Hermite polynomials in two complex variables: Mathematical properties

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OUTLINE

- $z^n/\sqrt{n!}$
- $H_n(z)$
- $H_{m,n}(z_1, z_2)$

HERMITE POLYNOMIALS $H_{m,n}$

The Hermite polynomials $H_{m,n}$, $m, n = 0, 1, \ldots$, in two complex variables z_1 and z_2 can be defined as

$$H_{m,n}(z_1, z_2) \stackrel{\text{def}}{=} \sum_{k=0}^{\min\{m,n\}} \binom{m}{k} \binom{n}{k} (-1)^k k! z_1^{m-k} z_2^{n-k}$$

POLYNOMIALS VERSUS FUNCTIONS DEFINED BY POLYNOMIALS

For complex z we define

$$H_{m,n}(z,\bar{z}) = \sum_{k=0}^{\min\{m,n\}} \binom{m}{k}\binom{n}{k}(-1)^k k! z^{m-k}\bar{z}^{n-k}.$$

$H_{m,n}$ are polynomials in z and $\bar{z}$ (they are not polynomials in a single variable $z \in \mathbb{C}$) with real coefficients but as functions they are of a single complex variable z.

Usually called:
Dattoli: INCOMPLETE HERMITE POLYNOMIALS;
Gazeau, Ghanmi, Fan, Klauder: COMPLEX HERMITE POLYNOMIALS;
Wünsche: LAGUERRE POLYNOMIALS IN TWO VARIABLES

POLYNOMIALS VERSUS FUNCTIONS DEFINED BY POLYNOMIALS

If $z = x + \mathrm{i}y$ then $H_{m,n}(z, \bar{z})$ may be written down as

$$\widetilde{H}_{m,n}(x, y) = \sum_{k=0}^{\min\{m,n\}} \sum_{i=0}^{m-k} \sum_{j=0}^{n-k} \frac{m!n!}{k!i!j!} \frac{\mathrm{i}^{m+k-i-j} x^{n-k-j+i} y^{m-k-i+j}}{(m-k-i)!(n-k-j)!}.$$

- Now, $\widetilde{H}_{m,n}$ becomes polynomials in two variables x and y with complex coefficient.
- Orthogonal with respect to the measure $\exp(-x^2 - y^2)$, $x, y \in \mathbb{R}$.

EXAMPLE OF COMPLEX POLYNOMIALS; $z^n/\sqrt{n!}$ and $H_n(z)$

- the monomials $\Phi_n(z) = z^n/\sqrt{n!}$ which are orthogonal with respect to the *rotationally* invariant measure $\exp(-z\bar{z})$. The monomials $z^n/\sqrt{n!}$ is an orthonormal basis in $\mathcal{H}_{\mathrm{Barg},1}$.
(important for the physics; V. Bargmann, Commun. Pur. Appl. Anal., 1961).

The Segal-Bargmann transforamtion

$$\mathcal{L}^2(\mathbb{R}^2, \mathrm{d}q\,\mathrm{d}p) \quad \stackrel{A}{\longleftrightarrow} \quad \mathcal{H}_{\mathrm{hol},1}(\mathbb{C}, \mathrm{e}^{-z\bar{z}}\,\mathrm{d}z),$$

A is the unitary operator.

the space $\mathcal{L}^2(\mathbb{R}^2, \mathrm{d}q\,\mathrm{d}p)$ of square integrable functions

the Bargmann space $\mathcal{H}_{\mathrm{Barg},1}(\mathbb{C}, \mathrm{e}^{-z\bar{z}}\,\mathrm{d}z)$ of analytical functions

EXAMPLE OF COMPLEX POLYNOMIALS; $z^n/\sqrt{n!}$ and $H_n(z)$

- the complex Hermite polynomials in one variable $H_n(z)$ (van Eijndhoven-Meyers polynomials; S. J. L. van Eijndhoven and J. L. H. Meyers, J. Math. Anal. Appl., 1990) is orthogonal with respect to the *non-rotationally* invariant measure $\exp[-\frac{(1-\alpha)^2}{4\alpha}(z^2+\bar{z}^2)-\frac{1-\alpha^2}{2\alpha}z\bar{z}]$, $0<\alpha<1$.

ORTHONORMAL VAN EIJNDHOVEN-MEYERS FUNCTIONS $h_{\alpha,n}(z)$, $z \in \mathbb{C}$

Orthonormal van Eijndhoven-Meyers' functions are defined as

$$h_{\alpha,n}(z) \stackrel{\text{def}}{=} b_n(\alpha)^{-1/2}\, \mathrm{e}^{-z^2/2}\, H_n(z), \qquad H_n(z) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2z)^{n-2k}}{k!(n-2k)!},$$

$$b_n(\alpha) \stackrel{\text{def}}{=} \frac{\pi\sqrt{\alpha}}{1-\alpha} \left(2\frac{1+\alpha}{1-\alpha}\right)^n n!,$$

where $z = x + \mathrm{i}\, y$, $x, y \in \mathbb{R}$, $0 < \alpha < 1$, and $n = 0, 1, \ldots$.

- $(h_{\alpha,n})_{n=0}^{\infty}$ is an orthonormal basis in $\mathcal{H}^{(\alpha)}$,
- the space $\mathcal{H}^{(\alpha)}$ is a reproducing kernel Hilbert space with the kernel

$$K_\alpha(z,w) = \sum_{n=0}^{\infty} h_{\alpha,n}(z)\overline{h_{\alpha,n}(w)}$$

$$= \frac{1-\alpha^2}{2\pi\alpha}\, \mathrm{e}^{-\frac{1+\alpha^2}{4\alpha}(z^2+\bar{w}^2)+\frac{1-\alpha^2}{2\alpha} z\bar{w}}, \qquad z, w \in \mathbb{C}.$$

ORTHONORMAL VAN EIJNDHOVEN-MEYERS FUNCTIONS $h_{\alpha,n}(z)$, $z \in \mathbb{C}$

Szafraniec, Contemporary Mathematics, 1998
Gazeau & Szafraniec, JPA (2011)

With $\mathcal{D}_\alpha \stackrel{\text{def}}{=} \mathrm{lin}(h_{\alpha,n})_{n=0}^{\infty}$ (the linear span of Hermite functions), we may say that the operators S_α^+ and S_α^- defined as

$$S_\alpha^+ f(z) = \sqrt{\frac{1-\alpha}{2(1+\alpha)}} \left(z f(z) - \frac{\mathrm{d}}{\mathrm{d}z} f(z) \right)$$

$$S_\alpha^- f(z) = \sqrt{\frac{2(1+\alpha)}{1-\alpha}} \left(z f(z) + \frac{\mathrm{d}}{\mathrm{d}z} f(z) \right), \quad z \in \mathbb{C}, \quad f \in \mathcal{D}_\alpha$$

are the creation and annihilation operators acting on $\mathcal{H}^{(\alpha)}$ and their commutation relation,

$$S_\alpha^- S_\alpha^+ - S_\alpha^+ S_\alpha^- = I_{\mathcal{H}^{(\alpha)}},$$

is still satisfied on $\mathcal{D}_\alpha$.

TRANSFORMS

Ali, Górska, Horzela & Szafraniec, JMP (2014)

- **From $\mathcal{H}^{(\alpha)}$ to $\mathcal{H}_{\rm hol}$:** $\Phi_n = A h_{\alpha,n}$ with

$$A(z, \bar{w}) = \sum_{n=0}^{\infty} \Phi_n(z) \overline{h_{\alpha,n}(w)}$$

The operator A is unitary, namely $\Phi_n = A h_{\alpha,n}$ is isometric and surjective mapping.

- **From $\mathcal{H}_{\rm hol}$ to $\mathcal{H}^{(\alpha)}$:** $h_{\alpha,n} = \bar{A}\Phi_n$.

HERMITE POLYNOMIALS IN TWO COMPLEX VARIABLES $H_{m,n}(z_1, z_2)$, $z_1, z_2 \in \mathbb{C}$

The Hermite polynomials $H_{m,n}$, $m, n = 0, 1, \ldots$ may come from the generating function

$$\sum_{m,n=0}^{\infty} \frac{s^m t^n}{m!n!} H_{m,n}(z_1, z_2) = \mathrm{e}^{z_1 s + z_2 t - st}, \quad z_1, z_2 \in \mathbb{C}.$$

The generating function cannot be factorized for two one-variable functions which depend on s or t. It means that in some sense $H_{m,n}$ is *entanglment* from two van-Eijndhoven-Meyers polynomials H_m.

The Hermite polynomials in two variables can be given as

$$H_{m,n}(z_1, z_2) = \frac{\mathrm{d}^m}{\mathrm{d}s^m} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \, \mathrm{e}^{z_1 s + z_2 t - st} \Big|_{s=0, t=0}.$$

ALGEBRAIC PROPERTIES OF $H_{m,n}(z_1, z_2)$, $z_1, z_2 \in \mathbb{C}$

The algebraic properties of $H_{m,n}$ are similar but not the same as the standard properties of the (standard) Hermite polynomials of x, $x \in \mathbb{R}$.

Raising and lowering operational formula:

$$H_{m+1,n}(z_1, z_2) = (z_1 - \partial_{z_2})H_{m,n}(z_1, z_2), \quad \partial_{z_2} H_{m,n}(z_1, z_2) = nH_{m,n-1}(z_1, z_2)$$
$$H_{m,n+1}(z_1, z_2) = (z_2 - \partial_{z_1})H_{m,n}(z_1, z_2) \quad \partial_{z_1} H_{m,n}(z_1, z_2) = mH_{m-1,n}(z_1, z_2)$$

Triple recurrence relation for $H_{m,n}$:

$$H_{m+1,n}(z_1, z_2) = z_1 H_{m,n}(z_1, z_2) - nH_{m,n-1}(z_1, z_2)$$
$$H_{m,n+1}(z_1, z_2) = z_2 H_{m,n}(z_1, z_2) - mH_{m-1,n}(z_1, z_2)$$

ORTHOGONALITY OF $H_{m,n}(z_1, z_2)$, $z_1, z_2 \in \mathbb{C}$

Let us express $H_{m,n}(z_1, z_2)$ in terms of two van Eijndhoven-Meyers polynomials $H_r(z)$, $z \in \mathbb{C}$

$$H_{m,n}(z_1, z_2) = 2^{-(m+n)} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} \mathrm{i}^{m-k} (-\mathrm{i})^{n-l} \times H_{k+l}\left(\tfrac{z_1+z_2}{2}\right) H_{m+n-k-l}\left(\tfrac{z_1-z_2}{2\,\mathrm{i}}\right).$$

Then, we use the orthogonal relation for van Eijndhoven-Meyers polynomials:

$$\int_{\mathbb{C}} H_r(z) \overline{H_s(z)}\, \mathrm{e}^{-\frac{(1-\alpha)^2}{4\alpha}(z^2+\bar{z}^2) - \frac{1-\alpha^2}{2\alpha} z\bar{z}}\, \mathrm{d}z = b_r(\alpha)\, \delta_{r,s}$$

ORTHOGONALITY OF $H_{m,n}(z_1, z_2)$, $z_1, z_2 \in \mathbb{C}$

Thus, the orthogonal relation for $H_{m,n}(z_1, z_2)$ has the form

$$\int_{\mathbb{C}^2} H_{m,n}(z_1, z_2)\overline{H_{p,q}(z_1, z_2)} g_\alpha(z_1, z_2)\, \mathrm{d}z_1\, \mathrm{d}z_2 = c_{m,n}(\alpha)\delta_{m,p}\delta_{n,q},$$

where

$$c_{m,n}(\alpha) = 2^{-(m+n)} b_m(\alpha) b_n(\alpha)$$

and $g_\alpha(z_1, z_2)$ is built form the product of two measure appropriate for van Eijdhoven-Meyers polynomials for the variables $(z_1 + z_2)/2$ and $(z_1 - z_2)/(2\,\mathrm{i})$. Thus, $g_\alpha(z_1, z_2)$ has the form

$$g_\alpha(z_1, z_2) = \exp[-\tfrac{1-\alpha}{4}|\bar{z}_2 + z_1|^2 - \tfrac{1}{4}(\tfrac{1}{\alpha} - 1)|\bar{z}_2 - z_1|^2].$$

ORTHONORMAL HERMITE FUNCTIONS $h_{m,n}^{(\alpha)}(z_1, z_2)$, $z_1, z_2 \in \mathbb{C}$

Let the orthonormal Hermite functions $h_{m,n}^{(\alpha)}(z_1, z_2)$, $m, n = 0, 1, \ldots$ be

$$h_{m,n}^{(\alpha)}(z_1, z_2) \stackrel{\text{def}}{=} [c_{m,n}(\alpha)]^{-1/2} \exp\left(-\frac{z_1 z_2}{2}\right) H_{m,n}(z_1, z_2),$$

where $0 < \alpha < 1$ is a parameter.

- $(h_{m,n}^{(\alpha)})_{n=0}^{\infty}$ is an orthonormal basis in $\mathcal{K}^{(\alpha)}$,
- the space $\mathcal{K}^{(\alpha)}$ is a reproducing kernel Hilbert space with the kernel

$$K_\alpha(z_1, z_2, w_1, w_2) = \sum_{n=0}^{\infty} h_{m,n}^{(\alpha)}(z_1, z_2)\overline{h_{m,n}^{(\alpha)}(w_1, w_2)}$$
$$= \frac{(1-\alpha^2)^2}{4\pi^2\alpha^2} \, e^{-\frac{1+\alpha^2}{4\alpha}(z_1 z_2 + \bar{w}_1 \bar{w}_2) + \frac{1-\alpha^2}{4\alpha}(z_1 \bar{w}_1 + z_2 \bar{w}_2)}$$

with $z_1, z_2, w_1, w_2 \in \mathbb{C}$.

RELATING $\mathcal{K}^{(\alpha)}$ TO THE BARGMANN SPACE

Recall that in two variable Bargmann space $\mathcal{H}_{\mathrm{Barg},2}$ the sequence

$$\Phi_{m,n}(z_1, z_2) \stackrel{\text{def}}{=} \frac{z_1^m}{\sqrt{m!}} \frac{z_2^n}{\sqrt{n!}}, \qquad z_1, z_2 \in \mathbb{C}, \quad m, n = 0, 1, \ldots.$$

of monomials is an orthonormal basis.

- **From $\mathcal{K}^{(\alpha)}$ to $\mathcal{H}_{\mathrm{Barg},2}$:** $\Phi_{m,n} = Bh_{m,n}^{(\alpha)}$ with

$$B(z_1, z_2, \bar{w}_1, \bar{w}_2) = \sum_{m,n=0}^{\infty} \Phi_{m,n}(z_1, z_2)\overline{h_{m,n}^{(\alpha)}(w_1, w_2)}.$$

  The operator B is unitary, i.e. $\Phi_{m,n} = Bh_{m,n}^{(\alpha)}$ is an isometric and surjective transformation.

- **From $\mathcal{H}_{\mathrm{Barg},2}$ to $\mathcal{K}^{(\alpha)}$:** $h_{m,n}^{(\alpha)} = \bar{B}\Phi_{m,n}$.

CREATION AND ANNIHILATION OPERATORS

Put $\mathcal{D}_2^{(\alpha)} \stackrel{\text{def}}{=} \mathrm{lin}(h_{m,n}^{(\alpha)})_{m,n=0}^{\infty}$ the linear span of Hermite functions. For $f \in \mathcal{D}_2^{(\alpha)}$ define the following four operators as

$$(a_{1,+}^{(\alpha)} f)(z_1, z_2) \stackrel{\text{def}}{=} \left(\frac{1-\alpha}{1+\alpha}\right)^{1/2} \left(\frac{z_1}{2} - \frac{\partial}{\partial z_2}\right) f(z_1, z_2),$$

$$(a_{1,-}^{(\alpha)} f)(z_1, z_2) \stackrel{\text{def}}{=} \left(\frac{1+\alpha}{1-\alpha}\right)^{1/2} \left(\frac{z_2}{2} + \frac{\partial}{\partial z_1}\right) f(z_1, z_2),$$

$$(a_{2,+}^{(\alpha)} f)(z_1, z_2) \stackrel{\text{def}}{=} \left(\frac{1-\alpha}{1+\alpha}\right)^{1/2} \left(\frac{z_2}{2} - \frac{\partial}{\partial z_1}\right) f(z_1, z_2),$$

$$(a_{2,-}^{(\alpha)} f)(z_1, z_2) \stackrel{\text{def}}{=} \left(\frac{1+\alpha}{1-\alpha}\right)^{1/2} \left(\frac{z_1}{2} + \frac{\partial}{\partial z_2}\right) f(z_1, z_2),$$

with $z_1, z_2 \in \mathbb{C}$.

LIMIT $\alpha \to 1-$

The orthogonal relation for the Hermite functions for $z_1 = \frac{w_1}{\sqrt{1-\alpha}}$ and $z_2 = \frac{w_2}{\sqrt{1-\alpha}}$, $w_1, w_2 \in \mathbb{C}$, in the limit $\alpha \to 1-$ goes to the orthogonal relation in the Bargmann space[1]:

$$\int_{\mathbb{C}^2} h^{(\alpha)}_{m,n}\Big(\frac{w_1}{\sqrt{1-\alpha}}, \frac{w_2}{\sqrt{1-\alpha}}\Big)\overline{h^{(\alpha)}_{p,q}\Big(\frac{w_1}{\sqrt{1-\alpha}}, \frac{w_2}{\sqrt{1-\alpha}}\Big)}\,\tilde{g}_\alpha(w_1, w_2)(1-\alpha)^{-2}\,\mathrm{d}w_1\,\mathrm{d}w_2$$

$$\xrightarrow{\alpha\to 1-} \int_{\mathbb{C}} \frac{w_1^m}{\sqrt{m!}}\frac{\bar{w}_1^p}{\sqrt{p!}}\,\mathrm{e}^{-w_1\bar{w}_1}\,\mathrm{d}w_1 \int_{\mathbb{C}} \frac{w_2^n}{\sqrt{n!}}\frac{\bar{w}_2^q}{\sqrt{q!}}\,\mathrm{e}^{-w_2\bar{w}_2}\,\mathrm{d}w_2,$$

where $\tilde{g}_\alpha(w_1, w_2) = \exp[-\frac{w_1 w_2 + \bar{w}_1\bar{w}_2}{2(1-\alpha)}]g_\alpha(\frac{w_1}{\sqrt{1-\alpha}}, \frac{w_2}{\sqrt{1-\alpha}})$;

[1] $\lim_{t\to 0} t^{m+n} H_{m,n}(\frac{z_1}{t}, \frac{z_2}{t}) = z_1^m z_2^n$.

LIMIT $\alpha \to 0+$

For $z_1 = u + \mathrm{i}\sqrt{\alpha}v$, $z_2 = \bar{u} + \mathrm{i}\sqrt{\alpha}\bar{v}$, $u, v \in \mathbb{C}$ one has the following limit formulae

$$\int_{\mathbb{C}^2} h_{m,n}^{(\alpha)}(u+\mathrm{i}\sqrt{\alpha}v, \bar{u}+\mathrm{i}\sqrt{\alpha}\bar{v})\overline{h_{p,q}^{(\alpha)}(u+\mathrm{i}\sqrt{\alpha}v, \bar{u}+\mathrm{i}\sqrt{\alpha}\bar{v})}\,\mathrm{e}^{\alpha u\bar{u}-v\bar{v}}\,\alpha\,\mathrm{d}u\,\mathrm{d}v \xrightarrow{\alpha\to 0+} \int_{\mathbb{C}} h_{m,n}(u,\bar{u})\overline{h_{p,q}(u,\bar{u})}\,\mathrm{d}u,$$

where

$$h_{m,n}(u,\bar{u}) \stackrel{\text{def}}{=} \frac{1}{\pi\sqrt{m!n!}}\,\mathrm{e}^{-u\bar{u}/2}\,H_{m,n}(u,\bar{u}), \qquad u \in \mathbb{C}.$$

FAN, KLAUDER

REPRODUCING KERNEL

We recall the form of the reproducing kernel built for the orthonormal Hermite function in two variable

$$h_{m,n}^{(\alpha)}(z_1, z_2) = [c_{m,n}(\alpha)]^{-1/2}\, e^{-z_1 z_2/2}\, H_{m,n}(z_1, z_2),$$

$$[c_{m,n}(\alpha)]^{-1/2} \sim \left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{m+n}{2}} \frac{1}{\sqrt{m!n!}}$$

which has the form

$$K_\alpha(z_1, z_2, w_1, w_2) = \frac{(1-\alpha^2)^2}{4\pi^2\alpha^2}\, e^{-\frac{1+\alpha^2}{4\alpha}(z_1 z_2 + \bar{w}_1 \bar{w}_2) + \frac{1-\alpha^2}{4\alpha}(z_1 \bar{w}_1 + z_2 \bar{w}_2)}$$

for $z_1, z_2, w_1, w_2 \in \mathbb{C}$.

(i) In the limit of $\alpha \to 1-$ the kernel $K_\alpha(z_1, z_2, w_1, w_2)$ tends to zero,

(ii) whereas, for $\alpha \to 0+$ the kernel $K_\alpha(z_1, z_2, w_1, w_2)$ goes to infinity.

LIMIT BEHAVIOR OF THE KERNEL

It can be shown that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^{\frac{m+n}{2}}}{m!n!}H_{m,n}(z_1,z_2)\overline{H_{m,n}(w_1,w_2)}$$

$$\xrightarrow{s\to 1}\pi\exp\left(\frac{z_1z_2+w_1w_2}{2}\right)\delta(z_1-w_1,z_2-w_2).$$

REMARK
The same happened for the van Eijndhoven-Meyers polynomials and the (standard) Hermite polynomials.

All of these are because the singularities mentioned in **(i)** and **(ii)**.

Here, *S. Twareque Ali's idea*:

The series is convergent for $0 < s < 1$ *and the Dirac delta appears only in the limit.*

applies.

VAN EIJNDHOVEN-MEYERS POLYNOMIALS; $\alpha \to 1-$

The functions $k_n^{(\alpha)}(z) \stackrel{\text{def}}{=} b_n(\alpha)^{-1/2} H_n(\frac{z}{\sqrt{1-\alpha}})$, $z \in \mathbb{C}$, form an orthonormal basis in $\tilde{\mathcal{H}}_\alpha$. The orthonormality can now be written explicitly as

$$\int_{\mathbb{C}} k_m^{(\alpha)}(z) \overline{k_n^{(\alpha)}(z)} \exp\left[-x^2 - \frac{1}{\alpha} y^2\right] \mathrm{d}x\mathrm{d}y = \delta_{m,n}.$$

$\tilde{\mathcal{H}}^{(\alpha)}$ is a reproducing kernel Hilbert space with the kernel

$$\begin{aligned} K^{(\alpha)}(z,w) &\stackrel{\text{def}}{=} \sum_{n=0}^{\infty} k_n^{(\alpha)}(z) \overline{k_n^{(\alpha)}(w)} \\ &= \frac{1+\alpha}{2\pi\sqrt{\alpha}} \exp\left[-\frac{1-\alpha}{2\alpha}(z^2 + \bar{w}^2) + \frac{1+\alpha}{2\alpha} z\bar{w}\right], \quad z, w \in \mathbb{C}. \end{aligned}$$

In the limit of $\alpha \to 1-$ the kernel $K^{(\alpha)}(z,w)$ goes to $\exp(z\bar{w})/\pi$.

VAN EIJNDHOVEN-MEYERS POLYNOMIALS; $\alpha \to 0+$

Another modification of the Hermite functions [2]

$$\hat{k}_n^{(\alpha)}(z) \stackrel{\text{def}}{=} \frac{1}{\sqrt{\pi n!}} \left[\frac{1-\alpha}{2(1+\alpha)}\right]^{n/2} H_n\left(\frac{x + \mathrm{i}\, y\sqrt{\alpha}}{\sqrt{1-\alpha}}\right), \quad z = x + \mathrm{i}\, y.$$

The functions $\hat{k}_n^{(\alpha)}$ satisfy the orthogonality relation

$$\int_{\mathbb{C}} \hat{k}_n^{(\alpha)}(z)\overline{\hat{k}_m^{(\alpha)}(z)}\, \mathrm{e}^{-|z|^2}\, \mathrm{d}x\, \mathrm{d}y = \delta_{n,m}, \quad z = x + \mathrm{i}\, y.$$

and gives the kernel

$$\hat{K}^{(\alpha)}(z,w) = \frac{1+\alpha}{2\pi}\, \mathrm{e}^{-\frac{1-\alpha}{2\alpha}[(x+\mathrm{i}y\sqrt{\alpha})^2 + (u-\mathrm{i}v\sqrt{\alpha})^2] + \frac{1+\alpha}{2\alpha}(x+\mathrm{i}\, y\sqrt{\alpha})(u-\mathrm{i}\, v\sqrt{\alpha})}$$

[2] They are no longer holomorphic, in fact they are polynomials in two real variables with complex coefficients.

HERMITE POLYNOMIALS IN TWO COMPLEX VARIABLES; $\alpha \to 1-$

We star with the polynomials

$$k_{m,n}^{(\alpha)}(z_1, z_2) \stackrel{\text{def}}{=} \frac{1}{\pi\sqrt{\alpha m! n!}} \left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{m+n}{2}} H_{m,n}\left(\frac{z_1}{\sqrt{1-\alpha}}, \frac{z_2}{\sqrt{1-\alpha}}\right),$$

which are orthogonal with respect to the measure $\exp\left[-\frac{1}{4}|\bar{z}_2 + z_1|^2 - \frac{1}{4\alpha}|\bar{z}_2 - z_1|^2\right]$. They form the orthonormal basis in $\mathcal{H}_2^{(\alpha)}$ with the reproducing kernel

$$K_2^{(\alpha)}(z_1, z_2, w_1, w_2) = \frac{(1+\alpha)^2}{4\pi^2\alpha^2}\, e^{-\frac{1-\alpha}{4\alpha}(z_1 z_2 + \bar{w}_1 \bar{w}_2) + \frac{1+\alpha}{4\alpha}(z_1 \bar{w}_1 + z_2 \bar{w}_2)},$$

which in the limit of $\alpha \to 1-$ gives the reproducing kernel in the two variable Bargmann space.

HERMITE POLYNOMIALS IN TWO COMPLEX VARIABLES; $\alpha \to 0+$

Following Twareque Ali's idea we modified the two variable complex Hermite polynomials

$$\hat{k}^{(\alpha)}_{m,n}(z,w) = \frac{1}{\pi\sqrt{m!n!}}\left(\frac{1-\alpha}{1+\alpha} - \frac{\epsilon}{2}\right)^{\frac{m+n}{2}} H_{m,n}\left(\frac{z-\sqrt{\alpha}w}{\sqrt{1-\alpha}}, \frac{\bar{z}+\sqrt{\alpha}\bar{w}}{\sqrt{1-\alpha}}\right),$$

$z, w \in \mathbb{C}$, whose reproducing kernel in the limit of $\alpha \to 0+$ is equal to

$$\begin{aligned}\hat{K}(z_1, w_1, z_2, w_2) &= \pi^{-2} \sum_{m,n=0}^{\infty} \frac{(1-\frac{\epsilon}{2})^{m+n}}{m!n!} H_{m,n}(z,\bar{z})\overline{H_{m,n}(w,\bar{w})} \\ &= \frac{4}{\pi^2(4-\epsilon)}\, e^{\frac{3-\epsilon}{4-\epsilon}(z\bar{z}+w\bar{w}) - \frac{1}{4-\epsilon}(z\bar{w}+\bar{z}w)}\, \frac{1}{\epsilon}\, e^{-\frac{(z-w)(\bar{z}-\bar{w})}{\epsilon}} .\end{aligned}$$

THANKS FOR YOUR ATTENTION
(MERCI)