Enhanced Quantization:
The *Right* Way to Quantize *Everything*

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Something Unusual

L. Landau, E.M. Lifshitz, *Quantum mechanics: Non-relativistic theory*, 3rd ed., Pergamon Press, 1977, page 3.

"Thus quantum mechanics occupies a very unusual place among physical theories: it contains classical mechanics as a limiting case, yet at the same time it requires this limiting case for its own formulation."

L & L rule

Classical & Quantum

quantum

$$\hbar > 0$$

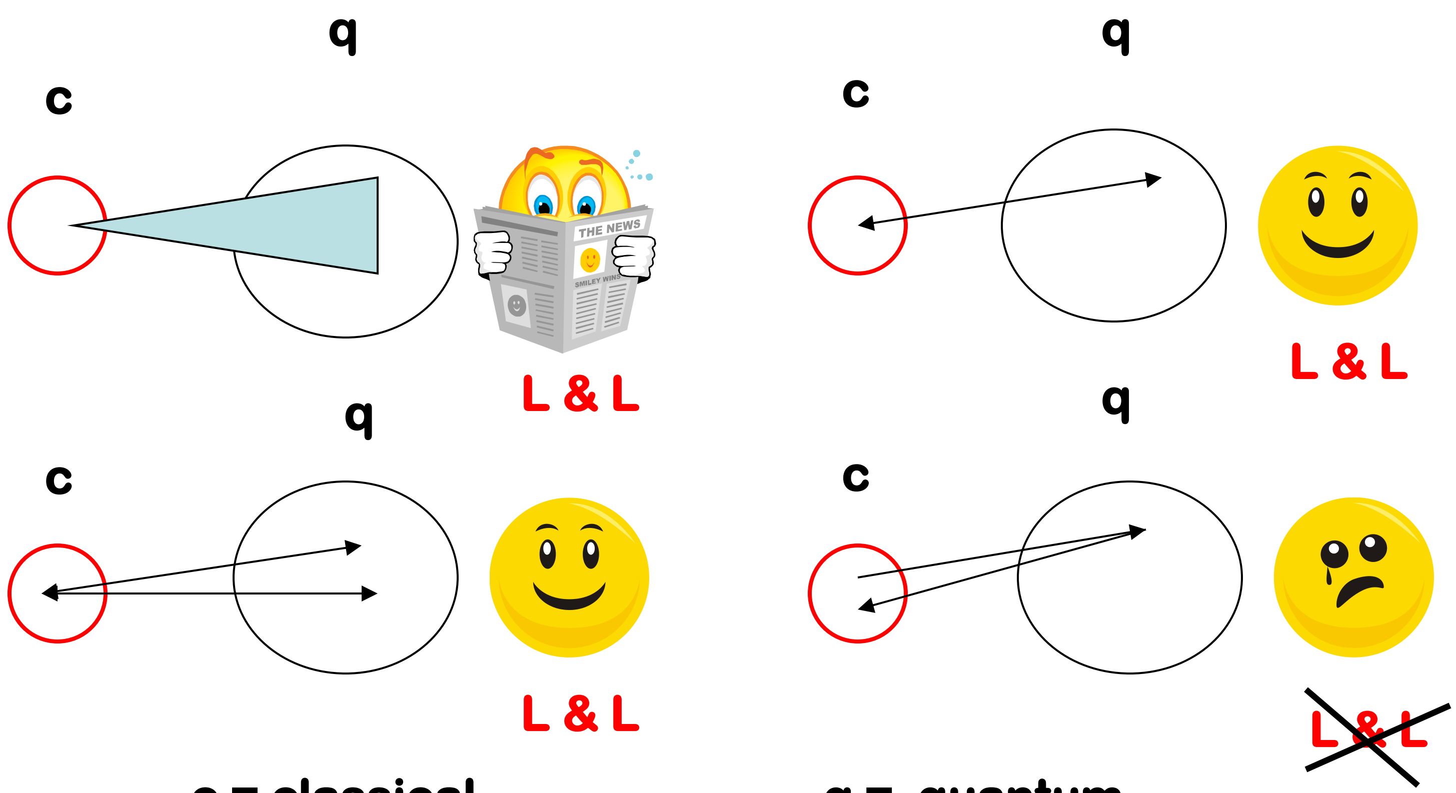
classical

$$\hbar = 0$$

$$\text{Planck's constant} \quad h = 6.62606957 \times 10^{-38} \ \text{Joule} \cdot \text{sec}$$

$$\hbar = h / 2\pi$$

Class. & Quant. Possibilities



c = classical **q = quantum**

“ Triviality of φ_n^4 ”

- **Covariant scalar models (imaginary time)**

$$I = \int \left(\tfrac{1}{2}\{[\nabla\varphi(x)]^2 + m^2\varphi(x)^2\} + \lambda\varphi(x)^4\right)\ d^n x \ ; \quad n \geq 5$$

``It is known that self-interacting scalar fields with a quartic non-linearity do not exist in dimension five or more. (The proofs apply to field theories with a single, scalar field.)’’

A. Jaffe and E. Witten (Nov. 2005)

http://www.claymath.org/sites/default/files/yangmills.pdf

~~**L & L rule**~~

that the linear operator H introduced in the preceding section is the energy of the system in quantum mechanics.

In classical mechanics a dynamical system is defined mathematically when the Hamiltonian is given, i.e. when the energy is given in terms of a set of canonical coordinates and momenta, as this is sufficient to fix the equations of motion. In quantum mechanics a dynamical system is defined mathematically when the energy is given in terms of dynamical variables whose commutation relations are known, as this is then sufficient to fix the equations of motion, in both Schrödinger's and Heisenberg's form. We need to have either H expressed in terms of the Schrödinger dynamical variables or H_t expressed in terms of the corresponding Heisenberg dynamical variables, the functional relationship being, of course, the same in both cases. We call the energy expressed in this way the *Hamiltonian* of the dynamical system in quantum mechanics, to keep up the analogy with the classical theory.

A system in quantum mechanics always has a Hamiltonian, whether the system is one that has a classical analogue and is describable in terms of canonical coordinates and momenta or not. However, if the system does have a classical analogue, its connexion with classical mechanics is specially close and one can usually assume that the Hamiltonian is the same function of the canonical coordinates and momenta in the quantum theory as in the classical theory.† There would be a difficulty in this, of course, if the classical Hamiltonian involved a product of factors whose quantum analogues do not commute, as one would not know in which order to put these factors in the quantum Hamiltonian, but this does not happen for most of the elementary dynamical systems whose study is important for atomic physics. In consequence we are able also largely to use the same language for describing dynamical systems in the quantum theory as in the classical theory (e.g. to talk about particles with given masses moving through given fields of force), and when given a system in classical mechanics, can usually give a meaning to 'the same' system in quantum mechanics.

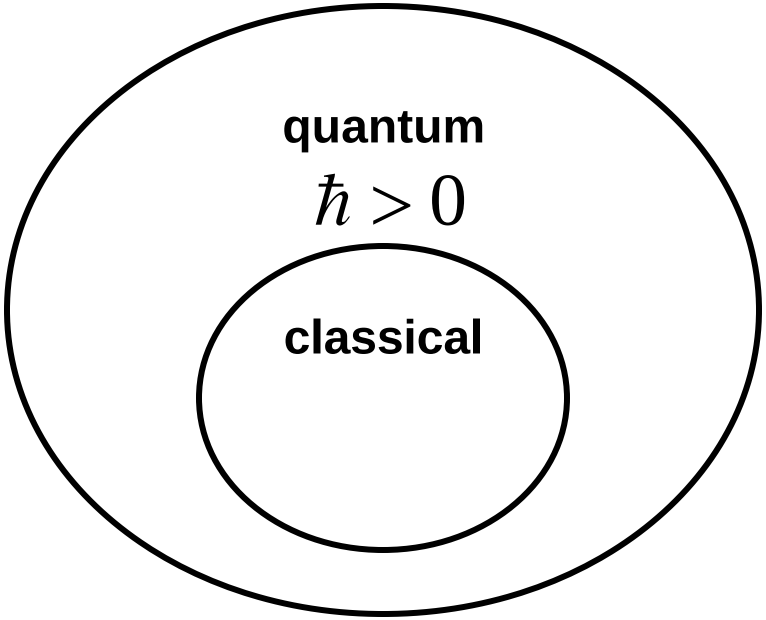
Equation (13) holds for v_t any function of the Heisenberg dynamical variables not involving the time explicitly, i.e. for v any constant

† This assumption is found in practice to be successful only when applied with the dynamical coordinates and momenta referring to a Cartesian system of axes and not to more general curvilinear coordinates.

DIRAC

The Principles of Quantum Mechanics

Classical $\subset$ Quantum



List of Topics

1 Classical/Quantum connection

"Enhanced Quantization"

Canonical & Affine quantization

Enhanced classical theories

2 A toy model of gravity

3 Two soluble examples

Action Principle Formulations

Classical action: $A_C = \int [\, p(t)\dot{q}(t) - H_c(p(t), q(t))]\, dt$

Variation: $\delta A_C = 0$ yields: $\dot{q} = \partial H_c / \partial p, \;\; \dot{p} = -\partial H_c / \partial q$

Solution: $p(t),\; q(t)$ given $p(0), q(0) \in \mathbf{R}^2$

Quantum action: $A_Q = \int \{ \langle \psi(t) | [i\hbar \partial / \partial t - \mathcal{H}] | \psi(t) \rangle \}\, dt$

Variation: $\delta A_Q = 0$ yields $i\hbar \partial |\psi\rangle / \partial t = \mathcal{H} |\psi\rangle$

Solution: $|\psi(t)\rangle$ given $|\psi(0)\rangle \in \mathbf{H}$

VERY DIFFERENT

Restricted Action Principle

Quantum action: $A_Q = \int \{ \langle \psi(t) | [i\hbar \partial / \partial t - \mathcal{H}] | \psi(t) \rangle \} \; dt$

Possible restrictions: $|\psi(t)\rangle \rightarrow |\psi_{\mathbf{E}}(t)\rangle \quad [\equiv \mathbf{E} \, |\psi(t)\rangle]$

Variation: $\delta A_Q = 0$ yields $\quad [i\hbar \partial |\psi_{\mathbf{E}}(t)\rangle / \partial t = \mathcal{H}_{\mathbf{E}} |\psi_{\mathbf{E}}(t)\rangle]$

$$[\mathcal{H}_{\mathbf{E}} \equiv \mathbf{E} \, \mathcal{H} \, \mathbf{E}\,]$$

Solution: $|\psi_{\mathbf{E}}(t)\rangle$ given $|\psi_{\mathbf{E}}(0)\rangle \in \mathbf{H}$

(1) Nature of $\{ |\psi_{\mathbf{E}}(t)\rangle \}$: $\underline{subspace \in \mathbf{H}}$ **(part space)**

(2) Nature of $\{ |\psi_{\mathbf{E}}(t)\rangle \}$: $\underline{subset \in \mathbf{H}}$ **(Gaussians)**

Unification of Classical and Quantum (1)

Quantum action: $A_Q = \int \langle \psi(t) | [i\hbar\partial/\partial t - \mathcal{H}] | \psi(t) \rangle \; dt$

Restricted variation: $|\psi(t)\rangle \to |\,\mathbf{?}\,(t)\,\rangle \in \mathbf{S} \subset \mathbf{H}$

sun

Macroscopic **variations of** **Microscopic** **states:**

Basic state: $\langle x | \eta \rangle = \eta(x)$

Translated basic state: $\langle x | \eta; q \rangle = \eta(x-q)$

Translated Fourier state: $\langle \kappa | \eta; p \rangle = \tilde{\eta}(\kappa - p)$

Coherent states: $\langle x | p,q \rangle = e^{ip(x-q)/\hbar}\,\eta(x-q)$

$$|p,q\rangle \equiv e^{-iqP/\hbar} e^{ipQ/\hbar} |\eta\rangle \;\; ; \quad |\eta\rangle = |0\rangle \;\; ; \quad (Q+iP)|0\rangle = 0$$

 s.a.

Unification of Classical and Quantum (2)

$$\text{Quantum action}: \quad A_Q = \int \langle \psi(t) | [i\hbar \partial / \partial t - \mathcal{H}] | \psi(t) \rangle \; dt$$

$$\text{Restricted variation}: \quad | \psi(t) \rangle \rightarrow | \, p(t), q(t) \rangle$$

subset

$$\text{New action}: \quad A_R = \int \langle \, p(t), q(t) | [i\hbar \partial / \partial t - \mathcal{H}] | \, p(t), q(t) \rangle \; dt$$

$$A_R = \int [\, p(t) \dot{q}(t) - H(\, p(t), q(t))] \; dt$$

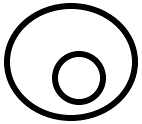
NOTE: THIS EQUATION APPEARS JUST LIKE THE CLASSICAL ACTION ! ! ! ! !

Unification of Classical and Quantum (2)

$$\text{Quantum action:}\quad A_Q = \int \langle \psi(t) | [i\hbar\partial/\partial t - \mathcal{H}] | \psi(t) \rangle \; dt$$

$$\underline{\text{Restricted variation:}}\quad \boxed{|\psi(t)\rangle \rightarrow |\, p(t), q(t) \rangle}$$ **subset**

$$\text{New action:}\quad A_R = \int \langle p(t), q(t) | [i\hbar\partial/\partial t - \mathcal{H}] | \, p(t), q(t) \rangle \; dt$$



$$\underline{A_R = \int [\, p(t)\dot{q}(t) - H(\,p(t), q(t))] \; dt}$$

CLASSICAL MECHANICS *IS* QUANTUM MECHANICS RESTRICTED TO A CERTAIN TWO DIMENSIONAL SURFACE IN HILBERT SPACE

Canonical Transformations

Restricted quantum action :

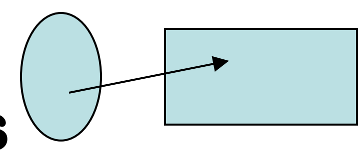
$$A_R = \int \langle p,q | [i\hbar\partial/\partial t - \mathcal{H}(P,Q)] | p,q \rangle \; dt$$

$$= \int [\, p\dot{q} - H(p,q)] \; dt$$

Canonical transformations : $\quad p\ dq = \tilde{p}\ d\tilde{q} + d\tilde{G}(\tilde{p},\tilde{q})$

$$\underline{|\,p,q\rangle} = |\,p(\tilde{p},\tilde{q}),q(\tilde{p},\tilde{q})\rangle \equiv \underline{|\,\tilde{p},\tilde{q}\rangle}$$

PS



H

Restricted quantum action :

$$A_R = \int \langle \tilde{p},\tilde{q} | [i\hbar\partial/\partial t - \mathcal{H}(P,Q)] | \tilde{p},\tilde{q} \rangle \; dt$$

same quantum

$$= \int [\, \tilde{p}\dot{\tilde{q}} + \dot{\tilde{G}}(\tilde{p},\tilde{q}) - \tilde{H}(\tilde{p},\tilde{q})] \; dt$$

Cartesian Coordinates

Classical/Quantum connection:

$$\underline{H(p,q)} \equiv \langle p,q | \mathcal{H}(P,Q) | p,q \rangle \quad , \quad [\,(Q+iP)|0\rangle = 0\,]$$

$$= \langle 0 | \mathcal{H}(P+p, Q+q) | 0 \rangle = \underline{\mathcal{H}(p,q)} + \mathcal{O}(\hbar; p,q)$$

Physical meaning: $[\,\langle 0|P|0\rangle = 0 \;,\; \langle 0|Q|0\rangle = 0\,]$

$$\boxed{\langle p,q|P|p,q\rangle = p \;;\quad \langle p,q|Q|p,q\rangle = q}$$

Fubini-Study metric: $[D_R^2 = \min_\alpha \| \, |\psi\rangle - e^{i\alpha}|\phi\rangle \, \|^2]$

$$\boxed{2\hbar\ [\ \| d|p,q\rangle \|^2 - |\langle p,q|\ d|p,q\rangle|^2] = dp^2 + dq^2}$$

$$(2/\hbar)\ [dp^2\langle \Delta Q^2\rangle + dp\ dq\langle\{\Delta Q, \Delta P\}\rangle + dq^2\langle \Delta P^2\rangle]$$

BIG DEAL YIPPEE BIG DEAL HOORAY BIG DEAL

CONVENTIONAL QUANTIZATION RECOVERED

Is There More?

- Are there ***other*** **two-dimensional sheets of normalized Hilbert space vectors** that may be used in **restricting the quantum action** and which lead to an ***enhanced classical canonical formalism***?

YES !

Affine Variables

Affine variables: $q = q\{q, p\} = \{q, pq\} \equiv \{q, d\}$

$$i\hbar Q = Q[Q, P] = [Q, QP] = [Q, D]; \; D \equiv (PQ + QP)/2$$ **s.a.**

Affine coherent states: $(q > 0 \;\; ; \;\; Q > 0)$

$$|p, q\rangle = e^{ipQ/\hbar} e^{-i\ln(q)D/\hbar} |\eta\rangle; \qquad [(Q-1) + iD/\tilde{\beta}]|\eta\rangle = 0$$

Overlap function:

$$\langle p', q' | p, q\rangle = \{\tfrac{1}{2}[\sqrt{q'/q} + \sqrt{q/q'} + i\sqrt{q'q}\,(p'-p)/\tilde{\beta}]\}^{-2\tilde{\beta}/\hbar}$$

Resolution of unity:

$$I = \int |p, q\rangle\langle p, q| \; dp\, dq / 2\pi\hbar / (1 - 1/2\tilde{\beta}/\hbar)$$

also $(q < 0 \, , \, Q < 0) \cup (q > 0 \, , \, Q > 0)$

Affine Quantization (1)

Quantum action :

$$A_Q \equiv \int \langle \psi(t) | [i\hbar\partial/\partial t - \mathcal{H}'(D,Q)] | \psi(t) \rangle \; dt$$

Restricted action : $| \psi(t) \rangle \rightarrow | p(t), q(t) \rangle$ **subset**

$$A_R \equiv \int \langle p(t), q(t) | [i\hbar\partial/\partial t - \mathcal{H}'(D,Q)] | p(t), q(t) \rangle \; dt$$

$$= \int [-q(t)\dot{p}(t) - H(p(t), q(t))] \; dt$$

Canonical transformation : $| \tilde{p}, \tilde{q} \rangle = | p, q \rangle$

$$A_R \equiv \int \langle \tilde{p}(t), \tilde{q}(t) | [i\hbar\partial/\partial t - \mathcal{H}'(D,Q)] | \tilde{p}(t), \tilde{q}(t) \rangle \; dt$$

$$= \int [-\tilde{q}(t)\dot{\tilde{p}}(t) + \dot{\tilde{G}}'(\tilde{p}(t), \tilde{q}(t)) - \tilde{H}(\tilde{p}(t), \tilde{q}(t))] \; dt$$

Affine Quantization (2)

Classical/Quantum connection:

$$\underline{H'(pq,q)} \equiv \langle p,q|\mathcal{H}'(D,Q)|p,q\rangle \quad , \quad [(Q-1)+iD/\tilde{\beta}]|\eta\rangle = 0$$

$$= \langle\eta|\mathcal{H}'(D+pqQ, qQ)|\eta\rangle = \underline{\mathcal{H}'(pq,q)} + \mathcal{O}(\hbar; p,q)$$

Physical meaning: $\quad [\,\langle\eta|D|\eta\rangle = 0 \;,\quad \langle\eta|Q|\eta\rangle = 1\,]$

$$\boxed{\langle p,q|D|p,q\rangle = pq \;;\quad \langle p,q|Q|p,q\rangle = q}$$

Fubini-Study metric:

$$2\hbar\;[\;\|\,d|p,q\rangle\|^2 - |\langle p,q|\,d\,|p,q\rangle|^2] = \underline{\tilde{\beta}^{-1}q^2dp^2 + \tilde{\beta}\,q^{-2}dq^2}$$

Poincare half plane: <u>geodesically complete</u>

The Q/C Connection : Summary

- ***The classical action arises by a restriction of the quantum action to coherent states***
- Canonical quantization uses P and Q which must be **self adjoint**
- Affine quantization uses D and Q which are **self adjoint** when $Q > 0$ (and/or $Q < 0$)
- ***Both canonical AND affine quantum versions are consistent with classical, canonical phase space variables p and q***

- ***Now for a few applications!***

Toy Model

- A toy model of gravity has singularities

$$A_C = \int_0^T [-q(t)\dot{p}(t) - q(t)p(t)^2]\, dt \ , \quad q(t) > 0$$

$$\dot{p}(t) = -p(t)^2$$

$$p(t) = p_0(1 + p_0 t)^{-1} \quad , \quad q(t) = q_0(1 + p_0 t)^2$$

- Canonical quantum corrections
- Affine quantum corrections
- *Affine quantization resolves singularities!*

Toy Model

Classical action.: $A_C = \int[-q\dot{p} - qp^2]dt \;;\qquad q > 0$

Solution : $p(t) = p_0(1+p_0 t)^{-1} \,,\qquad q(t) = q_0(1+p_0 t)^2$

Canonical quant.: $\langle p,q|PQP|p,q\rangle = qp^2 + qa^2 \,;\; a^2 = \hbar/2$

Solution : $p(t) = a\cot(a(t+\tau)) \,,\quad q(t) = (E_0/a^2)\sin(a(t+\tau))^2$

Affine quant.: $\langle p,q|DQ^{-1}D|p,q\rangle = qp^2 + \hbar^2 C/q$

Solution : $p(t) = \dfrac{(t+\tau)}{(t+\tau)^2 + K} \,,\quad q(t) = M[(t+\tau)^2 + K] > 0$

$$\hbar^2 C = \langle\eta|DQ^{-1}D|\eta\rangle \,;\; K = \hbar^2 C/4E_0^2 \,;\; M = 4E_0$$

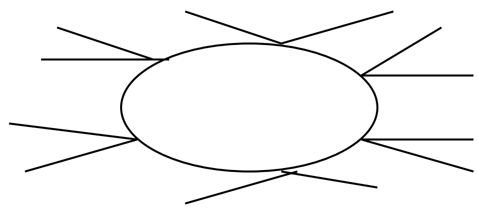
$$e^{iqP/\hbar} Q e^{-iqP/\hbar} = Q + q \;;\quad e^{i\ln(q)D/\hbar} Q e^{-i\ln(q)D/\hbar} = qQ$$

Ultralocal Models

- Ultralocal scalar field models $[x \in \mathbf{R}^s]$

$$A = \int \{ \tfrac{1}{2} [\dot{\varphi}(t,x)^2 - m_0^2 \varphi(t,x)^2] - g_0 \varphi(t,x)^4 \} dt\ dx$$

- Non-renormalizable quantum theory



$$(p_0^2 + \cancel{\vec{p}^2} + m_0^2)^{-1}$$

- Also a **trivial** (=free) theory
- ***Affine quantization is the key idea***
- ***But first, two important remarks***

Free & Pseudofree Theories - 1

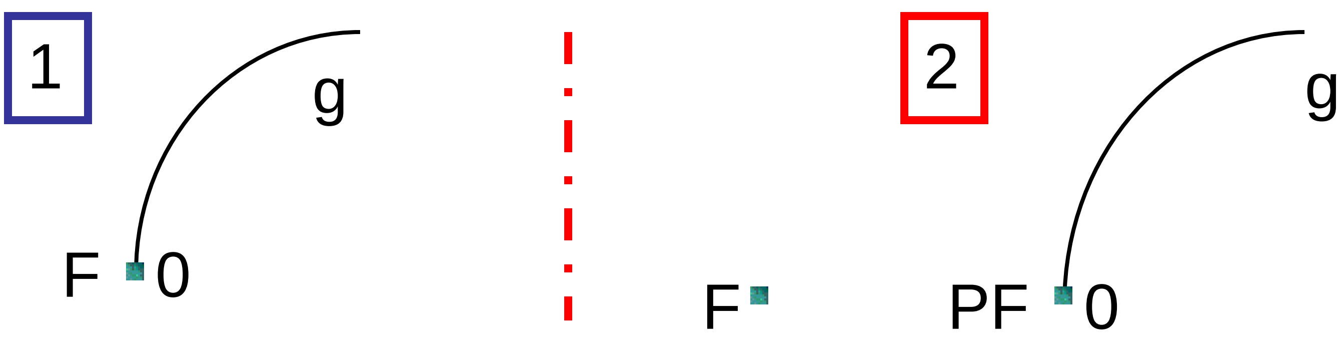
Classical action:

$$A_g(x) = A_0(x) + gA_I(x) \; ; \quad \lim_{g\to 0} A_g = ?$$

Example 1: $\quad A_g = \int \{ \tfrac{1}{2}[\dot{x}(t)^2 - x(t)^2] - gx(t)^4 \} \, dt$

Example 2: $\quad A_g = \int \{ \tfrac{1}{2}[\dot{x}(t)^2 - x(t)^2] - gx(t)^{-4} \} \, dt$

Moral: $\quad \lim_{g\to 0} A_g \equiv A'_0 \; ; \quad (1)\; A'_0 = A_0 \; , \quad (2)\; A'_0 \neq A_0$



Free & Pseudofree Theories - 2

Free action: $A_0 = \int \{ \tfrac{1}{2}[\dot{x}^2 - x^2]\} dt$

Interacting action: $A_g = \int \{ \tfrac{1}{2}[\dot{x}^2 - x^2] - g x^{-4}\} dt$

Pseudofree action: $\lim_{g\to 0} A_g \equiv A'_0 \neq A_0$

Free quantum propagator:

$$K_f(x'',T;x',0) = N_0 \int e^{-\int \{ \frac{1}{2}[\dot{x}^2 + x^2]\} dt} Dx$$

$$= \Sigma_{n=0}^{\infty} h_n(x'') h_n(x') e^{-(n+1/2)T}$$

Pseudofree quantum propagator:

$$K_{pf}(x'',T;x',0) = \lim_{g\to 0} N_g \int e^{-\int \{ \frac{1}{2}[\dot{x}^2 + x^2] + g x^{-4}\} dt} Dx$$

$$= \theta(x''x') \Sigma_{n=0}^{\infty} h_n(x'') h_n(x') [1 - (-1)^n] e^{-(n+1/2)T}$$

Ultralocal Scalar Models (1)

Classical action: $\quad [x \in \mathbf{R}^s]$

$$A = \int \{\pi\dot{\phi} - \tfrac{1}{2}[\pi^2 + m_0^2\phi^2] - g_0\phi^4\} dt\, dx$$

Free quantum model on a lattice:

$$C(f) = M \int e^{i\Sigma_k f_k \phi_k a^s - m_0 \Sigma_k \phi_k^2 a^s} \Pi_k d\phi_k \rightarrow e^{-\int f(x)^2 dx / 4m_0}$$

Interacting model on a lattice: **[C.L.T.]**

$$C(f) = M \int e^{i\Sigma_k f_k \phi_k a^s - \Sigma_k Y(\phi_k^2, a)} \Pi_k d\phi_k \rightarrow e^{-\int f(x)^2 dx / 4\mathbf{m}}$$

Non-renormalizable AND *Trivial*

Central Limit Theorem

$$C(h) = \text{"}\mathfrak{N}\int \exp\{i\int h(x)\varphi(x)\;dx - \int G(\varphi(x)^2)\;dx\}\;\Pi_x d\varphi(x)\text{"}$$

$$\tilde{C}_K(h) = \tilde{N}\int \exp\{\sum_{k=1}^{K}[ih_k\varphi_k\Delta - G(\varphi_k^2)\Delta]\}\prod_{k=1}^{K} d\varphi_k$$

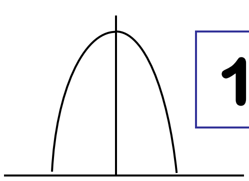
$$C_K(h) = N\prod_{k=1}^{K}\int \exp[ih_k u - G(\Delta^{-2}u^2)\Delta - F(\hbar,u^2,\Delta)]\;du$$

$$= \prod_{k=1}^{K}[1 - h_k^2\langle u^2\rangle/2! + h_k^4\langle u^4\rangle/4! - h_k^6\langle u^6\rangle/6! + \cdots]$$

Lim $K \to \infty$ requires $\langle u^2\rangle = O(\Delta)$, which can be done in TWO basic ways:

1) $\langle u^{2m}\rangle = O(\Delta^m)$, leads to $C(h) = \exp[-A\int dx\; h(x)^2]$ $(C.L.T.)$

2) $\langle u^{2m}\rangle = O(\Delta)$, leads to $C(h) = \exp(-\int dx \int \{1-\cos[uh(x)]\}\; W(u)\; du)$



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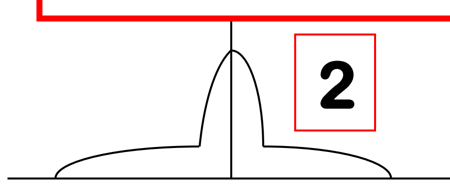

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Ultralocal Scalar Models (2)

Afine quantization: $[\hat{\phi}(x), \hat{\kappa}(y)] = i\delta(x-y)\hat{\phi}(x)$

$$\pi(x)^2 = \{\pi(x)\phi(x)\}\phi(x)^{-2}\{\phi(x)\pi(x)\} \equiv \kappa(x)\phi(x)^{-2}\kappa(x)$$

$$\rightarrow \hat{\kappa}(x)\hat{\phi}(x)^{-2}\hat{\kappa}(x) = \hat{\pi}(x)^2 + \tfrac{3}{4}\hbar^2\delta(0)^2\hat{\phi}(x)^{-2}$$

using $\hat{\kappa}(x) \equiv \tfrac{1}{2}[\hat{\pi}(x)\hat{\phi}(x) + \hat{\phi}(x)\hat{\pi}(x)]$

Lattice Hamiltonian: $F \equiv (\tfrac{1}{2} - ba^s)(\tfrac{3}{2} - ba^s)a^{-2s}$

$$\mathcal{H} = \tfrac{1}{2}\Sigma_k\{-\hbar^2 a^{-2s}\partial^2/\partial\phi_k^2 + m_0^2\phi_k^2 + 2g_0\phi_k^4 + \hbar^2 F\phi_k^{-2} - E_0\}a^s$$

includes a novel counter term !

Ultralocal Scalar Models (3)

Interacting model ground state distribution :

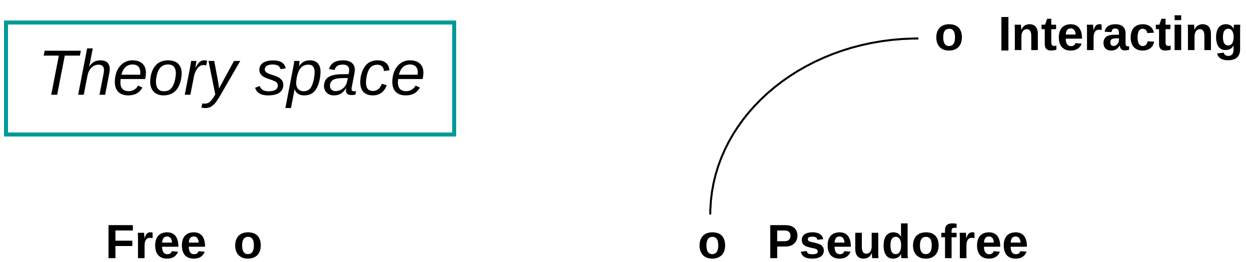
$$C(f)=\int\Pi_k(ba^s)\{e^{if_k\phi_k a^s-Y(\phi_k^2,b,a)}\,|\,\phi_k\,|^{-(1-2ba^s)}\;d\phi_k\}$$

$$\to e^{-b\int dx\int[1-\cos(f(x)\lambda)]\exp[-y(\lambda^2,b)]\;d\lambda/|\lambda|}$$

Pseudofree distribution : $\quad (g_0 \to 0)$

$$C_{pf}(f)=e^{-b\int dx\int[1-\cos(f(x)\lambda)]\exp[-bm\lambda^2]\;d\lambda/|\lambda|}$$

$2m\,,0$



Ultralocal Scalar Models (4)

$$A_g = \int\{\tfrac{1}{2}[\dot{\phi}(t,x)^2 - m_0^2\phi(t,x)^2] - g_0\phi(t,x)^4\}dt\, d^s x$$

Free: $I_p = M\int[\Sigma'_k\phi_k^2 a^s]^p e^{-m_0\Sigma'_k\phi_k^2 a^s}\Pi'_k d\phi_k = O(N'^p) \to \infty$

Hyper-spherical coordinates: $\phi_k = \kappa\eta_k\ ;\quad \kappa^2 = \Sigma'_k\phi_k^2$

$$I_p = M\int[\kappa^2 a^s]^p e^{-m_0\kappa^2 a^s}\kappa^{(N'-1)}d\kappa\, d\mu(\eta) = O(N'^p) \to \infty$$

Pseudofree: $\kappa^{(N'-1)} \to \kappa^{(R-1)}\ ;\quad R = 2ba^s N' < \infty$

$$J_p = M'\int[\Sigma'_k\phi_k^2 a^s]^p e^{-m_0\Sigma'_k\phi_k^2 a^s}\Pi'_k[\phi_k^2]^{-(1-2ba^s)/2}\Pi'_k d\phi_k < \infty$$

NO DIVERGENCES

Ultralocal Models : Summary

- Canonical quantization of interacting ultralocal scalar fields is ***perturbatively non-renormalizable*** and ***rigorously trivial***
- Affine quantization of interacting ultralocal scalar fields is ***rigorously nontrivial*** & ***NO*** divergences
- Ultralocal scalar models involve **discontinuous perturbations** for which interacting theories are continuously connected to a pseudfree theory and ***not*** to their own free theory
- *Hyper-spherical radius variable measure is a simple key to solution* : $\kappa^{N'-1} \to \kappa^{R-1}$; $R < \infty$

Rotationally Sym. Models

- Rotationally symmetric models $[\vec{p}^{\,2} \equiv \vec{p}\cdot\vec{p}]$

$$H(\vec{p},\vec{q}) = \tfrac{1}{2}[\vec{p}^{\,2} + m_0^2\vec{q}^{\,2}] + \lambda_0(\vec{q}^{\,2})^2 \quad , \qquad \vec{p} = \{p_n\}_{n=1}^{N}$$

- Free quantum models for $N \leq \infty$
- Interacting quantum models for $N < \infty$

$$H(\vec{p},\vec{q}) \equiv \langle \vec{p},\vec{q}|\mathcal{H}|\vec{p},\vec{q}\rangle$$

- ***Reducible operator representation is the key***

$$H(\vec{p},\vec{q}) \equiv \langle \vec{p},\vec{q}|\mathcal{H}(\vec{P},\vec{Q},\cdots)|\vec{p},\vec{q}\rangle$$

Rotationally Sym. Models (1)

$$\text{Phase space coordinates}: \ \vec{p} = (p_1, \ldots, p_N) \quad , \quad \vec{q} = (q_1, \ldots, q_N)$$

$$H(\vec{p}, \vec{q}) = \tfrac{1}{2}[\vec{p}^{\,2} + m_0^2 \vec{q}^{\,2}] + \lambda_0 (\vec{q}^{\,2})^2 \qquad ; \quad N \leq \infty$$

$$\text{Invariant under} \quad \vec{p} \to O\vec{p} \quad , \quad \vec{q} \to O\vec{q} \quad ; \quad O \in \mathrm{SO}(N, \mathrm{R})$$

$$\text{Basic invariants}: \ X \equiv \vec{p}^{\,2} \quad , \quad Y \equiv \vec{p} \cdot \vec{q} \quad , \quad Z \equiv \vec{q}^{\,2}$$

$$\text{Constants of motion}: \quad E \quad , \quad \vec{L}^2 = (\vec{p} \times \vec{q})^2 = XZ - Y^2$$

$$\text{Quantization}: \quad \vec{p} \to \vec{P} \quad , \quad \vec{q} \to \vec{Q} \quad ; \ [Q_j, P_k] = i\hbar \delta_{jk}$$

$$\text{Hamiltonian}: \quad \mathcal{H} = \tfrac{1}{2} : \vec{P}^2 + m_0^2 \vec{Q}^2 : + \lambda_0 : (\vec{Q}^2)^2 : \quad , \quad N < \infty$$

$$\textit{When} \quad N \to \infty \quad , \quad \textit{it is necessary that} \quad \lambda_0 = \lambda / N$$

Rotationally Sym. Models (2)

Schroedinger equation with a *real unique ground state*

$\varphi_N(\vec{x})$ with *full rotational symmetry*: $\psi_N(r) = \varphi_N(\vec{x})$.

Fourier transformation of the ground state distribution

$$C_N(\vec{p}) \equiv \int e^{i\vec{p}\cdot\vec{x}/\hbar}\varphi_N(\vec{x})^2 d\vec{x}$$

$$= \int e^{ipr\cos(\theta)/\hbar}\psi_N(r)^2 r^{N-1}\sin(\theta)^{N-2} dr\, d\theta\, d\Omega_{N-2}$$

$$\cong K_N \int e^{-p^2 r^2/2\hbar^2(N-2)}\psi_N(r)^2 r^{N-1} dr\, d\Omega_{N-2}$$

$$\to \int_0^\infty e^{-bp^2/2\hbar} f(b)\, db \quad ; \quad \left[\int_0^\infty f(b)\, db = 1\right]$$

sym

Uniqueness: $f(b) = \delta(b - 1/2\mathrm{m})$ **A free theory!**

Result: $C_\infty(\vec{p}) = e^{-p^2/4\mathrm{m}\hbar}$;

Magic Dots

$$\vdots\ e^{i(\alpha P+\beta Q)/\hbar}\ \vdots \equiv e^{i(\alpha P+\beta Q)/\hbar} \Big/ \left\langle \eta \middle| e^{i(\alpha P+\beta Q)/\hbar} \middle| \eta \right\rangle$$

$$\left\langle \eta \middle| \vdots\ e^{i(\alpha P+\beta Q)/\hbar}\ \vdots \middle| \eta \right\rangle = 1 \quad ; \qquad \left| p,q \right\rangle \equiv e^{i(pQ-qP)/\hbar} \left| \eta \right\rangle$$

$$\left\langle p,q \middle| \vdots\ e^{i(\alpha P+\beta Q)/\hbar}\ \vdots \middle| p,q \right\rangle \equiv \left\langle \eta \middle| \vdots\ e^{i(\alpha(P+p)+\beta(Q+q))/\hbar}\ \vdots \middle| \eta \right\rangle$$

$$\equiv e^{i(\alpha p+\beta q)/\hbar}$$

$$\left\langle p,q \middle| \vdots\ H(P,Q)\ \vdots \middle| p,q \right\rangle \equiv H(p,q)$$ form OR operator?

operator needs: $\left\langle p,q \middle| \{ \vdots\, H(P,Q)\, \vdots \}^2 \middle| p,q \right\rangle < \infty$

Rotationally Sym. Models (3)

$$(m_0\vec{Q}+i\vec{P})\;|0\rangle=0\;\;;\;\;|\vec{p},\vec{q}\rangle=\exp[i(\vec{p}\cdot\vec{Q}-\vec{q}\cdot\vec{P})/\hbar]\;|0\rangle$$

G

$$\langle\vec{p},\vec{q}|\tilde{\mathcal{H}}|\vec{p},\vec{q}\rangle=\langle\vec{p},\vec{q}|\{\tfrac{1}{2}:\vec{P}^2+m_0^2\vec{Q}^2:+w:(\vec{P}^2+m_0^2\vec{Q}^2)^2:\}|\vec{p},\vec{q}\rangle$$

$$=\tfrac{1}{2}(\vec{p}^{\,2}+m_0^2\vec{q}^{\,2})+w(\vec{p}^{\,2}+m_0^2\vec{q}^{\,2})^2\quad;\quad N\leq\infty$$

TEST

$$[m(\vec{Q}+\zeta\vec{S})+i\vec{P}]\;|0;\zeta\rangle=[m(\vec{S}+\zeta\vec{Q})+i\vec{R}]\;|0;\zeta\rangle=0\;\;;\;\;0<\zeta<1$$

G

$$|\vec{p},\vec{q};\zeta\rangle\equiv\exp[i(\vec{p}\cdot\vec{Q}-\vec{q}\cdot\vec{P})/\hbar]\;|0;\zeta\rangle$$

$$\langle\vec{p},\vec{q};\zeta|\mathcal{H}'|\vec{p},\vec{q};\zeta\rangle=\langle\vec{p},\vec{q};\zeta|\{\tfrac{1}{2}:\vec{P}^2+m^2(\vec{Q}+\zeta\vec{S})^2:$$

$$+\tfrac{1}{2}:\vec{R}^2+m^2(\vec{S}+\zeta\vec{Q})^2:+v:[\vec{R}^2+m^2(\vec{S}+\zeta\vec{Q})^2]^2:\}|\vec{p},\vec{q};\zeta\rangle$$

$$=\tfrac{1}{2}(\vec{p}^{\,2}+m^2\vec{q}^{\,2})+\tfrac{1}{2}\zeta^2m^2\vec{q}^{\,2}+v\zeta^4m^4(\vec{q}^{\,2})^2$$

$$\equiv\tfrac{1}{2}(\vec{p}^{\,2}+m_0^2\vec{q}^{\,2})+\lambda_0(\vec{q}^{\,2})^2\quad;\quad N\leq\infty$$

REAL

Rot. Sym. Models : Summary

- Conventional quantization works if N is finite but leads to ***triviality*** if N is infinite
- *Enhanced quantization applies even for* ***reducible*** *operator representations*
- Using the ***Weak Correspondence Principle*** $H(\vec{p},\vec{q}) = \langle \vec{p},\vec{q} | \mathcal{H} | \vec{p},\vec{q} \rangle$ a ***nontrivial*** quantization results if N is *finite* or N is *infinite* --- *with* ***NO*** *divergences !*
- *Class. & Quant. formalism is similar for all N*

Canonical *vs*. Enhanced

- Canonical quantization requires Cartesian coordinates, but ***WHY*** is not clear
- Canonical quantization works well for many problems, but ***NOT*** for all problems

---

- Enhanced quantization clarifies coordinate transformations and Cartesian coordinates
- Enhanced quantization can yield canonical results -- ***OR*** provide proper results when canonical quantization fails

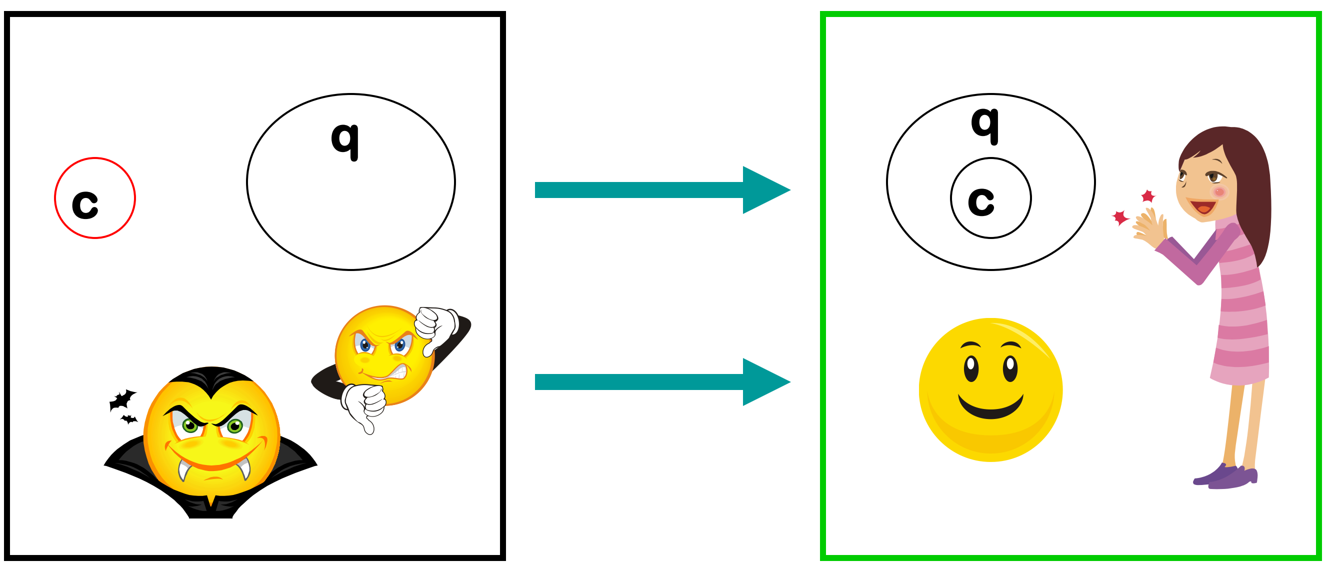
Other Enh. Quant. Projects

- Ultralocal and Rotationally Symmetric models completely solved with Enh. Quant. (*done*)
- Covariant scalar models φ_n^4 (*done*)
- Nonrenormalizable scalar fields (*done*)
- Simple models of affine quantization eliminating classical singularities (on going)
- Incorporating constrained systems within enhanced quantization (partially)
- Affine quantum gravity (partially)
- Extension to fermion fields (hints)

Main Message of Today

Class./Quan. Coexistence *IS*

Possible *AND IT IS* Beneficial



GOOD ***BETTER***

MERCI

Spin States & Enh. Quant.

$$\text{Spin coherent states:} \quad [S_1, S_2] = i\hbar S_3 \;\; ; \quad [\hbar > 0]$$

$$|\theta,\varphi\rangle = e^{-i\varphi S_3/\hbar} e^{-i\theta S_2/\hbar} |s,s\rangle \;\; ; \;\; S_3|s,m\rangle = m\hbar|s,m\rangle$$

$$A_R = \int \langle \theta(t),\varphi(t) | [i\hbar\partial/\partial t - \mathcal{H}(\mathbf{S})] | \theta(t),\varphi(t)\rangle \; dt$$

$$= \int [s\hbar \cos(\theta(t))\dot{\varphi}(t) - H(\theta(t),\varphi(t))] \; dt$$

$$= \int [p(t)\dot{q}(t) - H(p(t),q(t))] \; dt$$

$$\text{Fubini-Study metric:} \quad [\; p = (s\hbar)^{1/2}\cos(\theta) \;\; , \quad q = (s\hbar)^{1/2}\varphi \;]$$

$$d\sigma^2 = s\hbar[d\theta^2 + \sin(\theta)^2 d\varphi^2]$$

$$= (1 - p^2/s\hbar)^{-1} dp^2 + (1 - p^2/s\hbar) \; dq^2$$

References - 1

- ``Enhanced Quantization: A Primer'', *J. Phys. A: Math. Theor.* **45**, 285304 (8pp) (2012); arXiv:1204.2870
- ``Enhanced Quantum Procedures that Resolve Difficult Problems''; *Rev. Math. Physics* 27, 1530002 (43pp) (2015); arXiv: 1206.4017
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- ``Divergences in Scalar Quantum Field Theory: The Cause and the Cure'', *Mod. Phys. Lett. A* **27**, 1250117 (9pp) (2012); arXiv:1112.0803
- Ultralocal Model Scalar Quantum Fields: "*Beyond Conventional Quantization*" (Cambridge, 2000 & 2005)
- Covariant Scalar Models and Affine Quantum Gravity, "*Enhanced Quantization: Particles, Fields & Gravity*" (World Scientific, 2015)

"This is an interesting book that details a genuinely new approach to quantisation that has considerable technical advantages and also many interesting conceptual implications. The bibliography is extensive."

Mathematical Reviews Clippings

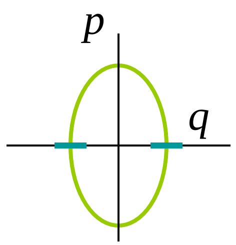
Does [Q,R]=0? (*No!*)

$$I = \int [p\dot{q} - \lambda(p^2 + q^4 - E)]dt$$

What values of (positive) E are allowed?

$$[(P^2 + Q^4) - E]\, |\psi\rangle = 0 \quad , \qquad \{E_n\}$$

$$\int \delta\{p^2 + q^4 - E\}\ e^{i\int p\dot{q}dt} DpDq$$



$$\int \delta\{p\}\ \Pi(4q^3)\ \delta\{p^2 + q^4 - E\}\ e^{i\int p\dot{q}dt} DpDq$$

$$= \int \Pi(4q^3)\ \delta\{q^4 - E\}\ Dq$$

$$= 1 \text{ or } 0 \quad , \qquad \textit{independent of } E\ !$$

Positive *or* Non-negative ?

$$\int_{q>0} e^{i\int[p\dot{q}-qp^2]dt} DpDq$$

$$= \int_{q>0} e^{i\int[\dot{q}^2/4q]dt} \Pi_t dq/\sqrt{2q}$$

Let $u^2 = 2q \;\;,\;\; du = dq/\sqrt{2q} \;\;,\;\; \dot{u}^2 = \dot{q}^2/2q$

$$= \int_{u\neq 0} e^{i\int \dot{u}^2/2dt} Du \quad (\,forget \;\; u=0)$$

$$= \int e^{i\int \dot{u}^2/2dt} Du = free\;\; particle!$$ Wrong answer!

$$= \int e^{i\int(\dot{u}_1^2+\dot{u}_2^2)/2dt} Du_1 Du_2 d\theta_{12}/2\pi$$ Right answer!

Enhanced Quantization Manifesto

Enhanced Quantization: Particles, Fields & Gravity, World Scientific, 2015

arXiv: 1206.4017

1204.2870 (basics)
1112.0803 (scalar fields)
1203.0691 (quantum gravity)
1308.4658 (overview)
1312.0814 (matrix models)