Complex Hermite polynomials - coherent states

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What should (has to) be done in the future?

„Polynomials” $H_{m,n}(z,\overline{z})$: mathematical properties

(**polynomials of two real variables with complex coefficients** called: *two index Hermite polynomials* (Hong-yi Fan&J.R.Klauder ;1994, V.Dodonov&V.I.Manko ;1994),*incomplete Hermite polynomials* (G.Dattoli; 2000-), *2D Hermite polynomials* (G.Dattoli, A.Ghanmi, M.Ismail; 2005-), *2D Laguerre polynomials* (A.Wünsche;1999-))

$$H_{m,n}(z,\overline{z}) = \sum_{l=0}^{\min\{m,n\}} \frac{m!n!}{l!(m-l)!(n-l)!}(-1)^l z^{m-l}\overline{z}^{n-l}$$

$$\exp(-st+zs+\overline{z}t) = \sum_{m,n=0}^{\infty} \frac{s^m t^n}{m!n!} H_{m,n}(z,\overline{z})$$

$$\frac{1}{\pi}\int_C dz \exp(-|z|^2) H_{m,n}(z,\overline{z})\overline{H}_{p,q}(z,\overline{z}) = m!n!\delta_{mp}\delta_{nq}$$

$$\frac{1}{\pi}\sum_{m,n}\frac{\exp(-|z|^2)H_{m,n}(z,\overline{z})\overline{H}_{m,n}(z',\overline{z}')}{m!n!} = \delta^{(2)}\left(z-\overline{z}'\right)$$

This will create a problem

„Polynomials" $H_{m,n}(z,\bar{z})$: physical applications I
(Hong-yi Fan, J.R.Klauder 1994)

$$|z\rangle_{F-K} \equiv \exp\left[-\frac{|z|^2}{2} + z\hat{a}_{(+)} - \bar{z}\hat{b}_{(+)} + \hat{a}_{(+)}\hat{b}_{(+)}\right]|00\rangle =$$

$$= \exp\left[-\frac{|z|^2}{2}\right]\sum_{m,n=0}^{\infty}\frac{H_{m,n}(z,\bar{z})}{\sqrt{m!n!}}|mn\rangle$$

$\hat{a}_{(+)}, \hat{b}_{(+)}$

standard boson creation operators for a bipartite system

Example of entangled states because:
- Not product states
- EPR states – eigenstates of relative coordinate and total momentum in a bipartite system

Not a coherent state because the latter have to be <u>normalizable</u> and are <u>never orthogonal</u> while we have

$${}_{F-K}\langle z' | z\rangle_{F-K} = \delta^{(2)}\left(z', z\right)$$

„Polynomials" $H_{m,n}(z,\bar{z})$: physical applications II
A „trick" leading to well defined coherent states

(N.Cotfas, J.-P. Gazeau, K. Górska; 2010)

A subset of $H_{m,n}(z,\bar{z})$, namely

$$H_{s+n,s}(z,\bar{z}) = s!(s+n)!\bar{z}^n \sum_{k=0}^{s} \frac{(-1)^{s-k} |z|^{2k}}{(s-k)!k!(n-k)!} =$$

$$= (-1)^s s!\bar{z}^n L_s^{(n)}(|z|^2)$$

with s fixed, is orthogonal with respect to the gaussian measure

$$\frac{1}{\pi}\int_C d^2 z e^{-|z|^2} H_{s+n,s}(z,\bar{z})\bar{H}_{s+m,s}(z,\bar{z}) = s!(s+n)!\delta_{nm},$$

and satisfies normalizability condition

$$\sum_{n=0}^{\infty} \frac{H_{s+n,s}(z,\bar{z})\bar{H}_{s+n,s}(z,\bar{z})}{s!(s+n)!} < \infty$$

This means that $\left\{ e^{-|z|^2/2} \frac{H_{s+n,s}(z,\bar{z})}{\sqrt{s!(s+n)!}} \right\}_{n=0}^{\infty}$

may be used for a construction of coherent states (and quantization).

Holomorphic Hermite polynomials $H_n(z)$:

(van Eijndhoven & Meyers, 1991)

Complex version of standard Hermite polynomials

$$H_n(z) = n!\sum_{k=0}^{[n/2]} \frac{(-1)^k (2z)^{n-2k}}{k!(n-2k)!} \qquad z = x+iy$$

1.Orthogonality with respect to a family of measures depending on $\alpha \in (0,1)$

$$\int_{R^2} H_n(x+iy)\bar{H}_n(x+iy) e^{-(1-\alpha)x^2-\left(\frac{1}{\alpha}-1\right)y^2} dxdy = \frac{\pi\sqrt{\alpha}}{1-\alpha}\left(2\frac{1+\alpha}{1-\alpha}\right)^n n!\delta_{nm,}$$

2.Functions

$$h_n^{(\alpha)}(z) = \left(\frac{1-\alpha}{\pi\sqrt{\alpha}}\right)^{1/2}\left(\frac{1-\alpha}{1+\alpha}\right)^{n/2}\frac{e^{-z^2}H_n(z)}{\sqrt{2n!}}$$

are an orthonormal basis in the Hilbert space(s) Ω of entire functions

$$\int_{R^2} dxdy e^{\alpha x^2-\frac{1}{\alpha}y^2} |f(x+iy)|^2 < \infty$$

which are Reproducing Kernel Hilbert Space(s) (RKHS) with the reproducing kernel(s)

$$K^{(\alpha)}(z,w) = \sum_n h_n^{(\alpha)}(z)\bar{h}_n^{(\alpha)}(w) = \frac{1-\alpha^2}{2\pi\alpha}\exp\left[-\frac{1+\alpha^2}{4\alpha}\left(z^2+w^2\right)+\frac{1-\alpha^2}{2\alpha}z\bar{w}\right]$$

What we have got is completely analogous to the „classical" Bargmann's approach for which monomials $\frac{z^n}{\sqrt{n!}}$ are orthonormal basis in the Hilbert space of functions integrable with respect to the gaussian measure and which is a Reproducing Kernel Hilbert Space with the Bargmann's reproducing kernel $K(z,w)=e^{z\bar{w}}$

What about coherent states in the situation just considered?

Holomorphic Hermite polynomials $H_n(z)$ coherent states
(J.-P. Gazeau and F.H.Szafraniec, 2012)

$$|z;\alpha\rangle = \frac{1}{\sqrt{N^{(\alpha)}(z)}} \sum_{n=0}^{\infty} \overline{h_n^{(\alpha)}} e_n$$

$$N^{(\alpha)}(z) = K^{(\alpha)}(z,z) = \sum_{n=0}^{\infty} \overline{h_n^{(\alpha)}}(z) h_n^{(\alpha)}(z) = \frac{\alpha^{-1}-\alpha}{2\pi} \exp\left[-\alpha x^2 + \alpha^{-1} y^2\right]$$

J.-P.G and FHSz used these states for the coherent states quantization of 2D non-commuttative harmonic oscillator but one can ask a question : *what are their properties and (physical) meaning?*

Coherent states built using holomorphic Hermite polynomials $H_n^{\alpha}(z)$ are squeezed states!

(S.T.Ali, K. Górska, A.Horzela and F.H.Szafraniec, 2014)

Intuition:

1. States of the form $|z;\alpha\rangle = \frac{1}{\sqrt{N^{(\alpha)}(z)}} \sum_{n=0}^{\infty} h_n^{(\alpha)}(z)\; e_n =$

$$\left(\frac{1-\alpha}{\pi\sqrt{\alpha}\,N^{(\alpha)}(z)}\right)^{1/2} \sum_{n=0}^{\infty} \left(\frac{1-\alpha}{1+\alpha}\right)^{n/2} \frac{e^{-z^2} H_n(z)}{\sqrt{2n!}}\; e_n \qquad (*)$$

have all the properties of coherent states - this is implied by the RKHS approach!

2. States (*) should have „something common" with squeezed states because
   2a. van Eijndhoven-Meyers measure is not rotationally invariant
   2b. the long-time known operational formula gives

$$\exp\left[-a\partial_z^2\right] z^n = a^{n/2} H_n\left(\frac{z}{\sqrt{a}}\right)$$

Squeezing operator

$$S(\xi) = \exp\left[\frac{1}{2}\left(\xi z^2 - \bar{\xi}\partial_z^2\right)\right]$$

The main results of the AGHSz paper reads:

For any $\xi \neq 0$ the squeezed basis vectors $e_n^{(\xi)}(z)$ coincide with the holomorphic Hermite polynomials multiplied by an exponential factor

$$e_n^{(\xi)}(z) = S(\xi)e_n(z) = \frac{\left(1-|\zeta|^2\right)^{1/4}\bar{\zeta}^{n/2}}{\sqrt{2^n n!}} e^{-z^2} H_n\left(\sqrt{\frac{1-|\zeta|^2}{2\bar{\zeta}}} z\right)$$

$$\zeta = \tanh(|\zeta|)\frac{\zeta}{|\zeta|} \in (-1,0)\cup(0,1)$$

which fully confirms intuitive arguments listed on the previous slide.

Holomorphic Hermite polynomials in two complex variables

$$H_{m,n}(z_1,z_2)=m!n!\sum_{k=0}^{\min(m,n)}\frac{(-1)^k z_1^{\,m-k} z_2^{\,n-k}}{k!(m-k)!(n-k)!}$$

$$z_1=x_1+iy_1$$
$$z_2=x_2+iy_2$$

whose generating function is

$$\exp(-st+z_1s+z_2t)=\sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}H_{m,n}(z_1,z_2)$$

may be considered as a complex generalization of the Fan-Klauder „polynomials"

$$H_{m,n}(z,\bar{z})=m!n!\sum_{l=0}^{\min\{m,n\}}\frac{(-1)^l z^{m-l}\bar{z}^{n-l}}{l!(m-l)!(n-l)!}$$

What are the properties of $H_{m,n}(z_1,z_2)$?

Van Eijndhoven-Meyers construction may be repeated step by step – we get

1.Orthogonality with respect to the family of $\alpha \in (0,1)$ dependent measures

$$\int_{C^2} H_{m,n}(z_1,z_2)\bar{H}_{p,q}(z_1,z_2)e^{-\frac{(1-\alpha)}{4}|\bar{z}_2+z_1|^2-\frac{1-\alpha}{4\alpha}|\bar{z}_2{}^2-z_1^2|}dz_1dz_2 = \frac{\pi\alpha}{(1-\alpha)^2}\left(\frac{1+\alpha}{1-\alpha}\right)^n m!n!\delta_{mp}\delta_{np}$$

2. We may introduce Hilbert space(s) of entire functions with bases

$$h^{(\alpha)}_{m,n}(z_1,z_2) = \left(\frac{1-\alpha}{\pi\sqrt{\alpha}}\right)\left(\frac{1-\alpha}{1+\alpha}\right)^{(m+n)/2}\frac{e^{-z_1z_2}H_{m,n}(z_1,z_2)}{\sqrt{m!n!}}$$

3. Reproducing kernel may be calculated explicitly for $\forall\alpha \in (0,1)$

$$K^{(\alpha)}(z_1,z_2,w_1,w_2) = \left(\frac{1-\alpha^2}{2\pi\alpha}\right)^2 \exp\left[-\frac{1+\alpha^2}{4\alpha}\left(z_1z_2+\bar{w}_1\bar{w}_2\right)+\frac{1-\alpha^2}{4\alpha}\left(z_1\bar{w}_1+z_2\bar{w}_2\right)\right]$$

Conlusion from our „mathematical work”

We have all „components” needed to construct coherent states using the RKHS approach: such obtained (bipartite) coherent states will satisfy the Gazeau-Klauder conditions:

-- continuity,

-- normalizabilty

-- resolution of unity,

moreover they lead to the unitary (generalized) Segal-Bargmann transform.

Observation:

Polynomials $H_{m,n}(z_1, z_2)$,as well as $H_{m,n}(z\ ,\bar{z}\ )$, cannot be represented in the product form – it can be deduced from their generating functions which, in both cases, do not factorize; it is also seen from the relation

$$H_{m,n}(z_1, z_2) =$$

Holomorphic Hermite polynomials

$$= 2^{-(m+n)} \sum_{k=0}^{m} \sum_{l=0}^{n} \frac{m!n!i^{m-k}(-i)^{n-l}}{k!(m-k)!l!(n-l)!} H_{k+l}\left(\frac{z_1 + z_2}{2}\right) H_{m+n-k-l}\left(\frac{z_1 - z_2}{2}\right)$$

The same statement is valid for functions $h_{m,n}^{(\alpha)}(z_1, z_2)$- so, constructing coherent states according to the RKHS recipe

$$|z_{1,}, z_2; \alpha\rangle = N^{-1/2}(z_{1,}, z_2) \sum_{m,n} h_{m,n}^{(\alpha)}(z_1, z_2)\ e_{mn}$$

we get states which exhibit both entanglement and squeezing.
In what follows we assume

$$|z_{1,}, z_2; \alpha\rangle = N^{-1/2}(z_{1,}, z_2) \sum_{m,n} h_{m,n}^{(\alpha)}(z_1, z_2)\ e_m \otimes f_n$$

What was said up to now is valid for $\alpha \in (0,1)$, what about the limits for $\alpha \to 1$ and $\alpha \to 0$?

Observation/remark: Orthogonality relations for functions

$$h_{m,n}^{(\alpha)}(z_1, z_2) = \left(\frac{1-\alpha}{\pi\sqrt{\alpha}}\right) \left(\frac{1-\alpha}{1+\alpha}\right)^{(m+n)/2} \frac{e^{-z_1 z_2} H_{m,n}(z_1, z_2)}{\sqrt{m!n!}}$$

become:

a) product of orthogonality relations for monomials $z_1{}^m / \sqrt{m!}$ and $z_2{}^n / \sqrt{n!}$ for $\alpha \to 1$, i.e. we end up with the 2D Bargmann case

b) orthogonality relation for $h_{m,n}(z, \overline{z})$ (Fan-Klauder 's EPR states) for $\alpha \to 0$

Theorem:

Bipartite coherent states

$$|\, z_{1,}, z_2; \alpha\rangle = N^{-1/2}(z_{1,}, z_2) \sum_{m,n} h_{m,n}^{(\alpha)}(z_1, z_2)\; e_m \otimes f_n$$

have the limits being:

a)product of two standard coherent states for $\alpha \to 1$ (in a weak sense)

b)entangled Fan and Klauder 's EPR states of for $\alpha \to 0$

What to do next ?

It’s needed to look for general properies of $|z_{1,}, z_2; \alpha\rangle$,
namely to answer the (standard) questions:

1. are they „*anihilation operator coherent states*” (AOCS)?
2. are they „minimal uncertainty coherent states” (MUCS)?
3. is it possible to generate them as „group theoretical coherent states”?
4. how to understand their meaning as entangled coherent states

and to study (probably) many their other properties, especially applicability in (real) physical problems

Partial answers known,
work in progress

Thank for your attention

$$|z\rangle_{F-K} \equiv \exp\left[-\frac{|z|^2}{2} + z\hat{a}_{(+)} - \bar{z}\hat{b}_{(+)} + \hat{a}_{(+)}\hat{b}_{(+)}\right]|00\rangle =$$

$$= \exp\left[-\frac{|z|^2}{2}\right]\sum_{m,n=0}^{\infty}\frac{H_{m,n}(z,\bar{z})}{\sqrt{m!n!}}|mn\rangle$$

Coherent states - three standard approaches

1.Anihilation operator coherent states (AOCS)

$$a||z\rangle\rangle = z||z\rangle\rangle$$

$$a\,|\,n\rangle = \sqrt{[n]}\,|\,n-1\rangle, \quad a^{+}\,|\,n\rangle = \sqrt{[n+1]}\,|\,n+1\rangle,$$

$$a\big|z\big\rangle = z\big|z\big\rangle \qquad [a,a^{+}] = [n+1]-[n]$$

2. Group theoretical coherent states (GTCS)

$$|\,\zeta\rangle = T(g_{\zeta})\,|\,0\rangle$$

$T(g_{\zeta})$ is an element of a group G (dynamical or in general arbitrary Lie group) parametrized by ζ

3.Minimal uncertainty coherent states (MUCS)

$$\sigma^{2}_{(z)}(x)\sigma^{2}_{(z)}(p) = \frac{\hbar^{2}}{4}$$

in general $\sigma^{2}_{(z)}(x) \neq \sigma^{2}_{(z)}(p)$

Harmonic oscillator – all approaches are equivalent

AOCS $$a|z\rangle = z|z\rangle, \quad a\,|\,n\rangle = \sqrt{n}\,|\,n-1\rangle,\; a^{+}\,|\,n\rangle = \sqrt{n+1}\,|\,n+1\rangle,$$
$$[a, a^{+}] = 1$$

GTCS $$|\,z\rangle = D(z)\,|\,0\rangle, \qquad D(z) = \exp(za^{+} - \bar{z}a)$$

MUCS $$\sigma^{2}_{(z)}(x)\sigma^{2}_{(z)}(p) = \frac{\hbar^{2}}{4}, \qquad \sigma^{2}_{(z)}(x) = \sigma^{2}_{(z)}(p)$$

give CS $$|\,z\rangle = e^{-\frac{1}{2}|z|^{2}} \sum_{n} \frac{z^{n}}{\sqrt{n!}} |n\rangle = e^{-\frac{1}{2}|z|^{2}} \sum_{n} \frac{z^{n}}{n!} (a^{+})^{n}\,|\,0\rangle,$$

which, *a posteriori*, satisfy a relation called the <u>*resolution of unity*</u>

$$\frac{1}{\pi}\int_{C} d(\mathrm{Re}\, z)d(\mathrm{Im}\, z)\,|\,z\rangle\langle z\,| = \sum_{n} |\,n\rangle\langle n\,| = I_{H}$$

The resolution of unity as a fundamental property of HO CS (*in fact introduced independently of them*)

1. „Continuous representation of QM” (J.R.Klauder 1963)
   i) normalizability, ii) continuity, iii) resolution of unity

2. „Analytic representation of QM” (V.Bargmann 1961)

2a) Reproducing property for Bargmann funtions

$$|f\rangle = \sum_n f_n |n\rangle \Rightarrow f_B(z) = e^{\frac{1}{2}|z|^2} \langle \bar{z} | f \rangle = \sum_n \frac{z^n}{\sqrt{n!}} f_n$$

Unitary Bargmann-Segal transform,

$$\frac{1}{\pi} \int_C d(\mathrm{Re}\, z') d(\mathrm{Im}\, z') K(z, z') f_B(z') = f_B(z)$$

$$K(z, z') = e^{\bar{z} z' - |z'|^2} = e^{-\frac{1}{2}\left(|z|^2 - |z'|^2\right)} \langle z | z' \rangle$$

2b) Mapping between Bargmann's and usual „x" representations

$$\langle x \mid f\rangle \equiv f(x) = \frac{1}{\pi}\int_C d(\operatorname{Re} z)\, d(\operatorname{Im} z) K_1(x,z) f_B(\bar{z})$$

$$K_1(x,z) = e^{-|z|^2/2} \sum_n \frac{z^n}{\sqrt{n!}} \langle x \mid n\rangle$$

3. Coherent states quantization (E.Lieb 1975, F.Berezin 1977)

$$f(q,p) \Rightarrow A_f = \frac{1}{\pi}\int_C d(\operatorname{Re} z)\, d(\operatorname{Im} z) f(z,\bar{z}) \mid z\rangle\langle z \mid$$

$$q = \frac{1}{\sqrt{2}}(z+\bar{z}), \qquad p = \frac{1}{\sqrt{2i}}(z-\bar{z})$$

Satisfying the resolution of unity has been considered so important property of coherent states that it has been proposed to make it , *a priori,* basic requirement put on any set of coherent states generalizing those of the harmonic oscillator.

(J.R.Klauder 1963, J-P.Gazeau & J.R.Klauder 1999, J-P.Gazeau 2009)

How to solve the resolution of unity for generalized coherent states ?

The answer is: One has to find a measure such that

$$\int \mu(dz)\,|\,z\rangle_{gen\ gen}\langle z\,| = I_H \qquad (*)$$

-<u>for GTCS </u>it is satisfied under natural conditions (unitarity, irreducibility, square integrability of the representation)

-<u>for other constructions</u>, generally ending up at wave-packet type expressions

$$|\,z\rangle = N^{-\frac{1}{2}}(|\,z\,|^2)\sum_n \varphi_n(z)\big|n\big\rangle \quad \text{e.g.} \quad |\,z\rangle = N^{-\frac{1}{2}}(|\,z\,|^2)\sum_n \frac{z^n}{\sqrt{\rho_n}}\big|n\big\rangle$$

condition (*) has to be checked each time.But even if one can show that a suitable measure exists, it is <u>neither easy to find it nor to answer the question how many solutions are admissible</u>!

Examples:

1. For harmonic oscillator CS the solution

$$\mu(dz) = \frac{1}{\pi} d(\mathrm{Re}\, z) d(\mathrm{Im}\, z)$$

is unique under the additional assumption that the measure is rotationally invariant. If it is not the case we have infinitely many (discrete) measures concentrated on sufficiently dense von Neumann lattices or another discrete subsets of the complex plane (Bargmann et al. 1971, M.Boon & J.Zak 1978, J.Zak 2003, A.Vourdas 2003, A.Vourdas et al. 2012)

2. For generalized CS of the type

$$|z\rangle = N^{-\frac{1}{2}}(|z|^2) \sum_n \frac{z^n}{\sqrt{\rho_n}} |n\rangle$$

the rotationally invariant solution is unique if the Stieltjes moment problem

$$\rho_n = \int_0^\infty dy \frac{W(y)}{N(y)} y^n, \qquad \mu(dz) = \frac{1}{\pi} W(|z|^2) d(\mathrm{Re}\, z) d(\mathrm{Im}\, z),$$

$$W(|z|^2) \geq 0$$

is solvable and determinate

Non-unique solutions

$$\rho_n = \Gamma\left(\alpha n + \beta\right) \qquad |\varepsilon| < 1, \quad m = \pm 1, \pm 2, \ldots, \quad |m| < \alpha/2$$

$$\frac{W(y)}{N(y)} = \frac{y^{\frac{\beta-\alpha}{\alpha}} e^{-y^{\frac{1}{\alpha}}}}{\alpha\Gamma\left(\beta\right)} \times \left[1 + \varepsilon \sin\left(\pi m\left(1 - \frac{\alpha}{\beta}\right) + y^{\frac{1}{\alpha}} \tan\left(\frac{\pi m}{\alpha}\right)\right)\right]$$

$$\rho_n = q^{-\alpha n^2 - \beta n} \qquad 0 < q < 1, \quad \alpha > 0$$

$$\frac{W(y)}{N(y)} = \frac{q^{\frac{\beta^2}{4\alpha}}}{2\sqrt{\pi\alpha \ln\left(1/q\right)}} y^{\frac{\beta - 2\alpha}{2\alpha}} e^{-\frac{\ln^2(y)}{4\alpha \ln(1/q)}} \times \qquad |\varepsilon| < 1$$

$$\times\left[1 + \varepsilon \sin\left(\frac{\pi m\left(2\alpha - \beta\right)}{2\alpha} + \frac{\pi m \ln(y)}{2\alpha \ln(1/q)}\right)\right]$$

J-M.Sixdieniers, K.A.Penson & A.Solomon 1999, K.A.Penson et al.2010

Does it exist an alternative approach being (as long as possible) free of introducing the measure and suitable $L^2(\mu)$ space from the very beginning?

Yes, it does if , as starting point, one takes the reproducing property and tries to formulate the problem using (consequently) theory of the reproducing kernel Hilbert spaces (RKHS).

F.H.Szafraniec, *Przestrzenie Hilberta z jądrem reprodukującym, Kraków 2004*
F.H.Szafraniec, Operator Theory:Advances and Applications, 114 (2000)253-263

The reproducing kernel Hilbert space-RKHS

Let X be a set. Suppose we are given a couple (H, K) where H is a Hilbert space of complex functions (with inner product denoted by $\langle \cdot, - \rangle$ and K is a complex function on $X \times X$. The function K is called a reproducing kernel of H and the space H a reproducing kernel Hilbert space with respect to K if

$$K_x \in H \qquad x \in X,\ f \in H \qquad (*)$$
$$f(x) = \langle f, K_x \rangle$$

where $K_x = K(\cdot, x)$ is called a kernel function. Formulae $(*)$ appeal to as a reproducing kernel property of the couple (H, K)

General properties of the RKHS (I)

(i) the kernel K must be necessarily positive definite

$$\sum_{i,j=0}^{N} K(x_i, x_j)\lambda_i \overline{\lambda}_j \geq 0 \qquad x_1, \ldots x_N \in X, \qquad \lambda_1, \ldots \lambda_N \in C$$

(ii) the linear functionals

$$\varphi_x : H \ni f \mapsto f\left(x\right) \in C$$

are continuous for any $x \in X$

General properties of the RKHS (II)

(iii) the kernel is uniquely determined by a space in the sense that if (H, K_1) and (H, K_2) are two RKHS couples then $K_1 = K_2$.
The kernel may be obtained from the formula

$$K(x,y) = \sum_{\alpha \in A} f_\alpha(x)\overline{f_\alpha(y)} \qquad x, y \in X$$

where $\{f_\alpha\}_{\alpha \in A}$ is any orthonormal basis of H.

(iv) the set $\{K_x; x \in X\}$ is total in H and, consequently, the space is uniquely determined by the kernel in the sense that if (H_1, K) and (H_2, K) are RKHS couples then $H_1 = H_2$.

Standard RKHS constructions

A. „*Coupling the kernel with a functional space*"

Suppose we are given a Hilbert space H of functions on X .If the linear functional $\phi_x : H \ni f \to f(x) \in C$ is continuous for any $x \in X$ then $K_H(x, y) = \phi_y^* \phi_x$ where ϕ_y^* stands for the adjoint of the operator ϕ_x becomes a kernel of H .

B. „*Coupling the functional space with a kernel*"

Suppose we are given a kernel $K : X \times X \to C$ and let us set $D_K = lin\{K_x : x \in X\}$.If K is positive definite then $\langle K_x | K_y \rangle = K(y, x)$ defines an inner product in D_K and the completion H_K of D_K can be still realized as a space of funtions. The resulting space H_K is a RKHS with the kernel K .

We shall follow the construction „B” – suppose that we are given a family $\{ f_\alpha \}_{\alpha \in A}$ of functions such that

$$\sum_{\alpha \in A} | f_\alpha(x) |^2 < \infty, x \in X$$

The kernel $K^f(x, y)$ defined by

$$K^f(x, y) = \sum_{\alpha \in A} f_\alpha(x) \overline{f_\alpha(y)} \qquad x, y \in X$$

is positive definite and following „B” results in the RKHS H^f.

Theorem 1

Let $\xi = \{\xi_\alpha\}_{\alpha \in A} \in l^2(A)$. Then, for every $x \in X$,

the series $\sum_{x \in A} \xi_\alpha f_\alpha(x)$ converges absolutely and

the function $f_\alpha : x \to \sum_{\alpha \in A} \xi_\alpha f_\alpha(x)$ is in H^f;

the series $\sum_{\alpha \in A} \xi_\alpha f_\alpha$ converges in H^f to f_ξ .

In particuliar, any function $f_\alpha, \alpha \in A$ belongs to H^f

and $\sum_{\alpha \in A} \overline{f_\alpha(x)} f_\alpha$ converges in H^f to K^f.

Theorem 2

The family $f = \{ f_\alpha \}_{\alpha \in A}$ is always complete in H^f.
Moreover, the following conditions are equivalent:

(i)
$\xi \in l^2(A)$ and $\sum_{\alpha \in A} \xi_\alpha f_\alpha(x) = 0$ for all $x \in X$ implies $\xi = 0$

(ii)
$\{ f_\alpha \}_{\alpha \in A}$ is orthonormal in H^f .

General construction

Step 1

Let's take:

-a separable Hilbert space H with fixed orthonormal basis $(e_n)_{n=0}^{d}$, $d = \dim H$,

--a sequence $(\Phi_n)_{n=0}^{d}$ of complex functions on X satisfying the conditions:

$$\sum_n |\Phi_n(x)|^2 < \infty \qquad x \in X$$

$(\xi)_{n=0}^{d} \in l^2(A)$ and $\sum_n \xi_n \Phi_n(x) = 0$ for all $x \in X$ implies $\xi = 0$.

Recalling „B" : $K(x, y) = \sum_n \Phi_n(y)\overline{\Phi}_n(x)$

Step 2

If $K(x, x) \neq 0$, $x \in X$, define a (prospectively coherent) state

$$|x\rangle = \sum_n K(x, x)^{-1/2}\overline{\Phi_n}(x) e_n$$

Step 3

Putting $K(x,x)=1$ does not change requirements of the Step1 but simplifies the Step 2

$$|x\rangle \Rightarrow c_x = \sum_n \overline{\Phi_n}(x)\, e_n$$

(Functions $\Phi(x)/\sum_n |\Phi(x)|^2$ satisfy Step 1 as well.)

Step 4

Because for any $h \in H$ the sequence $\{\langle e_n | h\rangle_H\}_{n=0}^{d} \in l^2$ the series $Bh = \sum_n \Phi_n \langle e_n | h\rangle$ converges in the H_K, we can write

$$H_K \ni (Bh)(x) = \sum_n \Phi_n(x) \langle e_n | h\rangle = \langle c_x | h\rangle_H$$

Analogue of the Bargmann-Segal transform

Step 5

$$\langle Bh \mid Bg \rangle_{H_K} = \sum_{n,m} \langle \Phi_n \mid \Phi_m \rangle_{H_K} \langle e_m \mid h \rangle_H \overline{\langle e_n \mid g \rangle_H} =$$

$$= \sum_n \langle e_n \mid h \rangle_H \overline{\langle e_n \mid g \rangle_H} = \langle h \mid g \rangle_H$$

which means that B establishes an isometry between H_K and H. Because $Be_k = \Phi_k$ for all k, it is surjective, hence unitary.

Step 6

$$B^{-1} K_x = B^{\dagger} K_x = c_x$$

makes the reciprocity between families $Bh, h \in H$ of the step 4 and $c_x, x \in X$ of the step 3 effective; any of these two can be viewed as an alternative of the other and deserve the name of the family of coherent states.

Step 7

$$\langle h|\int|x\rangle\langle x|\mu(dx)|g\rangle_{L^2(\mu)}=\int\langle h|x\rangle\langle x|g\rangle_{L^2(\mu)}\mu(dx)=$$

$$=\int\overline{(Bh)(x)}(Bg(x))\mu(dx)=\langle Bh|Bg\rangle_{L^2(\mu)}=$$

$$=\langle Bh|Bg\rangle_{H_K}=\langle h|g\rangle_H$$

i.e., the step 5 means *the resolution of unity.*

Examples

1. A „trivial” one - complex monomials

$$\Phi_n(z) = \frac{z^n}{\sqrt{n!}}$$

do fit to our scheme - but can we give another example?

The answer is ‘yes’ – such an example is provided by the complex (holomorphic) Hermite polynomials defined identically as the standard ones but considered as functions of the argument being a complex number.

Complex Hermite polynomials

$$H_n(x+iy) = n!\sum_{k=0}^{[k/2]} \frac{(-1)^k (2(x+iy))^{n-2k}}{k!(n-2k)!}$$

$$\int_{R^2} H_n(x+iy)\overline{H_m(x+iy)}\exp\left[-(1-s)x^2 - \left(\frac{1}{s}-1\right)y^2\right]dxdy = b_n(s)\delta_{nm}$$

$$b_n(s) = \frac{\pi\sqrt{s}}{1-s}\left(2\frac{1+s}{1-s}\right)^n n!, \qquad s<1$$

$$\frac{1-s}{\pi\sqrt{s}}\sum_n\left(\frac{1-s}{1+s}\right)^n \frac{H_n(z)\overline{H_n(w)}}{2^n n!} =$$

$$= \frac{1-s^2}{2\pi s}\exp\left[-\frac{\left(1-s^2\right)}{4s}\left(z^2+\overline{w^2}\right)+\frac{\left(1-s^2\right)}{4s}z\overline{w}\right] < \infty$$

van Eijnhoven & Meyers 1990

Relation to combinatorics? to combinatorial physics?

Let us give a problem: *find polynomials of a complex variable orthogonal with respect to the two dimensional gaussian measure*

$$\int_{R^2} P_n(x+iy)\overline{P_m(x+iy)}\exp\left[-\alpha x^2-\beta y^2\right]dxdy=c_n(\alpha,\beta)\delta_{nm}$$

assuming that P_n's *are Sheffer polynomials, i.e. generated by*

$$\sum_{n=0}^{\infty}P_n\left(x+iy\right)\frac{t^n}{n!}=g(t)\exp\left[(x+iy)f\left(t\right)\right]$$

$$g(0)\neq 0,\quad f(0)=0,\quad f'\left(0\right)\neq 0.$$

Then one finds two solutions: either monomials or complex Hermite polynomials. Taking another measure one can end up on other Sheffer polynomials which are often met as solutions to combinatorial problems!

Conlusions/Outlook

- „Replacing” the *resolution of unity* by reproducing property enables us to look at the problem of completeness of coherent states from another, broader, point of view emphasizing properties of functions $\Phi_n(z)$ used in their construction. They do not need be monomials any longer which is important because namely these functions carry probabilistic content of coherent states and their choice is related to the physical situation under consideration.
- Proposed approach provides us with new tools useful to study coherent states for which the *resolution of unity* cannot be effectively investigated using methods based on the moment problem, e.g. when the latter is indeterminate, or impossible to be formulated.