H_+$)

$$\cong L^2(F_+, F_-) \equiv \mathcal{M}$$

Hilbert-Schmidt

$$A, B \in u_{res}(F, F_+)$$

Lie algebra



$$s(A,B) = \frac{1}{4} \operatorname{tr}\left(J[J,A], [J,B]\right)$$

$$g^{\mathcal{M}}(\gamma, \delta) = 2 \operatorname{Re} \underset{\underset{\operatorname{tr}_F}{\parallel}}{\operatorname{tr}_{F_+}} (\gamma^* \delta)$$

Kähler metric

$$\Omega = 2 \operatorname{Im}(\gamma^* \delta) \sim s$$

Kähler form

$$\mathcal{M} \ni \begin{pmatrix} 0 & -\gamma^* \\ \gamma & 0 \end{pmatrix}$$

complex structure $\equiv$ mult by "i" in $L^2(F_+, F_-)$

Some details (Sato's approach)

$$X_A = \sum_{m,n} a_{mn} :\psi_{-m} \psi_n^*:$$

normal product

A fulfils $\boxed{a_{ij} = 0 \ \forall\, i,j \text{ with } |i-j| > N \text{ for some } N>0}$

$$[X_A, X_B] = X_{[A,B]} + \omega(A,B)$$

$= \sum a_{ij} b_{ji} (\theta_{\{i<0\}} - \theta_{\{j<0\}})$

$(= -\omega(B,A))$

"Schwinger term"

$$gl(\infty) = \{ X_A \text{ fulfilling } \square \} \oplus \mathbb{C}$$

⇝ symmetry group of Sato's Grassmannian

$$G = \{ e^{X_1} \dots e^{X_n} \mid X_k \in gl(\infty) \}$$

KP & KdV

$x_1, x_2, x_3 \ldots$ "time variables" $x_1, x_3, x_5 \ldots$

$L = \partial + f_1 \partial^{-1} + f_2 \partial^{-2} + \ldots$ $\quad L = (\partial^2 + u)^{\frac{1}{2}}$

$L w = k w$ $\quad L^2 = P$

$\frac{\partial w}{\partial x_j} = (L^j)_+ w$ linear equations $\begin{cases} P w = k^2 w \\ \frac{\partial w}{\partial x_\ell} = (L^\ell)_+ w \end{cases}$

$\Downarrow$ $\quad \Downarrow$ compatibility conditions

KP $\boxed{\frac{\partial L}{\partial x_j} = [(L^j)_+, L]}$ (Lax form) KdV $\boxed{\frac{\partial P}{\partial x_\ell} = [(L^\ell)_+, P]}$

$(L^2)_- = 0$ yields back

* Hirota: $\boxed{(D_1^4 + 3D_2^2 - 4 D_1 D_3)\tau \cdot \tau = 0}$ $\quad \boxed{(D_1^4 - 4 D_1 D_3)\tau \cdot \tau = 0}$

$$\boxed{\frac{3}{4}\frac{\partial^2 u}{\partial x_2^2} = \frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial u}{\partial x_3} - \frac{3}{2} u \frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3}}_{\boxed{}}\Big)} \qquad \boxed{} = 0$$

KP (original) $\quad$ KdV (original)

* Hirota derivatives

$$f(x_i + y_i)\, g(x_i - y_i) =$$

$$f \cdot g + y_1 (D_1 f \cdot g) + \dots$$

Hirota derivatives

$$\frac{y_1^2}{2} (\underbrace{D_2 f \cdot g}_{\text{single entity!}}) + \dots$$

example: $D_t D_x f \cdot f = 2 \left(\frac{\partial^2 f}{\partial t \partial x} - \frac{\partial f}{\partial t} \frac{\partial f}{\partial x} \right)$

$\boxed{u = 2 \frac{\partial^2}{\partial x^2} \log \tau}$ rewrite KdV and find

$$\boxed{(4 D_t D_x - D_x^4) \tau \cdot \tau = 0}$$

KdV

$u_t = u_{xxx} + 6 u u_x$

alternatively $D_{x_i} = \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x'_i} \right) \Big|_{x = x'}$

$$\boxed{(D_x^4 - 4 D_x D_t + 3 D_y^2) \tau \cdot \tau = 0}$$

KP

$$3 u_{yy} + \frac{\partial}{\partial x} (-4 u_t + u_{xxx} + 6 u u_x) = 0$$

Geometric portrait

$\ast$ Fermionic picture Fock space $\supset$ $G\,|vac\rangle$ (orbit) ↙ Sato's grassmannian

$\ast$ Bosonic picture $\mathbb{C}[x_1, x_2 \dots] \supset$ τ - functions
$\mathrel{|||}$
points of the orbit

$$G = \{ e^{X_1} \dots e^{X_k} / X_i \in gl(\infty) \}$$

$\equiv$ Coherent states (D'Ariano - Rasetti '84)

{KP} = orbit equations

in Hirota form

★ Coherent states for infinite dimensional Grassmannians (τ - functions (D'Ariano - Rasetti) (Previato, — 2011)

$$\tau_{\bar{W}}((w', v)) = \langle v, \xi_W \rangle, \quad v \in Det_{W'}$$

$$\mathcal{H}_+ \ni \xi_W \longmapsto \tau_{\bar{W}} \in \Gamma_{L^2}(Det^* \to Gr)$$

★ Boson - fermion correspondence (à la —, Valli '94)

ordinary notation

$$\tau(t, g) = \langle \Omega, e^{H(t)} g \Omega \rangle = \sum_Y \underbrace{c_Y(g)}_{} \chi_Y(t)$$

Plücker coordinates

Ω admissible basis pertaining to ϕ

$$g = e^{\sum t_i z^i} \qquad H(t) = \sum \frac{1}{i} \frac{\partial}{\partial t_i}$$

$$c_Y(g) = \chi_Y(\partial_t) \tau(t, g) \Big|_{t=0}$$

$\{\chi_Y\}$ Schur functions

★ Calabi's diastasis function

$$D(z, w) = f(z, z) + f(w, w) - f(z, w) - f(w, z)$$

f local Kähler potential (independent of it)

in our context

$$D([\tau], [\tau']) = \log \frac{\sum_Y |c_Y|^2 \sum_Y |c'_Y|^2}{|\sum c_Y \bar{c}_Y'|^2}$$

$$\equiv \log \frac{\|\tau\|^2 \|\tau'\|^2}{|\langle \tau, \tau' \rangle|^2}$$

$$D([\tau = 1], [\tau]) = \log \sum_Y |c_Y|^2 = \log \|\tau\|^2$$

FS - kähler potential

(Fubini - Study)

One has

$$\boxed{|\langle \xi_{W_1}, \xi_{\overline{W}_2}\rangle|^2 = |\langle \tau_{W_1}, \tau_{W_2}\rangle|^2 = e^{-D(W_1,W_2)}}$$

Fermion interpretation Boson interpretation

(—, G. Valli '93, '94)

★ $\boxed{\text{Segre embedding}}$ (—, Wurzbacher '98)

$$Gr \times Gr \hookrightarrow GR$$
$$(W_1, \overline{W}_2) \longmapsto W_1 \oplus \overline{W}_2^{\perp}$$

$Gr = Gr(H, H_+)$ $\quad GR = Gr(H_{\mathbb{C}}, W = H_+ \oplus \overline{H}_-)$

$$\pi_{(S,T')}(Y) = \langle W_1 \oplus \overline{W}_1^{\perp}, H_S \oplus \overline{H_{T'}^{\perp}} \rangle$$
$$= \langle W_1, H_S\rangle \langle W_1, H_T\rangle = \pi_S(W_1)\pi_T(W_1)$$

$$Gr \hookrightarrow GR$$
$$W \longmapsto W_1 \oplus \overline{W}_1^{\perp}$$

$$\pi_{(S,S')}(W_1 \oplus \overline{W}_1^{\perp}) = \pi_S(W_1)^2$$

(via the Pfaffian line bundle Pf)

restricted orthogonal group ("spin representation")

τ - function counterpart

$$\tau^2_{Gr} = \tau_{GR}|_{Gr}$$

$$\tau^2_{BKP} = \tau_{KP}|_{x_2 = x_4 = \ldots = 0}$$

orthogonal affine subalgebra B_∞ of $A_\infty = gl(\infty)$

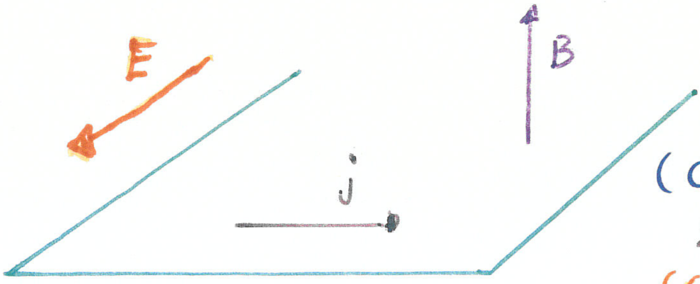
$\rightsquigarrow$ get $\boxed{\tau_{Segre} = \tau_{Gr} \otimes \tau_{Gr} = \tau_{GR}|_{Gr\times Gr}}$

$$\boxed{\mathrm{Det}|_{Gr\times Gr} = Pf^{\otimes 2}}$$

One also gets ★ Calabi-type rigidity theorems using coherent states & Wigner's theorem (—, G. Valli 92, — E. Previato 2011)

★ CCR aspects

★ FQHE & Laughlin wave functions
(—, 2015)



quantum theory of a
(Coulomb interacting)
2d - electron gas
(completely polarized)

low temperature
strong magnetic fields

ground state approximately

described by the Laughlin wave function

$$\Psi(z_1 \dots z_n) = \prod_{i<j} (z_i - z_j)^m \, e^{-\sum |z_i|^2}$$

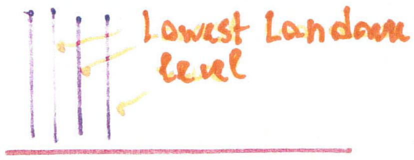
↖ ground state of a quantum harmonic oscillator

$\underset{\text{odd}}{m} = \frac{1}{\nu}$ ν: filling factor

$$\sigma_H = \nu \frac{e^2}{h}$$

★ Hall conductance

for a torus sample



ν = slope of a holomorphic vector bundle over a torus (Brillouin manifold) parametrising boundary conditions (Varnhagen '95)

elementary excitations: quasiparticles / holes
with charge $e^* = \pm e \nu$ (fractional!)
and anyon statistics $(-1)^{\nu}$ ⇝ braid group

⇝ generalised Laughlin wave functions

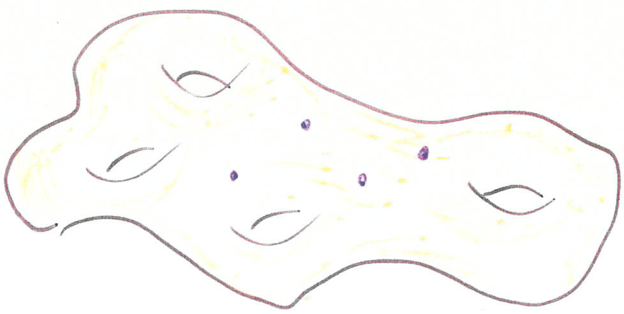
Thus

ν = statistical parameter = slope

Task:

analyse the above coincidence from a (RS) braid group theoretical perspective

sample: closed RS of genus $g \geq 1$



Brillouin (or spectral) manifold: $J(\Sigma_g)$

Jacobian

$\boxed{\text{unitary representations of } B(\Sigma_g, n) \text{ RS braid group} \xleftarrow{} \text{Weyl-Heisenberg} \xrightarrow{} \text{Projectively flat (HE) vector bundles } \mathcal{E} \to J(\Sigma_g)}$

Matsushima

$$\frac{\text{degree}}{\text{rank}} = \underset{\text{slope}}{\mu} = \nu g!$$

"slope-statistics" formula

get generalised Laughlin wave functions

★ Techniques

Finite-dimensional representations stem from the finite CCR

ex: $g = 1$ $\quad \nu = \frac{q}{r}$ $\quad (q>0, r>0, (q,r)=1)$

on $\mathbb{C}^r$: U_1, U_2 satisfying $U_1 U_2 = e^{-2\pi i \nu} U_2 U_1$

$$\hat{\rho}(a_1) = U_1 \qquad \hat{\rho}(b_1^{-1}) = U_2 \qquad U_j^r = I$$

$$U_1 = \begin{pmatrix} 1 & & 0 \\ & e(\nu) & \\ & & \ddots \\ 0 & & & e((r-1)\nu) \end{pmatrix} \qquad U_2 : e_i \mapsto e_{i-1} \quad \text{shift map}$$

$$e(x) := e^{2\pi i x}$$

then complete the definition accordingly

★ Standard Weyl-Heisenberg CCR

$$V(\vec{\beta})\, U(\vec{\alpha}) = e^{2\pi i \nu\, \vec{\alpha}\cdot\vec{\beta}}\, U(\vec{\alpha})\, V(\vec{\beta}) .$$

$\vec{\alpha}, \vec{\beta} \in \mathbb{R}^g$

both arising in the representation theory of Riemann Surface braid groups (Bellingeri's presentation)

$\rightsquigarrow$ geometric construction ★

* Theta functions and Canonical commutation relations (after M.S. '86)

L : h.l.b on an abelian variety X
$h^0(L)=1 \rightsquigarrow$ Riemann theta function ξ
via $\mathcal{L}.Q.$

($Z = iI$ for simplicity)

$$\mathcal{H} = \mathcal{L}^2(L, \langle , \rangle) \qquad L = \text{any } L_\xi$$

∇ Chern - Bott connection

$$\nabla_{\frac{\partial}{\partial q_i}}, \nabla_{\frac{\partial}{\partial p_i}} \qquad \frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i} \in \mathrm{Lie}(X) \cong \mathbb{R}^{2n}$$

$$\left[\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_j}\right] = 0$$

are skew-hermitian

$$\left(0 = \int_X \frac{\partial}{\partial q_i} \langle s', s \rangle = \int_X \langle \nabla_{\frac{\partial}{\partial q_i}} s', s \rangle + \int_X \langle s', \nabla_{\frac{\partial}{\partial q_i}} s \rangle \right)$$

Theorem (M.S. '86) $\quad Q_j = \frac{i}{\sqrt{2\pi}} \nabla_{\frac{\partial}{\partial q_i}}, \quad P_j = \frac{i}{\sqrt{2\pi}} \nabla_{\frac{\partial}{\partial p_j}}$

$j=1,2,\dots n$ realise a representation of the Heisenberg commutation relations

$$[Q_i, P_j] = \sqrt{-1}\, \delta_{kj} I$$

$$[Q_i, Q_j] = [P_i, P_j] = 0$$

Consequently, their induced parallel transport operators gives rise to a representation of the Weyl commutation relations

$$\bar{V}(\beta)\,\bar{U}(\alpha) = e^{i\alpha\beta}\,u(\alpha)\,\bar{V}(\beta)$$

$$U(\alpha) = e^{i\sum_j \alpha_j Q_j} \qquad V(\beta) = e^{i\sum_j \beta_j P_j}$$

The representation is irreducible

This is a consequence of the Riemann-Roch theorem n conjunction with the von Neumann uniqueness theorem

holomorphic structure

from $A_j = \frac{1}{\sqrt{2}}(Q_j + iP_j)$ $\sim \bar{\partial}$ annihilation operator ← crucial

$H^0(L)$ = ground state space of the quantum harmonic oscillator defined by

$$H = \frac{1}{2}\sum_j (P_j^2 + Q_j^2)$$

also: geometric description of coherent states

$$\left(\Psi_{\alpha,\beta} = e^{\frac{i}{2}\alpha\cdot\beta}\, u(\alpha)\, V(-\beta)\, \Psi_{0,0} \right)$$

ground state of the harmonic oscillator

via $\mathrm{Pic}^0(X)$

⇝ theory of Landau levels in QH effects

hol. vector bundles over Pic^0

("Brillouin manifold")

★ HE - vector bundles on Jacobians (Matsushima)

Illustration for $g = 1$

Start from a r-dimensional representation of the finite Weyl-Heisenberg group corresponding to a 2-lattice $\Gamma \subset \mathbb{C}$ giving rise to the torus $\mathbb{C}/\Gamma$:

$$\boxed{U(\gamma+\gamma') = U(\gamma)U(\gamma')\, e^{\frac{i}{2r} A(\gamma',\gamma)}}$$

$A = \operatorname{Im} R \qquad R$ hermitian form on $\mathbb{C}$ with

$$\left[\frac{1}{2\pi} A(\gamma,\gamma') \in \mathbb{Z} \right]$$

$$A = -2\pi q \cdot \omega = -2\pi q \cdot dx \wedge dy = -i\pi q \; dz \wedge d\bar{z}$$

$$U_1 U_2 = U_2 U_1 \, e^{-2\pi i \frac{q}{r}} \qquad \nu = \frac{q}{r}$$

$$R(z,w) = \pi q \cdot z\bar{w}$$

factor of automorphy (theta factor)

$$j(\gamma,z) = U(\gamma) \exp\left[\frac{1}{2r} R(z,\gamma) + \frac{1}{4r} R(\gamma,\gamma) \right]$$

yielding $\mathcal{E}_\nu \to \mathbb{C}/\Gamma$ stable bundle of rank r

$$\boxed{\mathcal{E}_\nu = (\mathbb{C}/\Gamma \times \mathbb{C}^r)/\Gamma}$$

action of Γ:

$$\gamma(z,\xi) := (\gamma + z, j(\gamma,z)\xi)$$

equipped with the hermitian metric $h = e^{-\frac{R}{2r}} I_r$ leading to a Chern-Bott connection ∇ with constant curvature $\Omega_\nabla = i \frac{A}{r} I_r = -2\pi i \, \nu \cdot e_1 \wedge e_2 \, I_r$

One has $h^0(\mathcal{E}_\nu) = q$ (Riemann-Roch-Hirzebruch)

$$\overset{\shortparallel}{\dim H^0(\mathcal{E}_\nu)} \leftarrow \text{holomorphic sections}$$

$r=1$ yields the q-level theta functions

$\rightsquigarrow$ ★ Weyl-Heisenberg again with multiplicity q via Chern-Bott connection

4 Noncommutative tori & Stable bundles (g = 1)

Prologue: Swan's theorem
sections of vector bundles ≡ finitely generated projective modules over the algebra of smooth functions on the base manifold

Noncommutative torus A_ϑ $\vartheta \in \mathbb{R}/\mathbb{Z}$
universal unital C^*-algebra generated by unitary operators U_1, U_2 satisfying

$$U_1 U_2 = e^{2\pi i \vartheta} U_2 U_1$$

(deformation of the standard commutative algebra of continuous functions on a torus - via Fourier)

$\mathbb{T}^2_\vartheta$: smooth sub-algebra

$$\sum a_{mn} U_1^m U_2^n \qquad \{a_{mn}\} \in \mathcal{S}(\mathbb{Z}^2)$$

rapidly decreasing

$$\delta_1(U^k) = 2\pi i\, n_1 U^k \qquad U^k = U_1^{n_1} U_2^{n_2}$$
$$\delta_2(U^k) = 2\pi i\, n_2 U^k$$

Natural (hermitian) $\mathbb{T}^2_\vartheta$ - right modules

$$\boxed{\mathcal{E}_{p,q} = \mathcal{S}(\mathbb{R}, \mathbb{C}^q)}$$

as a vector space, they do not depend on p
$(p > 0,\ q > 0,\ (p,q) = 1)$

$$\boxed{\mathrm{rk}(\mathcal{E}_{p,q}) := \tau_{\mathrm{End}(\mathcal{E}_{p,q})}(I) = p - \vartheta q \quad (> 0)}$$

(Murray - von Neumann)

$$\tau\left(a = \sum a_{mn} U_1^m U_2^n\right) = a_{00}$$
$$\tau(ab) = \tau(ba)$$

(right) module structure :

$\downarrow \mathcal{E}_{p,q}$

$$(V_1 \xi)(s) = e(s)\,\xi(s)$$

$$(V_2 \xi)(s) = \xi\left(s - (p/q - \theta)\right)$$

$$W_1 W_2 = \bar{e}(p/q)\, W_2 W_1 \qquad W_1^{\,q} = W_2^{\,q} = 1$$

finite W-H relations for $\mathbb{Z}/q\mathbb{Z}$

Then set :

$$\xi\, U_i := (V_i \otimes W_i)\,\xi \qquad i = 1, 2$$

+ Connes connection

$$(\nabla_1 \xi)(s) = 2\pi i \, \frac{q}{p - \theta q}\, s\, \xi(s)$$

$$(\nabla_2 \xi)(s) = \xi'(s) \qquad s \in \mathbb{R}$$

$$\left[\ \Omega = [\nabla_1, \nabla_2]\, e_1 \wedge e_2 = -2\pi i \, \frac{q}{p - \theta q}\, e_1 \wedge e_2\ \right]$$

Curvature

$$\left[\ c_1(\mathcal{E}_{p,q}) = \frac{1}{2\pi i}\, \mathrm{Tr}(-\Omega) = q \qquad \text{(integer!)}\ \right]$$

1st Chern class

* holomorphic structure (—, Polishchuk–Schwarz)

$$\bar{\nabla} := \nabla_1 + i \nabla_2$$

$\ker \bar{\nabla}$: noncommutative theta vectors (Schwarz, '00)
(q-dimensional)

$$\xi = \xi(s) = e^{-\pi \frac{q}{p-\theta q} s^2} \cdot v \qquad v \in \mathbb{C}^q$$

quantum harmonic oscillator ground state

$\ker \bar{\nabla}^* = \{0\}$ (from $[\bar{\nabla}, \bar{\nabla}^*] = 4\pi \frac{q}{p-\theta q} I$)

$$\operatorname{Ind}(\bar{\nabla}) = \dim(\ker \bar{\nabla}) - \dim(\ker \bar{\nabla}^*) = q$$

index formula

Upshot:

* Now take $p = r > 0$, $\theta = 0$. We get a "classical" rank r hermitian holomorphic vector bundle over a complex torus, which is stable, equipped with a hermitian connection with constant curvature, having slope

$$\mu(\mathcal{E}_{r,q}) = \frac{q}{r} \quad (= \gamma)$$

and such that

$$h^0(\mathcal{E}_{r,q}) = q$$

(for $r = q = 1$ one retrieves the theta line bundle)

(Also: "Schwarz = Laughlin")

★ "statistics" of theta vectors

Let $\nu = \frac{q}{r}$. Consider $\mathcal{E}_\nu$

Then set $\nu' = \frac{r}{q} = \frac{1}{\nu}$ & consider

$$W_1 W_2 = e^{-2\pi i \frac{r}{q}} W_2 W_1$$

★ realised on $H^0(\mathcal{E}_{\nu'}) \cong \mathbb{C}^q$

$$C : \mathcal{E}_\nu \to \mathbb{T}^2_{\nu'}$$

Connes

But the above representation determines an ∞-dim representation of $B(\Sigma_1, q)$ with "dual" statistics parameter $\sigma' = (-1)^{\nu'}$ and can be promoted to a stable bundle $\mathcal{E}_{\nu'}$ (Matsushima)

$$M : \mathbb{T}^2_\nu \to \mathcal{E}_\nu$$

Matsushima

$$\left[MC : \mathcal{E}_\nu \to \mathcal{E}_{\nu'} \qquad CM : \mathbb{T}^2_\nu \to \mathbb{T}^2_{\nu'} \right]$$

Matsushima - Connes duality

★ This can be interpreted via the Fourier - Mukai - Nahm transform

★ FMN via noncommutative theta vectors

$J(\Sigma_2)$ is a self-dual abelian variety parametrising flat line bundles thereon

$$\underset{x}{\cup\!\!\!\!\;\shortmid} \ \longmapsto \ \mathcal{P}_x$$

$$\mathcal{E}_x = \mathcal{E} \otimes \mathcal{P}_x$$ family of Matsushima bundles

$\mathcal{E}$: Matsushima

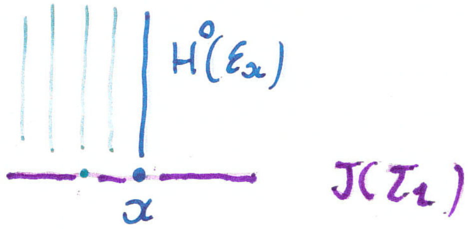
$= H^0(\mathcal{E}_x)$: fibre at x

$\ker \bar{\nabla}$

of a bona fide **holomorphic vector bundle** on $J(\Sigma_2)$,

$$\mathrm{FMN}(\mathcal{E}) \longrightarrow J(\Sigma_2)$$

equipped with a natural hermitian connection $\hat{\nabla}$



inherited from ∇ on $\mathcal{E}$

we elaborate on this via ncg

$$\bar{\tilde{\nabla}} = \tilde{\nabla}_1 + i\,\tilde{\nabla}_2 = \bar{\nabla} - 2\pi i z\, I$$

$$\tilde{\nabla}_1 = \nabla_1 - 2\pi i \alpha I \qquad\qquad z = \alpha + i\beta$$

$$\tilde{\nabla}_2 = \nabla_2 - 2\pi i \beta I \qquad\qquad \alpha, \beta \in [0,1]$$

coordinates on $J(\Sigma_1)$

theta vectors:

$$\left\{ \quad \xi_z \equiv \xi_{\alpha,\beta} = e^{-\pi \frac{r}{q}\left(\frac{q}{r}s - z\right)^2} v \qquad v \in \mathbb{C}^q \right.$$

★ coherent states

L^2-normalized theta vectors

$\int e^{-\gamma x^2} = \sqrt{\pi/\gamma}$

$\gamma > 0$

$$\tilde{\xi}_z = e^{-\pi \frac{r}{q}\beta^2} \left(\frac{2q}{r}\right)^{\frac{1}{4}} \xi_z$$

(discard ν temporarily)

+ Berry - Simon connection (Nahm transform)

$$z \mapsto A_z = \langle \tilde{\xi}_z, d\tilde{\xi}_z \rangle = \underbrace{\langle \tilde{\xi}_z, \partial_\alpha \tilde{\xi}_z \rangle}_{= -2\pi i \frac{r}{q} \beta d\alpha \cdot I_q} d\alpha + \underbrace{\langle \tilde{\xi}_z, \partial_\beta \tilde{\xi}_z \rangle}_{= 0} d\beta$$

from $\int e^{-\gamma x^2} x \, dx = 0$

+ projection $P_x : L^2 \to H^0(\mathcal{E}_x)$

+ curvature $\Omega := 2\pi i \frac{r}{q} d\alpha \wedge d\beta \, I_q = 2\pi i \, \nu' d\alpha \wedge d\beta \, I_q$

$$(Ch_0(\mathcal{E}), Ch_1(\mathcal{E})) = (r, q)$$

dualization

$$(Ch_0(FMN^*(\mathcal{E})), Ch_1(FMN^*(\mathcal{E}))) = (q, r)$$

+ Conclusion : $MC = FMN^*$ up to moduli

Also, one finds that if $\mathcal{E} \to J(\Sigma_g)$ is projectively flat, it can be the FMN transform of $\mathcal{E}' \to \Sigma_g$ only for $g = 1$

GEOMETRIC QUANTIZATION & LANDAU LEVELS revisited
(A. Galasso & M.S 2016)

Holomorphic geometric quantization of the harmonic oscillator

$M = \mathbb{R}^{2n} \cong \mathbb{C}^{n} \qquad n \geq 1 \qquad (\hbar = 1) \qquad z_j = x_j + i y_j$

$$\tilde{\omega} = \sum_{j=1}^{n} dx_j \wedge dy_j = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \frac{i}{2} \partial \bar{\partial} \mathcal{Y}$$

Kähler form

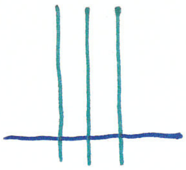
$$\mathcal{Y} := \sum_{j=1}^{n} \bar{z}_j z_j =: 2\hat{h}$$

$\hat{h}$ oscillator Hamiltonian

Kähler potential

$L = M \times \mathbb{C} \longrightarrow M$ trivial complex line bundle

$s_0 \equiv 1$ trivialising section



M

Hermitian metric $(1, 1) := e^{-\mathcal{Y}}$

$$(s = \hat{s} \cdot 1,\ s' = \hat{s}' \cdot 1) = \bar{\hat{s}}\, \hat{s}'\, e^{-\mathcal{Y}}$$

$$\tilde{\mathcal{H}} = L^2(M, \mu) \qquad \mu = e^{-\mathcal{Y}} dx_1 dy_1 \dots dx_n dy_n$$

Chern-Bott connection (in a holomorphic frame)

$$\nabla = d - \partial \mathcal{Y} = \nabla' + \nabla''$$

$$\nabla' = \partial - \partial \mathcal{Y}, \quad \nabla'' = \bar{\partial}$$

$$\nabla = d - \partial \mathcal{Y} = d - \sum_{j=1}^{n} \bar{z}_j dz_j$$

curvature: $\Omega = d(-\partial \mathcal{Y}) = -2i\tilde{\omega} = -2i\, d\hat{\theta}$

$\hat{\theta} = -\frac{i}{2} \sum_{j=1}^{n} \bar{z}_j dz_j$ symplectic potential work in $\hat{\mathcal{H}}$ (actually, suitable domains therein)

$$A_j := \underbrace{\nabla_{\frac{\partial}{\partial \bar z_j}}}_{} = \frac{\partial}{\partial \bar z_j} \qquad A_j^+ := \left(\nabla_{\frac{\partial}{\partial \bar z_j}}\right)^+ = -\frac{\partial}{\partial z_j} + \frac{\partial \varphi}{\partial z_j} = -\frac{\partial}{\partial z_j} + \bar z_j$$

annihilation & creation operators

CCR

$$[A_j, A_k] = [A_j^+, A_k^+] = 0$$

$$[A_j, A_k^+] = I \cdot \delta_{jk}$$

identification crucial in theta function theory (both classical and noncommutative) (M.S. 1986, M.S. 2015, A.Schwarz 2000)

multiplicity = dimension of the common kernel of the A_j (von Neumann uniqueness theorem 1931)

$$\mathcal{H} = \{ f \in \hat{\mathcal{H}},\ f \text{ holomorphic} \}$$

Bargmann-Fock space

already a Hilbert space

$\bar{\mathcal{H}}$ conjugate space

$$\hat{\mathcal{H}} \cong \mathcal{H} \otimes \bar{\mathcal{H}} \qquad \{ z_i^h \bar z_j^k \}$$

orthonormal basis for $\hat{\mathcal{H}}$ up to constants

$i,j = 1 \ldots n$

$h \geq 0 \quad k \geq 0$

Prequantum Hamiltonian

$$Q(\hat h) := -i \nabla_{X_{\tilde h}} + \tilde\theta(X_{\tilde h})$$

$$= -i X_{\tilde h} = \sum_{j=1}^{n} \left(z_j \frac{\partial}{\partial z_j} - \bar z_j \frac{\partial}{\partial \bar z_j} \right)$$

Hamiltonian vector field pertaining to $\hat h$

contraction $\rightsquigarrow$ $i_{X_{\tilde h}} \hat\omega + d\hat h = 0$

restrict to $\mathcal{H}$:

$$Q(\hat{h})\Big|_{\mathcal{H}} = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$$

Euler operator (number)

zero point energy missing

Set :

$$a_j := A_j\Big|_{\bar{\mathcal{H}}} = \frac{\partial}{\partial \bar{z}_j}$$

$$a_j^{+} := A_j^{+}\Big|_{\bar{\mathcal{H}}} = \bar{z}_j$$

CCR :
$$[a_i, a_j] = [a_i^{+}, a_j^{+}] = 0$$
$$[a_i, a_j^{+}] = \delta_{ij} I$$

Similarly one has CCR on $\mathcal{H}$:

$$b_j = \frac{\partial}{\partial z_j} \qquad b_j^{+} = z_j$$

★ Upshot: $\hat{\mathcal{H}} \equiv \mathcal{H}_A$ carries a representation π_A of the CCR defined via the Chern-Bott connection. π_A is **reducible**, with **infinite multiplicity** given by $\mathcal{H}$. The latter carries a CCR representation π_b, which is in turn **irreducible**. Similarly one gets π_a on $\bar{\mathcal{H}}$

$$\pi_A = \pi_b \otimes \pi_a \quad , \text{ acting on } \quad \mathcal{H} \otimes \bar{\mathcal{H}} = \mathcal{H}_b \otimes \mathcal{H}_a = \mathcal{H}_A = \tilde{\mathcal{H}}$$

$$z_i^{m_1} \bar{z}_j^{m_2} \longleftrightarrow (b_i^{+})^{m_1} (a_j^{+})^{m_2} |0\rangle \qquad a_i|0\rangle = b_i|0\rangle = 0$$

4 Charged particle in a constant magnetic field (on a plane) classical theory

1st approach $M = T^*\mathbb{R}^2 \cong \mathbb{R}^4$ phase space

$$\omega = dp_x \wedge dx + dp_y \wedge dy$$ symplectic form

$$h = \frac{1}{2}\left[(p_x - y)^2 + (p_y + x)^2\right]$$ Hamiltonian

$$\ell = x p_y - y p_x$$ z-component of angular momentum

Canonical transformation:

$$\begin{cases} P_1 = \frac{1}{\sqrt{2}}(x + p_y) \\ Q_1 = \frac{1}{\sqrt{2}}(y - p_x) \\ P_2 = \frac{1}{\sqrt{2}}(y + p_x) \\ Q_2 = \frac{1}{\sqrt{2}}(x - p_y) \end{cases}$$

$$h_j := \frac{1}{2}(P_j^2 + Q_j^2)$$

One gets:

$$h = h_1 = \frac{1}{2}[P_1^2 + Q_1^2] \qquad \ell = h_1 - h_2$$

$$\{h, \ell\} := \omega(X_h, X_\ell) = 0$$

h, ℓ: complete set of first integrals

Also set, for future use

$$z = \frac{1}{\sqrt{2}}[P_1 + iQ_1], \quad \xi = \frac{1}{\sqrt{2}}[Q_2 - iP_2]$$

$$\Rightarrow \quad h_1 = \xi\bar{\xi}, \quad h_2 = z\bar{z}, \quad \ell = \xi\bar{\xi} - z\bar{z}$$

we have a completely integrable system (as two harmonic oscillators)

2d Liouville tori parametrized by (h_1, h_2) or (h_1, ℓ)

$$\omega = dh_1 \wedge d\varphi_1 + dh_2 \wedge d\varphi_2$$ (h, φ) action-angle variables

2nd approach

equip $M = T^*\mathbb{R}^2$ with a new symplectic form

physical constants reinserted

$$\omega' = \underset{\text{old}}{\omega} + \frac{eB}{c}\, dx \wedge dy \qquad eB > 0$$

magnetic term

New Hamiltonian : $h' = \frac{1}{2m}\left(p_x^2 + p_y^2\right)$

(gauge invariant formulation)

New angular momentum $\ell' = \ell + \frac{eB}{2c}(x^2+y^2)$

Again $\{h', \ell'\} = 0$

we have $X_{h'} = \frac{1}{m}\left(p_x \partial_x + p_y \partial_y - \frac{eB}{c} p_x \partial_{p_y} + \frac{eB}{c} p_y \partial_{p_x}\right)$

$$X_{\ell'} = -y\,\partial_x + x\,\partial_y - p_y\,\partial_{p_x} + p_x\,\partial_{p_y}$$

★ Translations

Action on (M, ω')

$$\overset{\mathbb{R}}{\Downarrow} \quad \overset{\mathbb{R}}{\Downarrow} \quad G = (\mathbb{R}^2, +)$$

$$(x, y, p_x, p_y) \longmapsto (x + a, y + b, p_x, p_y)$$

$\mathfrak{g} = \mathbb{R}^2$ acting on M in a Hamiltonian fashion

$$\partial_x \equiv X_{t_x} \leftrightarrow t_x := p_x - \frac{eB}{c} y \qquad + \text{const.}$$

$$\partial_y \equiv X_{t_y} \leftrightarrow t_y := p_y + \frac{eB}{c} x \qquad + \text{const.}$$

$$\Rightarrow \quad \{t_x, h'\} = \{t_y, h'\} = 0 \qquad \leftarrow \text{dynamical symmetry group}$$

★ Rotations $\quad S^1 \cong U(1)$ acts on M as

$$\begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix} \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}$$

$\mathfrak{u}(1) \cong \mathbb{R}$ acts via

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mapsto -y\,\partial_x + x\,\partial_y - p_y\,\partial_{p_x} + p_x\,\partial_{p_y} = X_{\ell'}$$

$$\{\ell', h'\} = 0$$

From the general formula $\quad [X_f, X_g] = X_{\{f,g\}}$

one finds that $\quad [X_{t_x}, X_{t_y}] = X_{\{t_x, t_y\}} = X_{\tilde{c}} = 0$

$\ast$ "magnetic" central extension

$\rightsquigarrow$ Heisenberg group

Symmetry group of the classical system

the (magnetic central extension of) the translation group yields a symmetry of the classical system but it will break down at the quantum level

Nevertheless, rotational symmetry will survive quantization

* Geometric quantization I

Perform holomorphic geometric quantization on the first ("unprimed") system, get BF:

$$\mathcal{H} = \left\{ \mathcal{F} = \mathcal{F}(\mathbf{z}) \,/\, \mathcal{F} \text{ is holomorphic and } \int |\mathcal{F}(\mathbf{z})|^2 e^{-\mathbf{z}\bar{\mathbf{z}}}\, d\xi\, d\eta\, dx\, dy < \infty \right\}$$

$\mathbf{z} = (\zeta, z)$, $\zeta = \xi + i\eta$, $z = x + iy$

$\mathbf{z}\bar{\mathbf{z}} := \zeta\bar{\zeta} + z\bar{z}$

$$\mathcal{H} \cong \underbrace{\mathcal{H}_1 \otimes \mathcal{H}_2}_{\text{BF}}$$

h_1 and h_2 are directly (holomorphically / geometrically quantizable (their flows preserve the holomorphic polarization)

$\hat{h}_j \propto$ quantum harmonic oscillator

One has an obvious action of $S^1 \times S^1$ on $\mathbb{C}^2$

* orthonormal basis (up to constants)

$$\zeta^{m_1} z^{m_2} \leftrightarrow (a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2} |0\rangle \qquad m_i \geq 0$$

$$a_1 = \frac{d}{d\zeta} \qquad a_1^\dagger = \zeta \qquad a_1 a_1^\dagger - a_1^\dagger a_1 = I \qquad a_1|0\rangle = 0$$

acting on $\mathcal{H}_1$, and similar expressions for $a_2, a_2^\dagger$ acting on $\mathcal{H}_2$. Obviously $[a_1^\#, a_2^\#] = 0$

$\#$ creation or annihilation operator

One also has

$$\hat{\ell} = \hat{h}_1 - \hat{h}_2 = a_1^+ a_1 - a_2^+ a_2$$

quantized angular momentum (z- component)

Remark : The above picture is compatible with Bohr-Sommerfeld

M is foliated by Liouville tori $\Lambda \simeq S^1 \times S^1$

cohomological condition: $\left[\frac{\theta}{2\pi} \right] \in H^1(\Lambda, \mathbb{Z})$

θ: symplectic potential

$$\theta = m_1 d\varphi_1 + m_2 d\varphi_2$$

$$m_j \in \mathbb{N}^*$$

$\rightsquigarrow$ BS- tori (+ circles & a point)

covariantly constant sections $\nabla s = 0$ "WKB- wave functions"

$$\psi_{m_1, m_2} = e^{i(m_1\varphi_1 + m_2\varphi_2)}$$

$$\sim z^{m_1} \bar{z}^{m_2}$$

* Geometric quantization II

use the vertical polarization

$$P_m := T_m Q \subset T_m M \qquad Q = \mathbb{R}^2$$

P-wave functions

$$\mathcal{H}_P := \left\{ s : \langle s, s \rangle = \int_Q (s,s)\, dx\, dy < +\infty \right\}$$

(s,s) hermitian structure on $L = M \times \mathbb{C}$

obtained via $\theta = p_x dx + p_y dy + \underset{\substack{\| \\ eB/2c\hbar}}{B} \left[x dy - y dx + \frac{i}{4\pi} d(x^2+y^2) \right]$

$$d(s,s) = 2\pi i (\theta - \bar{\theta})(s,s) = -B\, d(x^2+y^2)\,(s,s)$$

in the trivialization $s_0 \equiv 1$ $\quad s \sim \psi$, $s' \sim \phi$, we get

$$(\psi, \phi) = \bar{\psi}\phi\, e^{-B(x^2+y^2)}$$

Quantizable classical observables (their flow preserves P)

$$f = \underbrace{v^i(q)}_{\text{v. field on } Q} p_i + \underbrace{u(q)}_{\text{smooth function on } Q}$$

★ The Hamiltonian is not quantizable

so quantize $h(4) = \mathrm{span}\{1, q_x, q_y, p_x, p_y\}$
Heisenberg algebra

and

extend it to the inhomogeneous symplectic algebra

$$hsp(4, \mathbb{R}) = \mathrm{span}\left\{ 1, q_i, p_j, q_i q_j, q_i p_j, p_i p_j \,\middle|\, i,j = x,y \right\}$$

via the squaring von Neumann rule

$$Q(p_j^2) = Q^2(p_j) \;; \quad Q(q_j^2) = Q^2(q_j)$$

we get

$$\hat{p}_x \psi = -i\hbar \partial_x \psi + \frac{eB}{2c}(y + ix)\psi$$

$$\hat{p}_y \psi \quad -i\hbar \partial_y \psi - \frac{eB}{2c}(x - iy)\psi$$

and

$$\hat{x}\psi = x\psi$$

$$\hat{y}\psi = y\psi$$

Passing to complex coordinates, we get

$$\hat{h} = -\frac{2\hbar^2}{m}\left(\partial_z - \frac{B}{2}\right)\partial_{\bar{z}} + \frac{\hbar eB}{2mc}$$

Schwartz functions

which is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2, \mathbb{C}) \subset L^2(\mathbb{C}, \mu)$ $\overset{\text{III}}{\mathcal{H}_B}$

via standard arguments (use Nelson's analytic vector theorem)

introduce $\hat{a} = -\frac{i}{\sqrt{B}}\partial_{\bar{z}}$ $\quad \hat{a}^{+} = -\frac{i}{\sqrt{B}}(\partial_z - B\bar{z})$

$$\boxed{\hat{h} = \frac{eB\hbar}{mc}\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right)}$$

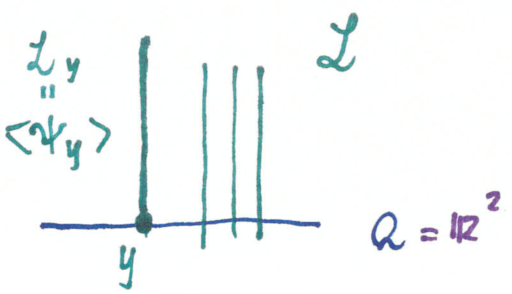
* Translational symmetry breaking

$$\hat{t}_x = -i\hbar\,(\partial_z + \partial_{\bar z}) + i\,\frac{eB}{4c}\,(z+\bar z)$$

$$\hat{t}_y = \hbar\,(\partial_z - \partial_{\bar z}) + \frac{eB}{4c}\,(z-\bar z)$$

$$[\hat{t}_{\substack{x\\y}}, \hat{h}] \neq 0$$

* Geometric interpretation



$\Psi_y \in$ ground state space of $\hat{h}_y$ (translated Hamiltonian)

$a_y \sim \bar\partial_y$ holomorphic structure

$$\bar\partial_y \Psi_y = 0$$

$$\Leftrightarrow \qquad b_y \Psi_y = 0$$

$\rightsquigarrow$ get a 1-dimensional space

and an index bundle (as in Fourier - Mukai - Nahm theory) carrying a natural connection (Nahm) with non trivial curvature:

Let $\xi \equiv \xi_{\alpha,\beta}(x) = (\bar U(\alpha)\bar V(\beta)\,\xi_0)(x) = \pi^{-\frac14}\exp\left[i\alpha x - \frac{(x-\beta)^2}{2}\right]$

$$\xi_0(x) = \pi^{-\frac14} e^{-\frac{x^2}{2}}$$

$$[\bar U(\alpha)\phi](x) = e^{i\alpha x}$$

$$[\bar V(\beta)\phi](x) = \phi(x-\beta)$$

$$U(\alpha)\,V(\beta) = e^{i\alpha\beta}\,V(\beta)\,U(\alpha)$$

CCR in Weyl form

$\xi_{\alpha\beta}$: standard coherent states

Kahm connection form

$$A = \langle \xi , d\xi \rangle$$

curvature

$$\Omega = dA = d \langle \xi , d\xi \rangle =$$

$$[\langle \partial_\alpha \xi, \partial_\beta \xi \rangle - \langle \partial_\beta \xi, \partial_\alpha \xi \rangle] \, d\alpha \wedge d\beta$$

$$= \quad 2i \ \mathrm{Im} \langle \partial_\alpha \xi, \partial_\beta \xi \rangle \, d\alpha \wedge d\beta$$

A routine computation yields

$$\Omega = -i \ d\alpha \wedge d\beta$$

$\rightsquigarrow$ "translational anomaly"

lack of commutativity with the Hamiltonian detected via the curvature of the coherent state line bundle

In contrast to translations, rotational symmetry survives quantization

$$\hat{\ell} = \hbar \left(z \partial_z - \bar{z} \partial_{\bar{z}} \right)$$

& $\quad [\hat{h}, \hat{\ell}] = 0$