Non-Hermitian coherent states for finite-dimensional systems

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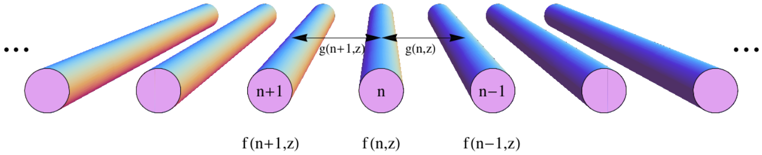
Outline

Optical waveguide arrays



- From mode-coupled theory for an infinite array:

$$-i\frac{d\mathcal{E}_n}{dz} = \omega f(n,z)\mathcal{E}_n + \lambda\left[g(n,z)\mathcal{E}_{n-1} + g(n+1,z)\mathcal{E}_{n+1}\right], \qquad n \in \mathbb{Z}$$

- If $g(1,z) = 0$, this set of equations uncouples into two semi-infinite sets: $n \leq 0$ and $n \geq 1$.
- If further $g(N,z) = 0$, for $N > 0$, it uncouples into three sets: $n \leq 0$, $1 \leq n \leq N$ and $n > N$.

Matrix description of optical waveguide arrays

In any case, the equations can be written as:

$$-i\frac{d}{dz}|\mathcal{E}(z)\rangle = \hat{H}(z)|\mathcal{E}(z)\rangle$$

where

$$|\mathcal{E}(z)\rangle = \sum_{j\in\mathcal{I}} \mathcal{E}_j(z)|j\rangle \qquad \mathcal{I}\subset\mathbb{Z}$$

and

$$|j\rangle = \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow j\text{-th} \\ \\ \\ \end{array}$$

The Hamiltonian $\hat{H}(z)$ reads:

$$\hat{H}(z) = \omega f(\hat{n}, z) + \lambda\left[g(\hat{n}, z)\hat{V}^\dagger + \hat{V}g(\hat{n}, z)\right]$$

with $\hat{n}|j\rangle = j|j\rangle$, $\hat{V}|j\rangle = |j-1\rangle$ and $\hat{V}^\dagger|j\rangle = |j+1\rangle$.

$$\hat{n} = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -1 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 0 & 2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$\hat{V} = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 0 & 1 & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$\hat{V}^\dagger = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots \\ \dots & 1 & 0 & 0 & 0 & \dots \\ \dots & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Symmetric waveguide arrays

If

$$\hat{H}(z) = \sum_{k=1}^{N} \alpha_k(z)\hat{A}_k$$

with $\hat{A}_k$ constant matrices closing a Lie algebra $\mathcal{G}$ with $[\hat{A}_i, \hat{A}_j] = \sum_{k=1}^{N} {c_{ij}}^k \hat{A}_k$, then the differential equation

$$-i\frac{d}{dz}|\mathcal{E}(z)\rangle = \hat{H}(z)|\mathcal{E}(z)\rangle$$

can be solved by group-theoretical methods (like Wei-Norman factorization):

$$|\mathcal{E}(z)\rangle = U(z)|\mathcal{E}(0)\rangle = \Pi_{k=1}^{N} e^{iu_k(z)\hat{A}_k}|\mathcal{E}(0)\rangle \equiv \hat{\rho}(g(z))|\mathcal{E}(0)\rangle$$

where the functions $u_k(z)$ satisfy non-linear first-order coupled differential equations involving the structure constants ${c_{ij}}^k$ and the coefficients $\alpha_k(z)$. $g(z) \in G$ and $\hat{\rho}$ is a representation of G.

Then $|\mathcal{E}(z)\rangle$ will be a coherent state of the Gilmore-Perelomov type!!

Unitarity versus non-unitarity

- We shall focus in a finite number of waveguide arrays: $\mathcal{I}$ finite.
- If the Lie algebra expanded by the A_k corresponds to a compact group (like SU(2)), $U(z)$ is unitary and the total power $P(z) = \sum_{j\in\mathcal{I}} |\mathcal{E}_j(z)|^2$ is conserved:

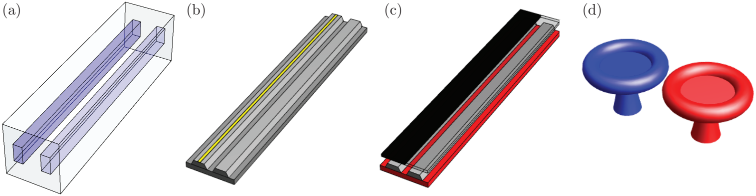
$$\frac{d}{dz}P(z) = 0$$

- If the Lie algebra expanded by the A_k corresponds to a non-compact group (like $SO(2,1)$ or $SO(3,1)$), $U(z)$ is not unitary and the total power is not conserved:

$$\frac{d}{dz}P(z) \neq 0$$

- Non-unitarity is caused by non-Hermiticity of the Hamiltonian, due to complex matrix elements of the Hamiltonian.
- Non-Hermitian Hamiltonians describe waveguides with losses or gain, or complex couplings (due, for instance, to torsion of the waveguides).

Non-Hermitian dimer



Consider the non-Hermitian coupling matrix

$$H_{nH} = \begin{pmatrix} \alpha_1(z) & \beta_1(z) \\ \beta_2(z) & \alpha_2(z) \end{pmatrix}, \qquad \alpha_i(z), \beta_j(z) \in \mathbb{C}$$

Setting the effective refractive indices relative to their average,

$$|\mathcal{E}(z)\rangle = e^{i \int_0^z \alpha_0(t) dt} |E(z)\rangle, \qquad \alpha_0(z) = \frac{1}{2} \left[\alpha_1(z) + \alpha_2(z) \right],$$

gives the traceless effective non-Hermitian coupling matrix,

$$H = \begin{pmatrix} \alpha(z) & \beta_1(z) \\ \beta_2(z) & -\alpha(z) \end{pmatrix}, \qquad \alpha(z) = \frac{1}{2} \left[\alpha_1(z) - \alpha_2(z) \right],$$

and the differential system,

$$-i\partial_z|E(z)\rangle = H(z)|E(z)\rangle.$$

Note that $H(z) \in sl(2,\mathbb{C}) \approx so(3,1)$.

Choosing the parameters appropriately, $H(z) \in su(1,1) \approx so(2,1)$.

$sl(2,\mathbb{C}) = so(3) \oplus i\, so(3)$, with basis $\{\hat{J}_x, \hat{J}_y, \hat{J}_z, i\hat{J}_x, i\hat{J}_y, i\hat{J}_z\}$

Varios conjugate $so(2,1)$ subalgebras can be found in $sl(2,\mathbb{C})$:

$\{\hat{K}_x, \hat{K}_y, \hat{K}_z\} \equiv \{i\hat{J}_x, i\hat{J}_y, \hat{J}_z\},\ \{i\hat{J}_y, i\hat{J}_z, \hat{J}_x\},\ \{i\hat{J}_z, i\hat{J}_x, \hat{J}_y\}, \ldots$

Experimentally, the easiest to realize is:

$\{\hat{K}_x, \hat{K}_y, \hat{K}_z\} \equiv \{i\hat{J}_y, i\hat{J}_z, \hat{J}_x\}$

It corresponds to a PT-symmetric waveguide array with balanced gain/loss.

Finite waveguide arrays with $SO(2,1)$ symmetry

The Hamiltonian for a finite waveguide array with underlying $SO(2,1) \approx SU(1,1)$ symmetry is

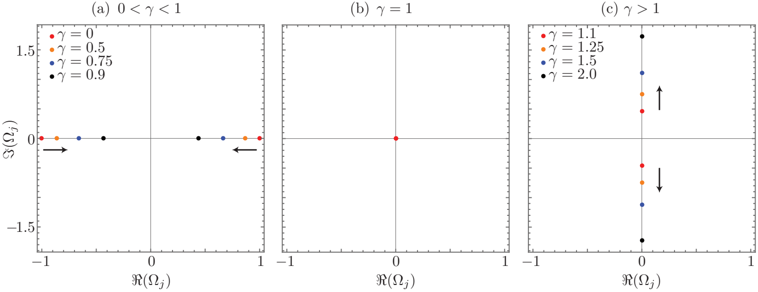
$$\hat{H}(z) = i\gamma(z)\hat{J}_z + \lambda(z)\hat{J}_x$$

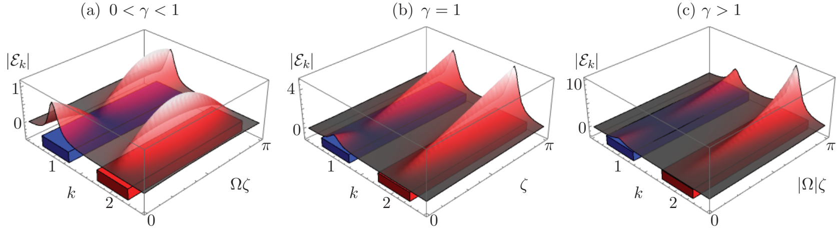
In this case there are $2j+1$ waveguides for the non-unitary representation with "spin" $j \in \frac{\mathbb{Z}}{2}$ of $SO(2,1) \approx SU(1,1)$.
Under the P symmetry ($n \to 2j - n$) and T symmetry ($i \to -i$, $z \to -z$) the Hamiltonian changes to:

$$\hat{H}(z) \to \hat{H}(z)^{PT} = -i\gamma(-z)\left(-\hat{J}_z\right) + \lambda(-z)\hat{J}_x = \hat{H}(z)$$

provided $\gamma(z)$ and $\lambda(z)$ are even. Therefore the system is PT-symmetric!!.
However, for some values of the parameters, the PT-symmetry can be spontaneously broken, in the sense that wavefunctions are not PT-symmetric

For the case $\gamma(z) = \gamma$ and $\lambda(z) = \lambda$, simple analytical expressions can be obtained. Three cases have to be considered, depending on wheather $\Omega = \sqrt{\lambda^2 - \gamma^2}$ is positive, zero or pure imaginary.




(a) $0 < \gamma < 1$
$|\mathcal{E}_k|$
1
0
1
k
2
0
$\Omega\zeta$
π
(b) $\gamma = 1$
$|\mathcal{E}_k|$
4
0
1
k
2
0
ζ
π
(c) $\gamma > 1$
$|\mathcal{E}_k|$
10
0
1
k
2
0
$|\Omega|\zeta$
π

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Non-Hermitian Coherent States

Let $\hat{\rho}$ be a non-unitary representation of a Lie group G with Lie algebra $\mathcal{G}$ in the Hilbert space $\mathcal{H}$.

Since $\hat{\rho}(g)^\dagger \neq \hat{\rho}(g)^{-1}$, it is crucial to introduce the contragredient (or dual) representation $\hat{\rho}^*(g) = \hat{\rho}(g^{-1})^\dagger$, which is neither unitary.

It verifies

$$\langle \hat{\rho}^*(g)\Psi, \hat{\rho}(g)\Phi \rangle = \langle \Psi, \Phi \rangle$$

If $\hat{\rho}$ is unitary then $\hat{\rho}^* = \hat{\rho}$.

Define a family of non-hermitian Gilmore-Perelomov Coherent States as usual:

$$|\Psi_g\rangle = \hat{\rho}(g)|\Psi\rangle \,, \quad \forall g \in G$$

for a suitable vector $|\Psi\rangle \in \mathcal{H}$.

We must introduce the dual family of Coherent States:

$$|\widetilde{\Phi}_g\rangle = \hat{\rho}^*(g)|\Phi\rangle\,, \quad \forall g \in G$$

for another suitable vector $|\Phi\rangle \in \mathcal{H}$.

Then

$$\langle\widetilde{\Phi}_g|\Psi_g\rangle = \langle\Phi|\Psi\rangle\,, \qquad \forall g \in G$$

Define the Analysis (or Sampling) operator for the set of coherent states:

$$\hat{T} : |\psi\rangle \mapsto \{\langle\Psi_g|\psi\rangle\}_{g\in G}$$

and for the dual set of coherent states:

$$\hat{\tilde{T}} : |\psi\rangle \mapsto \{\langle\tilde{\Phi}_g|\psi\rangle\}_{g\in G}$$

Introduce the Resolution operators

$$\hat{A} = \hat{T}^\dagger \hat{\tilde{T}} = \int_G d\mu(g)\hat{\rho}(g)|\Psi\rangle\langle\Phi|\hat{\rho}(g)^{-1}$$

$$\hat{\tilde{A}} = \hat{A}^\dagger = \hat{\tilde{T}}^\dagger \hat{T} = \int_G d\mu(g)\hat{\rho}^*(g)|\Phi\rangle\langle\Psi|\hat{\rho}(g)^\dagger$$

It holds

$$\hat{\rho}(g)\hat{A} = \hat{A}\hat{\rho}(g)\,, \quad \hat{\rho}^*(g)\hat{\tilde{A}} = \hat{\tilde{A}}\hat{\rho}^*(g)\,, \quad \forall g \in G$$

If the representation $\hat{\rho}$ is irreducible then $\hat{A} = \hat{\tilde{A}} = \lambda I_{\mathcal{H}}$.

Square integrability of the representation $\hat{\rho}$ is required (and admissibility conditions for the vectors $|\Psi\rangle$ and $|\Phi\rangle$), or a suitable restriction in the integration to a quotient space G/H or to a subset $C \subset G$, but in this case the resulting resolution operator need not verify $\hat{A} = \lambda I_{\mathcal{H}}$.

Define the Overlapping Kernel:

$$\begin{aligned} K(g', g) &= \langle \tilde{\Phi}_{g'} | \Psi_g \rangle = \langle \Phi | \hat{\rho}^*(g')^\dagger \hat{\rho}(g) \Psi \rangle \\ &= \langle \Phi | \hat{\rho}(g')^{-1} \hat{\rho}(g) \Psi \rangle = K(g'^{-1} g) \end{aligned}$$

Under suitable conditions $K(g', g)$ is a reproducing kernel defining a reproducing kernel Hilbert space.

Example: SU(1,1) Non-hermitian coherent states

Consider the realization of $su(1,1) \approx so(2,1)$
$\{K_x \equiv iJ_x, K_y \equiv iJ_y, K_z = J_z\}$
(note that is different to that of the non-hermitian dimer!!).

For the $2j+1$-dimesional unitary irreducible representation of $SU(2)$, we get a $2j+1$-dimensional non-unitary irreducible representation of $SU(1,1)$.

The representation can be chosen as:

$$\hat{\rho}(\zeta, \zeta^*, \gamma) = e^{\zeta K_+ - \zeta^* K_-} e^{i\gamma K_z}$$

For instance, the $j = 1/2$ case is:

$$\hat{\rho}(\zeta, \zeta^*, \gamma) = \frac{1}{\sqrt{1 - |\zeta|^2}} \begin{pmatrix} e^{i\gamma} & e^{i\gamma}\zeta \\ e^{-i\gamma}\zeta^* & e^{-i\gamma} \end{pmatrix}$$

The contragradient representation is:

$$\hat{\rho}^*(\zeta, \zeta^*, \gamma) = \frac{1}{\sqrt{1 - |\zeta|^2}} \begin{pmatrix} e^{i\gamma} & -e^{i\gamma}\zeta \\ -e^{-i\gamma}\zeta^* & e^{-i\gamma} \end{pmatrix}$$

Choosing $|\Psi\rangle = |\Phi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The coherent states are:

$$\begin{aligned} |\xi,\gamma\rangle &= \frac{1}{\sqrt{1-|\zeta|^2}} \begin{pmatrix} e^{i\gamma} \\ e^{-i\gamma}\xi^* \end{pmatrix} \\ \widetilde{|\xi,\gamma\rangle} &= \frac{1}{\sqrt{1-|\zeta|^2}} \begin{pmatrix} e^{i\gamma} \\ -e^{-i\gamma}\xi^* \end{pmatrix} \end{aligned}$$

The resolution operator is:

$$\hat{A} = \int_{\mathbb{D}\times S^1} \frac{d\zeta d\zeta^*}{(1-|\zeta|^2)^2} d\gamma \frac{1}{1-|\zeta|^2} \begin{pmatrix} 1 & e^{2i\gamma}\zeta \\ e^{-2i\gamma}\zeta^* & -|\zeta|^2 \end{pmatrix}$$

but it is divergent since the representation is not square integrable.

The overlapping kernel is given by:

$$K((\zeta',\gamma'),(\zeta,\gamma)) = \frac{1}{\sqrt{(1-|\zeta|^2)(1-|\zeta'|^2)}}(e^{i(\gamma-\gamma')} - e^{-i(\gamma-\gamma')}\zeta^*\zeta')$$

It is hermitian and positive definite, but it is not bounded. In fact, it coincides with the overlapping kernel for the coherent states associated with the representation of the discrete series with Bargmann index $k = -1/2$.

To overcome this we can:

- Consider the subgroup $(0,0,\gamma)$, leading to a unitary (reducible!) representation of $U(1)$.
- Consider a subgroup of elliptic elements in $SU(1,1)$.
- Fix $|\zeta| = r_0 < 1$.
- Take $0 \le r_m < |\zeta| < r_M < 1$.
- Consider discrete families like $\zeta = r_0 e^{i\frac{2\pi k}{N}}$, $k = 0,1,\ldots,N-1$.

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