Orthogonal polynomials attached to nonlinear coherent states (Pöschl-Teller oscillator)

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COHERENT STATES AND THEIR APPLICATIONS
A CONTEMPORY PANORAMA

November 14-18, 2016 CIRM, Marseille, France

Part 1

1. Nonlinear Coherent States
2. Example : The symmetric Pöschl-Teller oscillator

Nonlinear Coherent States (NLCS)

We summarize the construction of NLCS as in the book of S. T. Ali, J. P. Antoine and J. P. Gazeau, Coherent States, Wavelets, and their Generalizations, 1999.

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$$|z\rangle = \left(e^{z\bar{z}}\right)^{-1/2} \sum_{n=0}^{+\infty} \frac{\bar{z}^n}{\sqrt{n!}} |\varphi_n\rangle, \quad z \in \mathbb{C}, \tag{1}$$

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  $|\varphi_n\rangle$, $n = 0, 1, 2, ..., \infty$, is an orthogonal basis in an arbitrary Hilbert space $\mathcal{H}$.

- Let $\{x_n\}_{n=0}^{\infty}$, $x_0 = 0$, be a sequence of positive numbers with $\lim_{n\to+\infty} x_n = R^2$, $R > 0$, and we use the notation $x_n! = x_1 x_2 \cdots x_n$ and $x_0! = 1$.

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- For each $z \in \mathcal{D} \subseteq \mathbb{C}$, a generalization of (1) :

$$|z\rangle = \left(\mathcal{N}(z\bar{z})\right)^{-1/2} \sum_{n=0}^{+\infty} \frac{\bar{z}^n}{\sqrt{x_n!}} |\varphi_n\rangle, \tag{2}$$

$\mathcal{N}(z\bar{z}) = \sum\limits_{n=0}^{+\infty} \frac{|z|^{2n}}{x_n!}$ is a normalization factor. The vectors $|z\rangle$ are well defined for all z for which $\mathcal{N}(z\bar{z})$ converges, i.e. $\mathcal{D} = \{z \in \mathbb{C}, |z| < R\}$.

- Assume $\exists$ a measure $d\nu$ on $\mathcal{D}$ for which we have the resolution of identity :

$$\int_{\mathcal{D}} |z\rangle\langle z| d\nu(z,\bar{z}) = 1_{\mathcal{H}} \tag{3}$$

- Setting $d\nu(z,\bar{z}) = \mathcal{N}(z\bar{z})d\eta(z,\bar{z})$, then for (3) to be satisfied $d\eta$ should be of the form $d\eta(z,\bar{z}) = \frac{d\theta}{2\pi}d\lambda(\rho),\ z = \rho e^{i\theta}$ where $d\lambda$ solves the moment problem

$$\int_0^R \rho^{2n} d\lambda(\rho) = x_n!, \quad n = 0,1,2,\ldots. \tag{4}$$

Example 1 :

The sequence of positif numbers : $x_n = n, \quad n = 0,1,2,\ldots$.

• Here $R = \infty$ and the problem in (4) is the Stieljes moment problem

$$\int_0^{+\infty} \rho^{2n} d\lambda(\rho) = n!, \quad n = 0,1,2,\ldots. \tag{5}$$

• Thus $d\lambda(\rho) = 2e^{-\rho^2}\rho d\rho, \quad 0 \leq \rho < \infty$. We recover the canonical CSs

$$|z\rangle = (e^{z\bar{z}})^{-1/2} \sum_{n=0}^{+\infty} \frac{\bar{z}^n}{\sqrt{n!}} |\varphi_n\rangle, \tag{6}$$

Example 2

The sequence of positif numbers

$$x_n = n\,(2\sigma + n - 1)\,, \quad n = 0, 1, 2, 3, \ldots, \tag{7}$$

with $2\sigma = 1, 2, 3, \ldots$

- $R = \infty$ and the moment problem reads

$$\int_0^{+\infty} \rho^{2n} d\lambda(\rho) = n!(2\sigma)_n \tag{8}$$

$(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 = 1$, is the shifted factorial.

- $d\lambda(\rho) = \frac{2}{\pi} K_{2\sigma-1}(2\rho)\rho^{2-2\sigma} d\rho,\ 0 \leq \rho < \infty$, $K_\tau(.)$: the Macdonald function of order τ.

- The CSs are of Barut-Girardello type (*Commun. Mat. Phys.* 1971) :

$$|z,\sigma\rangle = \frac{|z|^{2\sigma-1}}{\sqrt{I_{2\sigma-1}(2|z|)}} \sum_{n=0}^{+\infty} \frac{\bar{z}^n}{\sqrt{n!(2\sigma)_n}} |\varphi_n\rangle, \quad z \in \mathbb{C}, \tag{9}$$

$I_\tau(.)$ is the modified Bessel function of the first kind and of order τ.

NLCS with a specific sequence of positive numbers x_n^γ

We deal with NLCS on $\mathbb{C}$, which interpolates between canonical CSs and a class of Barut Girardello CSs without specifying the Hamiltonian.

- For $\gamma \in [0, \infty)$, $x_0^\gamma = 0$, $x_1^\gamma = \Gamma(2\gamma + 1)$,

$$x_n^\gamma := \frac{n(n+\gamma)(n+2\gamma-1)}{n+\gamma-1}, \quad n = 2, 3, 4, \ldots\ , \tag{10}$$

  and

$$x_n^\gamma ! := n!(n+\gamma)(2\gamma+1)(2\gamma+2)\cdots(2\gamma+n-1) \tag{11}$$

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- Define NLCS by

$$|z;\gamma\rangle := (\mathcal{N}_\gamma(z\bar{z}))^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{\bar{z}^n}{\sqrt{x_n^\gamma!}} |\phi_n\rangle \,, \quad n = 0, 1, 2, \ldots \,, \tag{12}$$

  $|\phi_n\rangle$ an orthonormal basis of an abstract Hilbert space $\mathcal{H}$.

Proposition 1

Let $\gamma \in [0, \infty)$.

- *The normalization factor*

$$\mathcal{N}_\gamma (z\bar{z}) = 2 \ {}_1F_2 \left(\begin{matrix} \gamma \\ \gamma + 1, 2\gamma \end{matrix} \ \middle| \ z\bar{z} \right) \tag{13}$$

${}_1F_2(a; b, c; x)$ *is the hypergeometric function which converge for all* x.

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${}_1F_2(a; b, c; x)$ *is the hypergeometric function which converge for all* x.

- *The resolution of the identity*

$$\int_{\mathbb{C}} |z;\gamma\rangle \langle z;\gamma| \, d\mu_\gamma(z) = \mathbf{1}_{\mathcal{H}}, \tag{14}$$

where

$$d\mu_\gamma(z) = \frac{4}{\Gamma(2\gamma+1)} \; {}_1F_2 \left(\begin{array}{c} \gamma \\ \gamma+1, 2\gamma \end{array} \mid z\bar{z} \right) G^{30}_{13} \left(z\bar{z} \,\middle|\, \begin{array}{c} \gamma - 1 \\ 0, \; \gamma, 2\gamma - 1 \end{array} \right) d\mu(z), \tag{15}$$

$G^{30}_{13}(.)$ *is the Meijer's G-function and* $d\mu$ *being the Lebesgue measure on* $\mathbb{C}$.

On the proof

Assume that $d\mu_\gamma(z) = \mathcal{N}_\gamma(z\bar{z})h(z\bar{z})d\mu(z)$, h an auxiliary density function, consider $z = \rho e^{i\theta}$, $\rho > 0$ and $\theta \in [0, 2\pi)$:

$$d\mu_\gamma(z) = \mathcal{N}_\gamma(\rho^2)h(\rho^2)\rho d\rho\frac{d\theta}{2\pi}. \tag{16}$$

Using the expression

$$\mathcal{O}_\gamma = \int_{\mathbb{C}} |z;\gamma\rangle\langle\gamma;z|\, d\mu_\gamma(z) \tag{17}$$

thus

$$\begin{aligned}
\mathcal{O}_\gamma &= \sum_{n,m=0}^{+\infty}\left(\int_0^{+\infty}\frac{\rho^{n+m}h(\rho^2)\rho d\rho}{\sqrt{\sigma_\gamma(n)\sigma_\gamma(m)}}\left(\int_0^{2\pi}e^{i(n-m)\theta}\frac{d\theta}{2\pi}\right)\right)|\phi_n\rangle\langle\phi_m| \\
&= \sum_{n=0}^{+\infty}\frac{1}{n!(n+\gamma)(2\gamma+1)_{n-1}}\left(\int_0^{+\infty}\rho^{2n}h(\rho^2)\rho d\rho\right)|\phi_n\rangle\langle\phi_n| \\
&= \sum_{n=0}^{+\infty}\frac{1}{2n!(n+\gamma)(2\gamma+1)_{n-1}}\left(\int_0^{+\infty}r^n h(r)dr\right)|\phi_n\rangle\langle\phi_n|.
\end{aligned}$$

We need h such that

$$\int_0^{+\infty} r^n h(r)dr = 2n!(n+\gamma)(2\gamma+1)_{n-1}. \tag{18}$$

Recall (A. M. Mathai, R. K. Saxena 1973, p.67) :

$$\int_0^{+\infty} G_{pq}^{ml}\left(\omega t \,\middle|\, \begin{array}{c} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{array}\right) t^{s-1} dt = \frac{1}{\omega^s} \frac{\prod_{j=1}^{m} \Gamma(b_j+s) \prod_{j=1}^{l} \Gamma(1-a_j-s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-s) \prod_{j=l+1}^{p} \Gamma(a_j+s)} \tag{19}$$

G_{pq}^{ml} the Meijer's function such that $0 \leq l \leq p < q$; $0 \leq m \leq q$; $\omega \neq 0$; $c^* = m + l - \frac{p}{2} - \frac{q}{2} > 0$, $|\arg \omega| < c^* \pi$; $-\min \Re(b_j) < \Re(s) < 1 - \max \Re(a_k)$ for $j = 1, \cdots, m$ and $k = 1, \cdots, l$. For $\omega = 1$, $p = 1$, $q = 3$, $m = 3$, $l = 0$, $a_1 = \gamma$, $b_1 = 1$, $b_2 = \gamma + 1$, $b_3 = 2\gamma$ and $s = n$, (19) becomes

$$\int_0^{+\infty} G_{13}^{30}\left(r \,\middle|\, \begin{array}{c} \gamma \\ 1, \gamma+1, 2\gamma \end{array}\right) \frac{2r^{n-1}}{\Gamma(2\gamma+1)} dr = 2n!(n+\gamma)(2\gamma+1)_{n-1}. \tag{20}$$

Then

$$h(r) = \frac{2r^{-1}}{\Gamma(2\gamma+1)} G_{13}^{30}\left(r \,\middle|\, \begin{array}{c} \gamma \\ 1, \gamma+1, 2\gamma \end{array}\right). \tag{21}$$

Using (H.Srivastava and L.Manocha, A Treatise on Generating Functions,1984, p.46) :

$$y^{\sigma} G_{pq}^{ml}\left(y \,\middle|\, \begin{matrix}(a_p)\\(b_q)\end{matrix}\right) = G_{pq}^{ml}\left(y \,\middle|\, \begin{matrix}(a_p+\sigma)\\(b_q+\sigma)\end{matrix}\right), \tag{22}$$

(21) becomes

$$h(r) = \frac{2}{\Gamma(2\gamma+1)} G_{13}^{30}\left(r \,\middle|\, \begin{matrix}\gamma-1\\0,\gamma,2\gamma-1\end{matrix}\right). \tag{23}$$

thus

$$\mathcal{O}_{\gamma} = \sum_{n=0}^{+\infty} |\phi_n\rangle\langle\phi_n| = \mathbf{1}_{\mathcal{H}}. \tag{24}$$

since $\{|\phi_n\rangle\}$ is an orthonormal basis of $\mathcal{H}$. Then, we obtain (14). $\square$

Remark 1

- **Case** $\gamma = 0$: the sequence x_n^γ reduces to

$$x_n^0 = n^2 \tag{25}$$

and

$$x_n^0! = (n!)^2 \tag{26}$$

the NLCS are of Barut-Girardello type with $2\sigma = 1$. Results on overcompletness or undercompletness of discrete sets of CSs based on the use of theorems that relate the growth of analytic functions to the density of their zeros were obtained by (A.Voudras, K.A Penson, G.H.E. Duchamp and A.I. Solomon, *J.Phys. A : Math.Theor.* 2012).

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- **Case** $\gamma = 1$: The sequence x_n^γ reduces to

$$x_n^1 = (n+1)^2, \; x_n^1! = ((n+1)!)^2 \tag{28}$$

and will be associated with to the Pöschl-Teller oscillator

The symmetric Pöschl-Teller (SPT) oscillator

- The Hamiltonian is given by $H_\nu = -\frac{1}{2m_*}\frac{d^2}{d\theta^2} + V_\nu(\theta)$,

$$V_\nu(\theta) := \frac{\hbar^2\alpha^2}{2m_*}\frac{\nu(\nu-1)}{\cos^2\alpha\theta}, \tag{29}$$

  $-\pi/2\alpha \le \theta \le \pi/2\alpha$, $\hbar$ the Planck's constant, $\alpha > 0$ the range of the potential, m_* is the reduced mass of the particle, $\nu > 1$ is the potential strength and θ gives the relative distance from the equilibrium position.

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- The energy of a bound state

$$E_n^\nu = \frac{\hbar^2\alpha^2}{2m_*}(\nu+n)^2, \quad n = 0, 1, 2, ... \tag{30}$$

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- Wavefunctions of bound states

$$\langle\theta|\phi_n^\nu\rangle = \sqrt{\frac{\alpha n!(n+\nu)\Gamma(\nu)\Gamma(2\nu)}{\pi^{1/2}\Gamma(n+2\nu)\Gamma(\nu+1/2)}}\cos^\nu(\alpha\theta)\ C_n^\nu(\sin\alpha\theta) \tag{31}$$

give an orthonormal basis of $\mathcal{H}_\alpha = L^2\left([-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha}], d\theta\right)$, where $C_n^\nu(.)$ is the Gegenbauer polynomial.

Remark 2

- When $\nu \to 1$, the SPT oscillator becomes the Infinite Square Well Potential (ISWP) with barriers at $\theta = \pm\pi/2\alpha$, an wavefunctions of bound states reduce to

$$\langle \theta | \phi_n^1 \rangle = \sqrt{\frac{2\alpha}{\pi}} \cos(\alpha\theta)\ U_n(\sin \alpha\theta), \tag{32}$$

where $U_n(.)$ is the Chebychev polynomial.

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  where $U_n(.)$ is the Chebychev polynomial.
- Subtracting the zero point energy $\nu(\nu-1)\hbar\alpha^2/2$ and then taking limits $\nu \to \infty$, $\alpha \to 0$, such that $\alpha^2\nu = m\omega/\hbar$, potential, energy levels, wavefunctions becomes those for the harmonic oscillator (M.N.Nieto, *Phys. Rev A.* 1978).

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Definition 1 : NLCS of the SPT oscillator

Let $\gamma \in (0,\infty)$ *and* $\nu > 1$. *Define*

$$|z;\gamma,\nu\rangle := \left(\mathcal{N}_\gamma(z\bar{z})\right)^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{\bar{z}^n}{\sqrt{x_n^\gamma!}}\,|\phi_n^\nu\rangle \tag{33}$$

$\mathcal{N}_\gamma(.)$ *is a normalization factor,* $x_n^\gamma! = n!(n+\gamma)(2\gamma+1)(2\gamma+2)...(2\gamma+n-1)$ *and* $|\phi_n^\nu\rangle$ *are wavefunctions of the SPT oscillator.*

Proposition 2 : A closed form for NLCS

Choosing $\nu = \gamma$

$$\langle \theta | z; \gamma \rangle = 2^{\gamma-1} \sqrt{\frac{\alpha \Gamma(\gamma+1) \Gamma\left(\gamma+\frac{1}{2}\right)}{\pi^{1/2}}} \left({}_1F_2 \left(\begin{array}{c} \gamma \\ \gamma+1, 2\gamma \end{array} \mid z\bar{z} \right) \right)^{-1/2} \tag{34}$$

$$\times \bar{z}^{\frac{1}{2}-\gamma} \exp\left(\bar{z} \sin \alpha\theta\right) \; J_{\gamma-\frac{1}{2}}(\bar{z} \cos \alpha\theta) \sqrt{\cos \alpha\theta},$$

for every $\theta \in \left[-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha}\right]$.

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for every $\theta \in \left[-\frac{\pi}{2\alpha},\frac{\pi}{2\alpha}\right]$.

Corollary 1. The particular case $\gamma = \nu = 1$

This corresponds to the infinite square well (ISW) potential

$$\langle\theta|z\rangle = \sqrt{\frac{\alpha}{\pi}}\left(I_0(2|z|-1)\right)^{-\frac{1}{2}}\exp\left(\bar{z}\sin\alpha\theta\right)\sin(\bar{z}\cos\alpha\theta) \tag{35}$$

$I_0(.)$ *the modified Bessel function of the first kind.*

On the proof

Writing the expression of the wavefunction

$$\langle \theta | z; \gamma \rangle := \langle \theta | z; \gamma, \gamma \rangle = \left(\mathcal{N}_\gamma \left(z\bar{z} \right) \right)^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{\bar{z}^n}{\sqrt{x_n^\gamma !}} \langle \theta | \phi_n^\gamma \rangle \,. \tag{36}$$

To get a closed form of $\mathcal{S}(\theta) = \sum_{n=0}^{+\infty} \frac{\bar{z}^n}{\sqrt{x_n^\gamma !}} \langle \theta | \phi_n^\gamma \rangle$ we replace $\langle \theta | \phi_n^\gamma \rangle$ by expression (31)

$$\mathcal{S}(\theta) = \sqrt{\frac{\alpha \Gamma(\gamma+1)}{\pi^{1/2} \Gamma\left(\gamma + \frac{1}{2}\right)}} \cos^\gamma(\alpha\theta) \sum_{n=0}^{+\infty} \frac{\bar{z}^n}{(2\gamma)_n} C_n^\gamma(\cos \alpha\theta). \tag{37}$$

We use (A. P. Prudnikov, Yu. A. Brychkov, volume 3 1990, p.711) :

$$\sum_{k=0}^{+\infty} \frac{t^k}{(2\tau)_k} C_k^\tau(y) = \Gamma\left(\tau + \frac{1}{2}\right) e^{yt} \left(\frac{t}{2} \sqrt{1-y^2} \right)^{\frac{1}{2}-\tau} J_{\tau - \frac{1}{2}}\left(t\sqrt{1-y^2} \right) \tag{38}$$

$J_\tau(.)$ denotes the Bessel function of order τ. For $k = n,\ t = \bar{z}, \tau = \gamma$ and $y = \sin \alpha\theta$

$$\mathcal{S}(\theta) = 2^{\gamma - 1/2} \sqrt{\frac{\alpha \Gamma(\gamma+1) \Gamma\left(\gamma + \frac{1}{2}\right)}{\pi^{1/2}}} \bar{z}^{1/2-\gamma} \exp\left(\bar{z} \sin \alpha\theta\right) \ J_{\gamma - \frac{1}{2}}(\bar{z} \cos \alpha\theta) \sqrt{\cos \alpha\theta} \tag{39}$$

which gives (34).

For $\nu = 1$, the SPT potential becomes the ISW with eigenfunctions $\{\phi_n^1(\theta)\}$.
The result (35) is deduced by setting $\gamma = 1$ in (34) and using $\Gamma(3/2) = \sqrt{\pi}/2$ together with

$$ {}_1F_2(1,2,2;\zeta^2) = \frac{1}{\zeta^2}(I_0(2\zeta) - 1) \tag{40} $$

for $\zeta = |z|$

See Ref. A.P.Prudnikov, Yu.A.Brychkov, O.I.Marichev, Integrals and Series : More special Function, Gordon and Breach Science Publishers, 1990, p.600

Next by using the formula

$$ J_{1/2}(\xi) = \sqrt{\frac{2}{\pi\xi}} \sin \xi, \tag{41} $$

where $\xi = \bar{z} \cos \alpha\theta$

See Ref.(L. C. Andrews, Special function for Engineers and Applied Mathematicians, Macmillan Publishing compagny, London 1985, p.203).

A coherent states transform

- The resolution of identity gives a unitary map $B_\gamma : \mathcal{H}_\alpha \longrightarrow A_\gamma(\mathbb{C})$ defined by

$$B_\gamma[\phi](z) = (\mathcal{N}(z\bar{z}))^{1/2} \langle \phi | z, \gamma \rangle_{\mathcal{H}_\alpha} \tag{42}$$

where

$$A_\gamma(\mathbb{C}) = Hol(D) \cap L^2(D, d\nu_\gamma)$$

with

$$d\nu_\gamma(z, \bar{z}) = \frac{2}{\Gamma(2\gamma+1)} G_{13}^{30}\left(z\bar{z} \,\middle|\, \begin{matrix} \gamma - 1 \\ 0,\ \gamma, 2\gamma - 1 \end{matrix} \right) d\mu(z), \tag{43}$$

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$$d\nu_\gamma(z,\bar{z}) = \frac{2}{\Gamma(2\gamma+1)} G_{13}^{30}\left(z\bar{z} \,\middle|\, \begin{matrix} \gamma - 1 \\ 0, \ \gamma, 2\gamma - 1 \end{matrix} \right) d\mu(z), \tag{43}$$

- A sequential characterization of the space $A_\gamma(\mathbb{C})$:

$f(z) = \sum_{n\geq 0} a_n z^n \in \mathcal{A}_\gamma(\mathbb{C})$ if and only if the (a_n) satisfy

$$\frac{1}{\Gamma(2\gamma+1)} \sum_{n=0}^{+\infty} n!(n+\gamma)\Gamma(n+2\gamma)|a_n|^2 < +\infty. \tag{44}$$

Theorem 1.

Let $\gamma > 1$. The coherent states transform $\mathcal{B}_\gamma : \mathcal{H}_\alpha \to \mathcal{A}_\gamma(\mathbb{C})$ is given by

$$\mathcal{B}_\gamma[\varphi](z) = \sqrt{\alpha \frac{\Gamma(\gamma+1)\Gamma(\gamma+1/2)}{\pi^{1/2}}} \left(\frac{z}{2}\right)^{\frac{1}{2}-\gamma}$$

$$\times \int_{\frac{-\pi}{2\alpha}}^{\frac{\pi}{2\alpha}} \exp\left(z \sin \alpha\theta\right) J_{\gamma-1/2}(z \sin \alpha\theta) \sqrt{\cos \alpha\theta}\,\varphi(\theta) d\theta.$$

For $\gamma \to 1$, which corresponds to infinite square well (ISW) potential,

$$\mathcal{B}_1[\varphi](z) = \frac{\left(\frac{\alpha}{\pi}\right)^{1/2}}{z} \int_{\frac{-\pi}{2\alpha}}^{\frac{\pi}{2\alpha}} \exp\left(z \sin \alpha\theta\right) \sin(z \cos \alpha\theta) \varphi\left(\theta\right) d\theta, \quad z \in \mathbb{C}$$

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Remark 2

- A generalization of x_n^γ was considered by T. Sheecharan, Prasanta K. Panigrahi and J. Banerji (Phys. *Rev. A.* 2004), who provided an algebraic construction of the CSs for a class of potentials, belonging to hypergeometric functions classes.
- In a similar context H.B. Zhang, G.Y. Jiang, and S.X. Guo (*J. Math. Phys*, 2014) considered the Pöschl-Teller oscillator where they investigated nonclassical properties via statistics of corresponding photon-counting probability distributions.

Two sets of orthogonal polynomials attached to NLCS

S. T. Ali and M. E. H. Ismail, Some orthogonal polynomials arising from coherent states, *J. Phys. A : Math. Theor.* 2012

The first set of polynomials : $\{P_n\}$ attached to the measure $d\nu_\gamma$

- These polynomials are obtained by symmetrizing the measure

$$d\nu_\gamma(r) = \frac{2}{\Gamma(2\gamma+1)} G^{30}_{13}\left(r^2 \middle| \begin{array}{c} \gamma-1 \\ 0,\ \gamma, 2\gamma-1 \end{array} \right) rdr. \tag{45}$$

(Recall Eq.(4), where $d\lambda$ solves the moment problem

$$\int_0^R \rho^n d\lambda(\rho) = x_n!$$

here $d\lambda :\equiv d\nu_\gamma$).

- The obtained measure on $(-\infty, +\infty)$ is

$$d\eta_\gamma(t) := \frac{1}{2} d\nu_\gamma(|t|) \tag{46}$$

with the moments

$$\mu_{2n} = 2\int_0^\infty r^{2n} d\eta_\gamma(t) = x_n^\gamma!, \tag{47}$$

$$\mu_{2n+1} = 2\int_0^\infty r^{2n+1} d\eta_\gamma(t) = 0, \quad n = 0, 1, 2, \dots . \tag{48}$$

- A set of (monic) polynomials $\{P_n\}$ orthogonal with respect to $d\eta_\gamma$ are obtained by means of the Hankel determinant

$$P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \vdots & \vdots & . & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}, \quad \Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & . & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \tag{49}$$

$n = 0, 1, 2, \ldots.$

Our task is to identify and discuss properties of polynomials P_n in three cases of the parameter γ :

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- Case $\gamma = 0$

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- Case $\gamma = 1$
- Case $\gamma = \infty$

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- The first polynomials

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= x^2 - 1 \\ P_3(x) &= x^3 - 4x \\ P_4(x) &= x^4 - \frac{32}{3}x^2 + \frac{20}{3} \end{aligned}$$

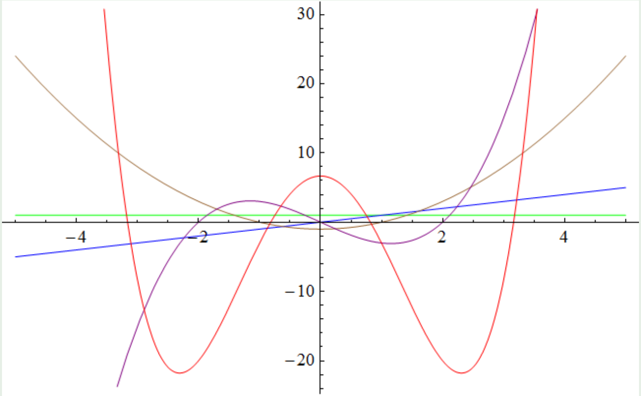

Figure: The polynomials P_0, P_1, P_2, P_3 and P_4

Some properties of $\{P_n\}$

- $\{P_n\}$ are symmetric with respect to the origin and satisfy

$$\int_{-\infty}^{+\infty} P_n(x)P_m(x)d\eta_0(x) = \xi_n\delta_{mn}, \tag{51}$$

$\xi_n > 0$ is a normalization constant and δ_{mn} is the Kronecher's symbol.

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- The polynomials $\widetilde{P}_n(x) := \frac{1}{\sqrt{\xi_n}} P_n(x)$ satisfy

$$x\widetilde{P}_n(x) = A_{n+1}\widetilde{P}_{n+1}(x) + A_n\widetilde{P}_{n-1}(x) \tag{52}$$

with the only information we known on the A_n :

$$\lim_{n\to\infty} \frac{A_n}{n} = \frac{\pi}{16}, \tag{53}$$

see W. V. Assche (*J. Comput. Appl. Math.* 1993).

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see W. V. Assche (*J. Comput. Appl. Math.* 1993).

- Precisely, if we set $V_n(x) := \widetilde{P}_{2n}(x)$, then (51) reads

$$2\int_0^{\infty} V_n(x)V_m(x)K_0(2\sqrt{x})dx = \delta_{mn}. \tag{54}$$

Calculations using $2\int_0^{+\infty} K_0(2\sqrt{x})x^n dx = (n!)^2$, provide us with constants ξ_n : $\xi_2 = 3,\ \xi_4 = 656/3,\ \xi_6 = 3681936/41$ leading to precise expressions of polynomials

$$V_1(x) = \frac{x-1}{\sqrt{3}}, \qquad V_2(x) = \sqrt{\frac{3}{41}}\left(\frac{1}{4}x^2 - \frac{8}{3}x + \frac{5}{3}\right) \tag{55}$$

$$V_3(x) = \sqrt{\frac{41}{2841}}\left(\frac{1}{36}x^3 - \frac{177}{164}x^2 + \frac{267}{41}x - \frac{131}{41}\right) \tag{56}$$

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Ditkin-Prudnikov problem (1967)

Let $V_0(x,k) = 1$, $V_1(x,k), ..., V_n(x,k)$, $k > 0$ an integer, be the OP system on $0 \leq x \leq \infty$, with respect to $\xi(x,k) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{-s}\Gamma^k(s)$, a, x, $\Re s > 0$. That is,

$$\int_0^\infty V_n(x,k)V_m(x,k)\xi(x,k)dx = \delta_{nm}.$$

Building the generating function, an analogue of Rodrigues formula, the recurrence relation for $V_n(x,k)$, $k \geq 2$, **is still an open problem**. Set $\gamma = \frac{2-k}{k-1}$. When $k = 2$ i.e., $\gamma = 0$ then $\xi(x,2) = 2K_0(2\sqrt{x})$ and $V_n(x,2) = V_n(x)$ connected to the $P_n(x)$.

Case $\gamma = 1$

- The generalized factorial $x_n^1! = ((n+1)!)^2, \; n = 0, 1, 2, \ldots$.

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- The measure $d\nu_1(r) = G_{02}^{20}\left(r^2 | 1, 1\right) rdr$.
- The symmetrized measure $d\eta_1(t) = \frac{1}{2} d\nu_1(|t|)$, $t \in (-\infty, +\infty)$.

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$$\mu_{2n} = \int_0^{+\infty} t^{n+\frac{1}{2}} K_0\left(2\sqrt{t}\right) dt = ((n+1)!)^2, \ \mu_{2n+1} = \int_0^{+\infty} t^{n+1} K_0\left(2\sqrt{t}\right) dt = 0. \tag{57}$$

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- The first polynomials

$$\begin{aligned} Q_0(x) &= 1, \\ Q_1(x) &= x \\ Q_2(x) &= x^2 - 4, \\ Q_3(x) &= x^3 - 9x \\ Q_4(x) &= x^4 - \frac{108}{5}x^2 + \frac{252}{5} \end{aligned}$$

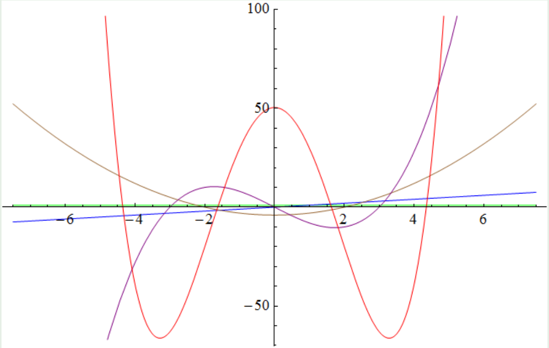

Figure: The polynomials Q_0, Q_1, Q_2, Q_3 and Q_4

$Q_n(x)$ can be connected with $P_n(x)$ by $xQ_{2n}(x) = P_{2n+1}(x)$, $n = 0, 1, 2, \ldots$.

Recalling that $\gamma = \frac{2-k}{k-1}$ then the case $\gamma = 1$ (which was involved in the infinite square well (ISW) potential) corresponds to the value $k = 3/2$. Then it may be useful to extend the Ditkin-Prudnikov problem to values $k \in]1, 2[$ since we don't known so much about the polynomials Q_n ?

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- The first polynomials

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$$1 - \left(\frac{x}{\gamma}\right)^2$$

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which are Laguerre polynomials on the variable $(x/\gamma)^2$.

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Ref. S. T. Ali and M. E. H. Ismail, *J. Phys A : Math. Theor* 2012.

The second set of polynomials : $\{\phi_n\}$ attached to shift operators

- Define the formal shift operator

$$a\phi_n = \sqrt{x_n}\phi_{n-1}, \quad a\phi_0 = 0, \quad a^*\phi_n = \sqrt{x_{n+1}}\phi_{n+1}, \quad n = 0, 1, 2, \ldots .$$

If $\sum_{n=0}^{\infty} \frac{1}{\sqrt{x_n}} = \infty$, then $Q = \frac{1}{\sqrt{2}}(a + a^*)$ is essentially self-adjoint then it has a unique self-adjoint extension (A. Odzijewicz, M. Horowski and A. Tereszkiewicz, *J. Phys. A : Math. Gen* 2001) denote again by Q.

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- Q acts on ϕ_n as

$$Q\phi_n = \sqrt{\frac{x_n}{2}}\phi_{n-1} + \sqrt{\frac{x_{n+1}}{2}}\phi_{n+1}. \tag{59}$$

and $\exists$ even measure dw such that Q acts on $L^2(\mathbb{R}, dw)$ as a multiplication on $\phi_n \in L^2(\mathbb{R}, dw)$:

$$x\phi_n(x) = \sqrt{\frac{x_n}{2}}\phi_{n-1}(x) + \sqrt{\frac{x_{n+1}}{2}}\phi_{n+1}(x), \quad n = 1, 2, \dots, \tag{60}$$

with $\phi_{-1} = 0$ and $\phi_0 = 1$ (dw comes from the spectral projectors, E_x, $x \in \mathbb{R}$, of Q, in the sense $dw(x) = \langle \phi_0 | E_x \phi_0 \rangle$.

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- The associated polynomials $\phi_n :\equiv \phi_n^{(1/2)}$ satisfy

$$x\phi_n^{(1/2)}(x) = \frac{n+1}{\sqrt{2}}\phi_{n+1}^{(1/2)}(x) + \frac{n}{\sqrt{2}}\phi_{n-1}^{(1/2)}(x). \tag{61}$$

with condition $\phi_{-1}^{(1/2)} = 0$ and $\phi_0^{(1/2)} = 1$.

Here our task is to write this formalism when $x_n = x_n^{\gamma}$, to identify and discuss properties of polynomials ϕ_n in three cases of the parameter γ :

- Case $\gamma = 0$
- Case $\gamma = 1$
- Case $\gamma = \infty$

Case $\gamma = 0$

- The generalized factoriel $x_n^0! = ((n)!)^2$ associated with Barut-Girardello CS ($x_n^{\sigma}! = n!\Gamma(n+2\sigma)$ where $\sigma = 1/2$). Here $x_n = n(n+2\sigma-1)$
- The associated polynomials $\phi_n :\equiv \phi_n^{(1/2)}$ satisfy

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with condition $\phi_{-1}^{(1/2)} = 0$ and $\phi_0^{(1/2)} = 1$.

- The first polynomials $\phi_0^{(1/2)}(x) = 1$, $\phi_1^{(1/2)}(x) = \sqrt{2}x$ and

$$\phi_2^{(1/2)}(x) = \frac{1}{2}(2x^2 - 1)$$

with the graphs

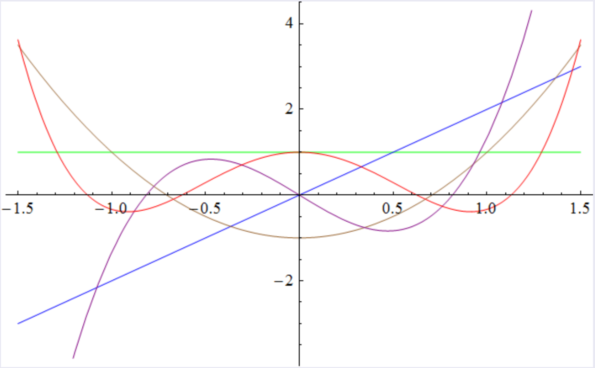

Figure: The polynomials $\phi_0, \phi_1, \phi_2, \phi_3$ and ϕ_4

Proposition 3

The $\phi_n^{(1/2)}$ *are a special case of Meixner-Pollaczek polynomials*

$$\phi_n^{(1/2)}(x) = P_n^{(1/2)}\left(\frac{x}{\sqrt{2}}, \frac{\pi}{2}\right) = i^n {}_2F_1\left(\begin{array}{c} -n, \frac{1}{2} + i\frac{x}{\sqrt{2}} \\ 1 \end{array} \mid 2\right) \tag{62}$$

given by a terminating Gauss hypergeometric ${}_2F_1$*-sum.*, with generating function

$$\sum_{n=0}^{+\infty} P_n^{1/2}\left(\frac{x}{\sqrt{2}}, \frac{\pi}{2}\right) = \frac{1}{\sqrt{1+t^2}} \exp\left(\sqrt{2}x \arctan t\right) \tag{63}$$

This is a partial answer ($2\sigma = 1$) to a question by S. T. Ali and M.E.H. Ismail about the ϕ_n? associated with $x_n = n(n + 2\sigma - 1)$ for $2\sigma \in \mathbb{N}^*$. By the way here is the answer :

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Proposition 4

For $2\sigma = 1, 2, 3...$, *the* ϕ_n *satisfy*

$$\sum_{n \geq 0} t^n \phi_n^{(\sigma)}(x) = \left(1 + t^2\right)^{-\sigma} \exp\left(\sqrt{2}x \arctan t\right)$$

so they are Meixner-Pollaczek polynomials

Proposition 3. We consider the normalized polynomials $q_n(x) := 2^{-\frac{1}{2}n} n! \phi_n^{(1/2)}(x)$ which satisfy

$$q_{n+1}(x) - xq_n(x) + \frac{1}{2}n^2 q_{n-1}(x) = 0. \tag{64}$$

Multiplying (64) by $t^n/n!$ and summing over n

$$\sum_{n=0}^{+\infty} q_{n+1}(x)\frac{t^n}{n!} - x\sum_{n=0}^{+\infty} q_n(x)\frac{t^n}{n!} + \frac{1}{2}\sum_{n=0}^{+\infty} nq_{n-1}(x)\frac{t^n}{(n-1)!} = 0. \tag{65}$$

Setting $G_x(t) := \sum_{n=0}^{+\infty} q_n(x)\frac{t^n}{n!}$, then (65) leads to

$$(t^2+2)\frac{d}{dt}G_x(t) + (t-2x)G_x(t) = 0, \tag{66}$$

which, by using the condition $G(0,0) = 1$, gives that

$$G_x(t) = \frac{\sqrt{2}}{\sqrt{2+t^2}} \exp\left(\sqrt{2}x \arctan\frac{t}{\sqrt{2}}\right).$$

If we particularize the generating function (R. Koekoek R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue) :

$$\sum_{n=0}^{+\infty} P_n^{(\lambda)}(u,\phi)t^n = \left(1 - e^{i\phi}t\right)^{-\lambda+iu}\left(1 - e^{-i\phi}t\right)^{-\lambda-iu}. \tag{67}$$

On the proof

by setting $\lambda = 1/2$, $\phi = \pi/2$ and $u = x/\sqrt{2}$, we then obtain

$$\sum_{n=0}^{+\infty} P_n^{(1/2)}\left(\frac{x}{\sqrt{2}}, \frac{\pi}{2}\right) t^n = (1 - it)^{-1/2 + ix/\sqrt{2}}(1 + it)^{-1/2 - ix/\sqrt{2}}. \tag{68}$$

Using the identity

$$\left(\frac{1 - it}{1 + it}\right)^{\frac{1}{2} iz} = \exp\left(z \arctan t\right), \tag{69}$$

for $z = \sqrt{2}x$, we get

$$\sum_{n=0}^{+\infty} P_n^{(1/2)}\left(\frac{x}{\sqrt{2}}, \frac{\pi}{2}\right) t^n = \frac{1}{\sqrt{1 + t^2}} \exp\left(\sqrt{2}x \arctan t\right). \tag{70}$$

By comparing (70) with (33), we arrive at (62).

Proof of **Proposition 4.** is deduce by the some method in the proof of **Proposition 3.**

Case $\gamma = 1$

- The generalized factorial $x_n^1! = ((n+1)^2)!$ which corresponds to the ISW potential.
- The associated polynomials ϕ_n with $\phi_{-1} = 0$ and $\phi_0 = 1$, satisfy

$$x\phi_n(x) = \frac{n+2}{\sqrt{2}}\phi_{n+1}(x) + \frac{n+1}{\sqrt{2}}\phi_{n-1}(x). \tag{71}$$

- The first polynomials $\phi_0(x) = 1$, $\phi_1(x) = \frac{\sqrt{2}}{2}x$, $\phi_2(x) = \frac{1}{3}(x^2 - 2)$ and

$$\phi_3(x) = \frac{\sqrt{2}}{12}\left(x^3 - \frac{13x}{2}\right)$$

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Proposition 5

The polynomials ϕ_n are the associated Meixner-Pollaczek polynomials

$$\phi_n^{(1/2,1)}(x) = P_n^{(1/2)}\left(\frac{x}{\sqrt{2}}, \frac{\pi}{2}, 1\right) \qquad (72)$$

orthogonal on $(-\infty, +\infty)$ with respect to

$$\omega^{\frac{1}{2}}(x,1) = \frac{e^{(2\phi-\pi)x}}{2\pi}\left|\Gamma\left(\frac{3}{2}+ix\right)\right|^2 \left|{}_2F_1\left(-\frac{1}{2}+ix, 1; \frac{3}{2}+ix; e^{2i\phi}\right)\right|^{-2} \qquad (73)$$

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Proposition 6

The generating function of $\phi_n(x)$ *is given by*

$$\sum_{n=0}^{+\infty} P_n^{\frac{1}{2}}\left(\frac{x}{\sqrt{2}}, \frac{\pi}{2}, 1\right) t^n = -\frac{1}{t(\sqrt{2}x - t)} + \frac{(t^2+1)\,{}_2F_1\left(2, 2; \frac{5+i\sqrt{2}x}{2}; \frac{1+it}{2}\right)}{t(\sqrt{2}x - t)(1+i\sqrt{2}x)(3+i\sqrt{2}x)}$$

$$+\frac{i\,{}_2F_1\left(1, 1; \frac{3+i\sqrt{2}x}{2}; \frac{1}{2}\right)}{1+i\sqrt{2}x}\,\frac{e^{\sqrt{2}x\,\mathrm{arctan}(t)}}{t\sqrt{t^2+1}}$$

where $|t| < 1$ and $x \in \mathbb{R}$.

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where $|t| < 1$ and $x \in \mathbb{R}$.

Corollary 1

We have the following expression of Hypergeometric function

$$ {}_2F_1\left(2, 2; \frac{5}{2}; \frac{1+it}{2}\right) = 3t(t^2+1)^{\frac{-3}{2}}\left(\frac{i\pi}{2} - \ln(t + \sqrt{t^2+1})\right) + \frac{3}{t^2+1} \tag{74}$$

Proposition 7

The difference equation of $P_n^{1/2}(\xi;\pi/2,1)$ is

$$[AT^{*4}+BT^{*3}+CT^{*2}+DT^{*}+EI]P_n^{1/2}(\xi,\pi/2,1)=0 \tag{75}$$

$T^*f(\xi)=f(\xi-i)$, $Tf(x)=f(x+1)$ and $If(x)=f(x)$ so
$T^*P_n(\xi)=\frac{(-i)^n}{(n+1)!}(Tm_n(\xi))(i\xi-1/2)$ where $m_n(.)$ is the Meixner polynomials

$$\begin{aligned}
A^* &= -2(n+2)\left(i\xi+\frac{7}{2}\right)^2 \\
B^* &= D^*=-2(n+1)(2n+5)-2 \\
B^* &= 2(n+2)[4(n^2+4n+3)+2(i\xi+3/2)^2+2(i\xi+3/2)^2 \\
&+ 2(i\xi-1/2)]-2(4n^2+5n-6) \\
E^* &= -2(n+2)\left(i\xi+\frac{1}{2}\right)^2.
\end{aligned}$$

Proposition 5. By the three-terms recurrence relation (H. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, vol2, 1953) :

$$(n+c+1)P^{\lambda}_{n+1}(x) = 2\left[(n+c+\lambda)\cos\phi + y\sin\phi\right]P^{\lambda}_{n}(x) - (n+c+2\lambda-1)P^{\lambda}_{n-1}(x), \quad (76)$$

where $P^{(\lambda)}_n(x) := P^{(\lambda)}_n(x,a,b,c)$ with conditions
$P^{(\lambda)}_{-1}(x) = 0,\ P^{(\lambda)}_{0}(x) = 1,\ 0 < \phi < \pi,\ 2\lambda + c > 0,\ c \geq 0$, or
$0 < \phi < \pi,\ 2\lambda + c \geq 1,\ c > -1$, for $a = 0,\ c = 1, \lambda = 1/2$ and $\phi = \pi/2$, we deduces the result. The weight function (73) is obtain from(H. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher Transcendental Functions, vol2, 1953, p.220).

Proposition 6. The generating function of $\phi_n(x)$, denoted
$G_{\phi(x)}(t) = G_{\phi}(x,t) := \sum\limits_{n\geq 0} \phi_n(x)t^n$, satisfy

$$(t^3 + t)G'_{\phi}(t) + (2t^2 - \sqrt{2}xt + 1)G_{\phi}(t) - 1 = 0 \quad (77)$$

with $G_{\phi(x)}(0) = 1$. Let a be the zero, if exist, of the function defined by
$F(t) := \int_a^t (s^2+1)^{-1/2} e^{-\sqrt{2}x\arctan(s)} ds$. Then, the solution of the equation (77)

$$G_{\phi(x)}(t) = \frac{e^{\sqrt{2}x\arctan(t)}}{t\sqrt{t^2+1}} F(t) \quad (78)$$

For $x=0$ we have

$$G_{\phi(0)}(t)=\frac{1}{t\sqrt{t^2+1}}\int_a^t \frac{1}{\sqrt{s^2+1}}ds \tag{79}$$

thus

$$G_{\phi(0)}(t)=\frac{arcsh(t)-arcsh(a)}{t\sqrt{t^2+1}} \tag{80}$$

where $arcsh(t)=ln(t+\sqrt{t^2+1})$. In the other hand we have

$$\begin{aligned} G_{\phi(0)}(t)=G_\phi(0,t) &= \sum_{n\geq 0}\phi_{2n}(0)t^{2n} && (81)\\ &= \frac{1}{2t}\sum_{n\geq 0}(-1)^n\frac{(n!)^2}{(2n+1)!}(2t)^{2n+1}. && (82)\end{aligned}$$

by using the formula, in [Prudnikov, vol.1, p.714] :

$$\sum_{n\geq 0}(-1)^n\frac{(n!)^2}{(2n+1)!}(x)^{2n+1}=4(4+x^2)^{-\frac{1}{2}}Arcsh(\frac{x}{2}) \qquad [|x|<2] \tag{83}$$

we get

$$G_{\phi(0)}(t)=\frac{arcsh(t)}{t\sqrt{t^2+1}} \qquad [|t|<1] \tag{84}$$

from (80) and (84) we conclude that $a=0$. Then

$$G_{\phi(x)}(t)=\frac{e^{\sqrt{2}x\arctan(t)}}{t\sqrt{t^2+1}}\int_0^t \frac{e^{-\sqrt{2}x\arctan(s)}}{\sqrt{s^2+1}}ds \qquad [|s|<1]. \tag{85}$$

$$G_{\phi(x)}(t) = \frac{e^{\sqrt{2}x\,\mathrm{arctan}(t)}}{t\sqrt{t^2+1}} \int_0^t \frac{e^{-\sqrt{2}x\,\mathrm{arctan}(s)}}{\sqrt{s^2+1}} ds \qquad [|s|<1]. \tag{86}$$

Now, we must calculate $F(t)$. To do that, we compute the exponential generating function $G_{q(x)}(t) = G_q(x,t) := \sum_{n\geq 0} \tilde{q}_n(x) t^n/n!$, of $\tilde{q}_n(x) = (x_n!/2^n)^{1/2}\phi_n(x)$. We have : *(i) The exponential generating function of polynomials $\tilde{q}_n(x)$ is given by*

$$G_{q(x)}(t) = C_1(x)\ {}_2F_1\left(2,2;\frac{5+i\sqrt{2}x}{2};\frac{2+i\sqrt{2}t}{4}\right) + C_2(x)(2x-t)(2+t^2)^{-\frac{3}{2}},$$
$$\times e^{\sqrt{2}x\,\mathrm{arctan}(\frac{t}{\sqrt{2}})}$$

where $x \in \mathbb{R}$, and

$$C_1(x) = \frac{1}{(1+i\sqrt{2}x)(3+i\sqrt{2}x)}, \qquad C_2(x) = \frac{2i}{1+i\sqrt{2}x}{}_2F_1(1,1;\frac{3+i\sqrt{2}x}{2};\frac{1}{2}).$$

(ii) The integral $F(t)$ is given by

$$F(t) = \left[\frac{(t^2+1){}_2F_1\left(2,2;\frac{5+i\sqrt{2}x}{2};\frac{1+it}{2}\right)}{(1+i\sqrt{2}x)(3+i\sqrt{2}x)} - 1\right]\frac{\sqrt{1+t^2}}{\sqrt{2}x-t}e^{-\sqrt{2}x\,\mathrm{arctan}(t)}$$
$$\times + \frac{i{}_2F_1\left(1,1;\frac{3+i\sqrt{2}x}{2};\frac{1}{2}\right)}{1+i\sqrt{2}x}$$

On the proof

(i) The normalized polynomials $\tilde{q}_n(x)$ satisfy the three-terms recurrence relation

$$x\tilde{q}_n(x) = \tilde{q}_{n+1}(x) + \frac{(n+1)^2}{2}\tilde{q}_{n-1}(x) \tag{87}$$

The generating function $G_{q(x)}(t)$ satisfy

$$(t^2+2)\frac{d^2}{dt^2}G_{q(x)}(t) + (5t-2x)\frac{d}{dt}G_{q(x)}(t) + 4G_{q(x)}(t) = 0, \tag{88}$$

by putting $t = i\sqrt{2}(1-2s)$ and $\psi(s) := G_{q(x)}(i\sqrt{2}(1-2s))$, Eq.(88) reduces to

$$s(1-s)\psi''(s) + \left(\frac{5+i\sqrt{2}x}{2} - 5s\right)\psi'(s) - 4\psi(s) = 0, \tag{89}$$

with parameters $a = b = 2$, and $c = \frac{5+i\sqrt{2}x}{2}$. Since c is not an integer, the solution of Eq.(89) is of the form (see [HBK], p. 256) :

$$\psi(s) = C_1 \; {}_2F_1\left(2,2;\frac{5+i\sqrt{2}x}{2};s\right) + C_2 s^{\frac{-3-i\sqrt{2}x}{2}} {}_2F_1\left(\frac{1-i\sqrt{2}x}{2},\frac{1-i\sqrt{2}x}{2};\frac{-1-i\sqrt{2}x}{2};s\right.$$

where C_1 and C_2 are constants. By taking account of our change of functions we get

$$G_{q(x)}(t) = C_1 \; {}_2F_1\left(2,2;\frac{5+i\sqrt{2}x}{2};\frac{2+i\sqrt{2}t}{4}\right) + C_2\left(\frac{2+i\sqrt{2}t}{4}\right)^{\frac{-3-i\sqrt{2}x}{2}}$$

$$\times\ {}_2F_1\left(\frac{1-i\sqrt{2}x}{2}, \frac{1-i\sqrt{2}x}{2}; \frac{-1-i\sqrt{2}x}{2}; \frac{2+i\sqrt{2}t}{4}\right), \tag{90}$$

we make appeal to the Euler's hypergeometric transformation

$${}_2F_1(a, b; c; x) = (1-x)^{c-a-b}{}_2F_1(c-a, c-b; c; x) \tag{91}$$

and using the identity (69) the expression of $G_{q(x)}(t)$ is given by

$$G_{q(x)}(t) = C_1(x)\ {}_2F_1\left(2, 2; \frac{5+i\sqrt{2}x}{2}; \frac{2+i\sqrt{2}t}{4}\right) + C_2(x)(2x-t)(2+t^2)^{-\frac{3}{2}} e^{\sqrt{2}x\,\mathbf{arctan}(\frac{t}{\sqrt{2}}}$$

in the other hand the conditions $G_x(0) = 1$ and $G'_x(0) = x$ gives

$$\begin{cases} {}_2F_1\left(2, 2; \frac{5+i\sqrt{2}x}{2}; \frac{1}{2}\right) C_1(x) + (\sqrt{2})^{-1} x C_2(x) = 1 \\ \frac{i2\sqrt{2}}{5+i\sqrt{2}x}\ {}_2F_1\left(3, 3; \frac{7+i\sqrt{2}x}{2}; \frac{1}{2}\right) C_1(x) + (2\sqrt{2})^{-1}(2x^2-1)C_2(x) = x \end{cases}$$

from this system of equation we get easily the quantities $C_1(x)$ and $C_2(x)$ given by

$$\begin{aligned} C_1(x) &= -\left[(2x^2-1){}_2F_1\left(2, 2; \frac{5+i\sqrt{2}x}{2}; \frac{1}{2}\right) - \frac{4\sqrt{2}ix}{5+\sqrt{2}ix}{}_2F_1\left(3, 3; \frac{7+i\sqrt{2}x}{2}; \frac{1}{2}\right)\right]^{-1} \\ C_2(x) &= \frac{2^{\frac{3}{2}} x {}_2F_1\left(2, 2; \frac{5+i\sqrt{2}x}{2}; \frac{1}{2}\right) - \frac{8i}{5+i\sqrt{2}x}{}_2F_1\left(3, 3; \frac{7+i\sqrt{2}x}{2}; \frac{1}{2}\right)}{(2x^2-1){}_2F_1\left(2, 2; \frac{5+i\sqrt{2}x}{2}; \frac{1}{2}\right) - \frac{4\sqrt{2}ix}{5+\sqrt{2}ix}{}_2F_1\left(3, 3; \frac{7+i\sqrt{2}x}{2}; \frac{1}{2}\right)}. \end{aligned}$$

In order to simplify $C_1(x)$ and $C_2(x)$ we applicate this formula [NIST Handbook of Mathematical Functions, p.388]

$$\begin{aligned} z(1-z)(a+1)(b+1){}_2F_1(a+2,b+2;c+2;z) &\quad + \\ (c-(a+b+1)z)(c+1){}_2F_1(a+1,b+1;c+1;z) &\quad - \quad c(c+1){}_2F_1(a,b;c;z)=0 \end{aligned}$$

then we get

$$C_1(x) = \frac{1}{(1+i\sqrt{2}x)(3+i\sqrt{2}x)} \tag{92}$$

$$C_2(x) = \frac{2i}{1+i\sqrt{2}x}{}_2F_1(1,1;\frac{3+i\sqrt{2}x}{2};\frac{1}{2}). \tag{93}$$

(ii) The relation between $G_\phi(x,t)$ and $G_q(x,t)$ is given by

$$\frac{d}{dt}\left(tG_\phi(x,t)\right) = G_q(x,\sqrt{2}t) \tag{94}$$

multiplying (86) by t and deriving, we get

$$G_\phi(x,t)+t\frac{d}{dt}G_\phi(x,t) = \frac{\sqrt{2}x-t}{(t^2+1)^{\frac{3}{2}}}e^{\sqrt{2}x\,\mathrm{arctan}(t)}\int_0^t \frac{e^{-\sqrt{2}x\,\mathrm{arctan}(s)}}{\sqrt{s^2+1}}ds+\frac{1}{t^2+1} \tag{95}$$

in the other hand we have

$$G_q(x,\sqrt{2}t) = \frac{{}_2F_1\left(2,2;\frac{5+i\sqrt{2}x}{2};\frac{1+it}{2}\right)}{(1+i\sqrt{2}x)(3+i\sqrt{2}x)}+\frac{i{}_2F_1\left(1,1;\frac{3+i\sqrt{2}x}{2};\frac{1}{2}\right)}{1+i\sqrt{2}x}\frac{\sqrt{2}x-t}{(t^2+1)^{\frac{3}{2}}}e^{\sqrt{2}x\,\mathrm{arctan}(t)}. \tag{96}$$

Case $\gamma \to \infty$

- The sequence of positive number $x_n^{\infty} = \gamma n$
- The polynomials ϕ_n have been obtained by S. T. Ali and M. E. H. Ismail (*J.Phys A : Math.* Theor 2012) and are the Hermite polynomials.

A Hamiltonian operator associated with polynomials $\{\phi_n\}$

V. V. Borzov, Orthogonal polynomials and generalized oscillator algebras, *Integral Transforms and Special Functions 2000*

Here, our aim is to attached "*à la Borzov*" a Hamiltonian operator to every set of orthogonal polynomials ϕ_n, we have discussed with respect to the parameter γ in the previous part.

Borzov's Method

- ρ a positive Borel measure on $\mathbb{R}$

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- $\{\phi_n(x)\}_{n=0}^{\infty}$ an orthonormal basis in the Hilbert space $F = L^2(\mathbb{R}, \rho(dx)) = F_1$
- Let $F_2 = L^2(\mathbb{R}, \rho(dy))$ a second copy of this Hilbert space.

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Borzov's Method

- ρ a positive Borel measure on $\mathbb{R}$
- $\{\phi_n(x)\}_{n=0}^{\infty}$ an orthonormal basis in the Hilbert space $F = L^2(\mathbb{R}, \rho(dx)) = F_1$
- Let $F_2 = L^2(\mathbb{R}, \rho(dy))$ a second copy of this Hilbert space.
- The Poisson Kernel on $F_1 \otimes F_2$ is defined by

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- If $\{\phi_n\}$ is complete in F and $|t| = 1$ then $K'_F = K^*_F = (K_F)^{-1}$

- The position operator Y_{F_2} is assumed to act by

$$y\phi_n(x) = b_{n-1}\phi_{n-1}(x) + b_n\phi_{n+1}(x) \tag{97}$$

with $b_{-1} = 0, \ \phi_0(x) = 1$, where $(b_n)_{n=0}^{\infty}$ is a given positive sequence

- The momentum operator is defined by

$$P_{F_1} = K_F^* Y_{F_2} K_F \tag{98}$$

- Finally, the Hamiltonian operator is defined by

$$H_{F_1}(t) = (X_{F_1})^2 + (P_{F_1}(t))^2 \tag{99}$$

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- The Hamiltonian operator

$$H = X^2 + P^2 = -\frac{d^2}{dx^2} + x^2 \tag{102}$$

  is the harmonic oscillator

Case $\gamma = 0$: Barut Girardello CS type ($2\sigma = 1$)

V. V. Borzov and E. V. Damaskinsky, The generalized coherent states for oscillators connected with Meixner and Meixner Pollaczek polynomials *(Theor. Math. Phys 2007)* have attached to the set of Meixner-Pollaczek polynomials $P_n^{(\sigma)}(x, \frac{\pi}{2})$ the following Hamiltonian operator

$$H = \frac{1}{\lambda^2}\cosh(i\lambda\partial_x) + \frac{1}{2}(x + i\lambda)xe^{i\lambda\partial_x} \tag{103}$$

$\sigma(\sigma - 1) = \lambda^{-4}$, that coincides with Hamiltonian of Linear Relativistic oscillator from the work N. M. Atakishiyev and S. K. Suslov, The Hahn and Meixner polynomials of an imaginary argument and some of their applications *(J. Phys. A. 1985)*

Case $\gamma = 1$

Attaching to the obtained set of associated Meixner-Pollaczek polynomials a Hamiltonian operator is under construction

Summarizing scheme

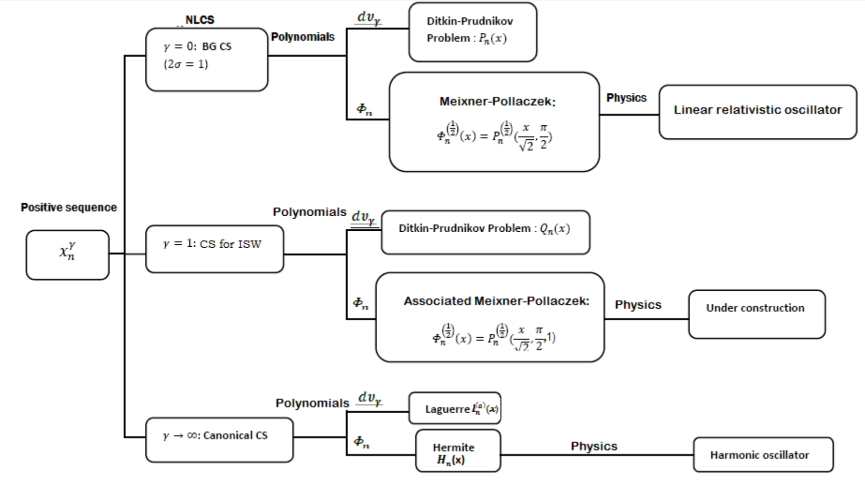

Figure: NLCS : Nonlinear coherent states, ISW : Infinite Square Well and BG : Barut-Girardello

Concluding remarks

- We have replaced the factorial $n!$ occurring in coefficients $z^n/\sqrt{n!}$ of the canonical coherent states by a specific generalized factorial $x_n^\gamma! = x_1^\gamma \cdots x_n^\gamma$, where x_n^γ is a sequence of positive numbers and $\gamma \in (0, \infty)$ being a parameter. The new coefficients are then used to consider a superposition of eigenstates of the Hamiltonian with a symmetric Pöschl-Teller (SPT) potential depending on a parameter $\nu > 1$. For equal parameters $\gamma = \nu$, we define the associated Bargmann-type transform and we derive some results on the infinite square well potential.

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- We have proceed by a general method (S. T. Ali, M. E. H. Ismail, *J. Phys A : Math. Theor.* 2012) to discuss, for some different values of γ, two sets of orthogonal polynomials that are naturally associated with these NLCS. One set of these polynomials, say $P_n(x)$, is obtained from a symmetrization of the measure which gives the resolution of the identity for the NLCS. Here, we can suggest a new generalization of these NLCS themselves by replacing the coefficients $z^n/\sqrt{x_n^\gamma!}$ by the obtained polynomials $P_n(x)$. In this direction, it's crucial to know some basic properties of these polynomials. However, for many values of γ, such properties are not known. As example, for $\gamma = 0$, the NLCS are of Barut-Girardello type and the resulting polynomials are related to the Ditkin-Prudnikov problem which is still open.

Concluding remarks

- The second set of orthogonal polynomials, say $\phi_n(x)$, arises from the shift operators associated to these coherent states. In this case, to polynomials $\phi_n(x)$ a Hamiltonian system could be associated (V. V. Borzov and E. V. Damaskinsky, *Integral Transforms Spec. Funct.* 2002). However, except having the three-terms recurrence relation, getting more information on the $\phi_n(x)$ is not so easy. Indeed, while dealing with an example cited in (S. T. Ali, M. E. H. Ismail, *J.Phys A : Math. Theor* 2012) the authors (D. Dai, W. Hu and X-S. Wang, *SIGMA* 2015) have obtained a uniform asymptotic expansion of $\phi_n(x)$ as n tends to infinity and they have concluded that their results suggest that the weight function associated with the $\phi_n(x)$ has an usual singularity which has never appeared for orthogonal polynomials in the Askey scheme.

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- Find some classification in Physics for these CS and try to characterize it by a priory Mathematical condition on the given sequence of positive number x_n

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Special Thanks