Workshop on CS, CIRM, Marseille 2016

Affine CS Quantization and Quantum Cosmology

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Outline

- **Covariant Integral Quantization**
- **Quantization of the Half-Plane : the Affine group**
- **Quantum Cosmology (QC)**
- **Summary**

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… Covariant Integral Quantization

- **Integral Q.** (IQ)[a] : framework to "quantize a set X" (fcts on X).
  Covariant IQ (CIQ) : Lie group implementation of IQ.
- The ingredients of CIQ
  - A Lie group $G \equiv X$ with left invariant measure $\mathrm{d}\mu(g)$,
  - A Unitary Irreducible Representation (UIR) $g \mapsto U_g$ of G in a Hilbert space $\mathcal{H}$.
  - A bounded self-adjoint operator M on $\mathcal{H}$.
- We define g-translations of M : $M(g) = U_g M U_g^{\dagger}$
- $\Longrightarrow$ Resolution of the identity (Schur's lemma)

$$\int_G M(g) \frac{\mathrm{d}\mu(g)}{c_M} = I_{\mathcal{H}}, \quad \text{for some } c_M \in \mathbb{R}$$

(a) "*Integral quantizations with two basic examples*", H. Bergeron, J. P. Gazeau, Annals of Physics, **344**, 43 (2014)

. . . Covariant Integral Quantization

- **The Quantization map** : $\boxed{f \text{ (fct on } G) \mapsto A_f \text{ (operator on } \mathcal{H})}$

$$f \mapsto A_f = \int_G f(g) M(g) \frac{\mathrm{d}\mu(g)}{c_M}$$

- **Covariance** : $\boxed{U_g A_f U_g^\dagger = A_{\mathcal{U}_g f}}$ with $(\mathcal{U}_g f)(g') = f(g^{-1}g')$
- **Expectation values** : ρ being a quantum density,

Quantum $\rightarrow$ $\boxed{\mathrm{Tr}\rho A_f = \int_G f(g)\, \mathrm{Tr}[\rho M(g)]\, \frac{\mathrm{d}\mu(g)}{c_M}}$ $\leftarrow$ classical-like

- **Consistent semi-classical portrait** $f \mapsto A_f \mapsto \check{f}$ (lower symbols) :

$$f(g) \mapsto \check{f}(g) = \mathrm{Tr}[M(g) A_f] = \int_G f(g') \mathrm{Tr}[M(g) M(g')]\, \frac{\mathrm{d}\mu(g')}{c_M}$$

Generally $\check{f} \neq f$: $\implies$ $\check{f}$ **is** $\hbar$ **corrected**. Consistency : $\lim_{\hbar \to 0} \check{f} = f$.

… CS Quantization

- **CS Quantization** : $\boxed{M = |\psi\rangle\langle\psi|}$

  where $|\psi\rangle$ is a unit vector which is “admissible" :

$$c_M \equiv c_\psi = \int_G |\langle\psi|U_g|\psi\rangle|^2 \, \mathrm{d}\mu(g) < \infty$$

- $\Longrightarrow \boxed{M(g) = |\psi_g\rangle\langle\psi_g|} \Longrightarrow$ **Coherent States** $\boxed{|\psi_g\rangle = U_g|\psi\rangle}$
- **Quantization map** :

$$f \mapsto A_f = \int_G f(g)\, |\psi_g\rangle\langle\psi_g| \, \frac{\mathrm{d}\mu(g)}{c_\psi}$$

- **Expectation values** : $\langle\phi|A_f|\phi\rangle = \int_G f(g)\, |\langle\psi_g|\phi\rangle|^2 \, \mathrm{d}\mu(g)/c_\psi$
- **Lower Symbols** : $f(g) \mapsto \check{f}(g) = \langle\psi_g|A_f|\psi_g\rangle$

. . . Special cases

- **Quantization of the plane** $\{(q,p) \in \mathbb{R}^2\}$: **Weyl-Heisenberg** group
  - $\{(q,p) \in \mathbb{R}^2\} \equiv$ phase space of a particle on the real line
  - $M = 2\mathcal{P}$ (parity $\equiv \mathcal{P}$) $\Longrightarrow$ Weyl-Wigner transform ("canonical quantization" $\Longrightarrow$ symmetric ordering),
  - $M = |\psi_0\rangle\langle\psi_0|$ (with $\langle x|\psi_0\rangle$ gaussian) $\Longrightarrow$ "standard" CSQ, (Berezin–Klauder–Toeplitz quantization $\Longrightarrow$ anti-normal ordering)
  - Other choices for M : generalized framework ($\Longrightarrow$ generalized ordering[(a)]).
- **Quantization of the half-plane** $\{(q,p) \in \mathbb{R}_+^* \times \mathbb{R}\}$
  - $\{(q,p) \in \mathbb{R}_+^* \times \mathbb{R}\} \equiv$ phase space of a particle on the half-line, or volume-expansion pair (Friedman, Bianchi models for gravity).
  - The "natural group" is **is not anymore** the Weyl-Heisenberg group but the **Affine group**.

(a) K.E. Cahill, R. Glauber, Phys. Rev. **177** (1969) 1857–1881

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. . . Half-Plane-Affine Group

- **Affine group** $\{(q,p) \in \mathbb{R}_+^* \times \mathbb{R}\}$:

$$\boxed{(q,p).(q_0,p_0) = (qq_0, p + \frac{p_0}{q})}$$, *left inv. measure* $\boxed{\mathrm{d}q\mathrm{d}p}$.

- **UIR :** $\mathcal{H} = L^2(\mathbb{R}^+, \mathrm{d}x) \ni \psi(x) \mapsto \boxed{U_{q,p}\psi(x) = \frac{e^{ipx}}{\sqrt{q}}\psi(x/q)}$.

- **Affine CS** (ACS) : $\boxed{|q,p\rangle = U_{q,p}|\psi_0\rangle}$ with

$\int_{\mathbb{R}_+} \mathrm{d}x\, |\psi_0(x)|^2 = 1$ and $c_{-1} = \int_{\mathbb{R}_+} \frac{\mathrm{d}x}{x} |\psi_0(x)|^2 < \infty$

- **Choice of the "fiducial vector"** ψ_0 (conditions not mandatory) :
  - $\psi_0(x) \in \mathbb{R}$, $\psi_0 \in \mathcal{S}(\mathbb{R}_+)$ (fcts of rapid decrease on $\mathbb{R}_+$)
  - $\implies \forall\gamma,\ c_\gamma = \int_{\mathbb{R}_+} \frac{\mathrm{d}x}{x^{2+\gamma}} |\psi_0(x)|^2 < \infty$

- **Quantization :** $f(q,p) \mapsto \boxed{A_f = \int_{\mathbb{R}_+\times\mathbb{R}} f(q,p)\, |q,p\rangle\langle q,p|\, \frac{\mathrm{d}q\mathrm{d}p}{2\pi c_{-1}}}$.

ACS Quantization : Main Features

Assumptions for the fiducial vector $\psi_0 : \psi_0(x) \in \mathbb{R}$, $\psi_0 \in \mathcal{S}(\mathbb{R}_+)$

- $\boxed{A_p = P \equiv -i\frac{d}{dx}}$ and $\boxed{A_q = \frac{c_0}{c_{-1}} Q}$ with $Q\phi(x) = x\phi(x)$
  - A_p is symmetric (using $\mathcal{S}(\mathbb{R}_+)$ as domain) but has **no self-adjoint extension** on $\mathcal{H} = L^2(\mathbb{R}_+, \mathrm{d}x)$.[(a)]
  - A_q is self-adjoint on $\mathcal{H}$.
- $\boxed{[A_q, A_p] = i(c_0/c_{-1})I_{\mathcal{H}}}$, [(a)] $\boxed{A_{q^\beta} = \frac{c_{\beta-1}}{c_{-1}} Q^\beta}$
- $\boxed{A_{p^2} = P^2 + K_\psi Q^{-2}}$ with $K_\psi = \int_{\mathbb{R}_+} \frac{\mathrm{d}u}{c_{-1}} u\, \psi_0'(u)^2 > 0$
  - A_{p^2} has a unique self-adjoint extension for $K \geq 3/4$, while P^2 has different possible self-adjoint extensions (choice of boundary conditions at $x = 0$).

(a) The rule $[Q, P] = i\lambda I$ holds true with self-adjoint Q <u>and</u> P, only if both have continuous spectrum $(-\infty, +\infty)$. Here Q self-adjoint $\geq 0 \implies P$ not self-adjoint.

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QC : advantages of ACS quantization

- Manageable quantum equations for isotropic and anisotropic models.
- Singularity resolution
- Unitary evolution (unambiguous and unique)
- Consistent (“natural”) semi-classical description

Remark1 (Klauder’s approach) : the use of an “affine" quantization instead of the Weyl-Heisenberg one is present in Klauder’s work on quantum gravity (see e.g. An Affinity for Affine Quantum Gravity, Proc. Steklov Inst. of Math. 272, 169-176 (2011) ; gr-qc/1003.2617 for recent references). But the procedure is not based on an “integral quantization".

Remark2 (Fanuel-Zonetti’s approach) : in M. Fanuel and S. Zonetti, EPL **101** (2013), [hep-th]/1203.4936, Fanuel & *al.* use affine coherent states, but only with a special choice of ψ_0 (“anti-normal ordering"). It is not an integral framework.

QC : Oscillatory Singularity[a]

Vacuum Bianchi IX (Mixmaster)

The metric ($q = a^{3/2}$ here) : $ds^2 = -(24)^2 \mathcal{N}^2 \mathrm{d}t^2 +$

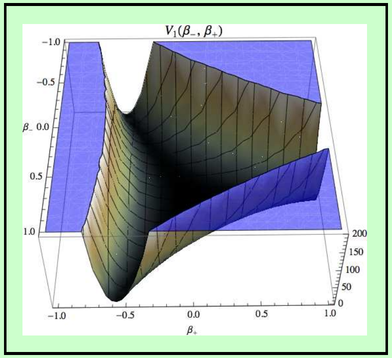
$$q^{4/3}\left(e^{2\beta_+ + 2\sqrt{3}\beta_-}\omega^1 \otimes \omega^1 + e^{2\beta_+ - 2\sqrt{3}\beta_-}\omega^2 \otimes \omega^2 + e^{2\beta_+}\omega^3 \otimes \omega^3\right)$$

The classical Hamiltonian "h"

- $$h = \mathcal{N}\,\mathcal{C}; \quad \mathcal{C} = \frac{9}{4}p^2 + 36 q^{2/3} - h_q^{\mathsf{anis}}$$
$$h_q^{\mathsf{anis}} = \frac{1}{q^2}\left(p_+^2 + p_-^2\right) + 12 q^{2/3} V(\beta_\pm)$$

- Constraint : $\frac{\partial h}{\partial \mathcal{N}} = \mathcal{C} = 0.$

Bianchi IX potential



(a) "*H. Bergeron, E. Czuchry, J.P. Gazeau, P. Malkiewicz, and W. Piechocki, Phys. Rev. D **92**, 061302(R) (2015) ; Phys. Rev. D, **92**, 124018 (2015) ; Phys. Rev. D **93**, 064080 (2016) ; Phys. Rev. D **93**, 124053 (2016)*

QC : Bianchi IX (adiabatic)

- **ACS (isotropic part) + Canonical quant. (anisotropic part)**

$$\mathcal{C} \mapsto A_{\mathcal{C}} = \frac{9}{4}\left(P^2 + \hbar^2 \frac{K_1}{Q^2}\right) + 36 K_3 Q^{2/3} - A_{h^{\mathsf{anis}}}(Q)$$

$$A_{h^{\mathsf{anis}}}(q) = \sum_n E_n(q) |e_n(q)\rangle\langle e_n(q)|$$

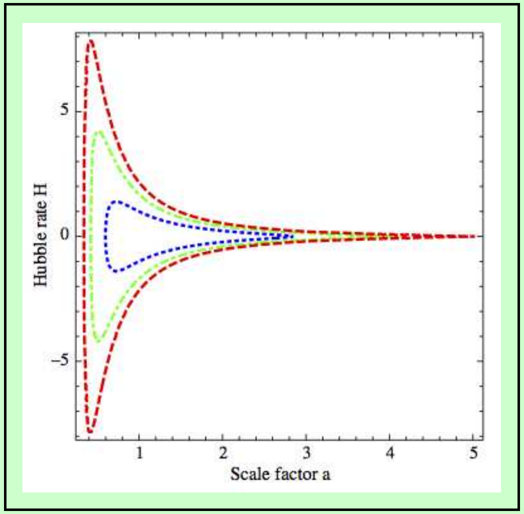
Klauder $\oplus$ Born-Oppenheimer approx. : $|\Psi_t\rangle = |q_t, p_t\rangle \otimes |e_N(q_t)\rangle$

Modified Friedman equation
(obtained from lower symbols)

$$H^2 + \frac{4\pi^2 G^2 \hbar^2}{c^4} \frac{K_4}{a^6} + \frac{K_5 c^2}{4a^2} = \frac{8\pi G}{3c^2 a^3} E_N(a^{3/2}) \simeq \frac{8\pi G \hbar}{3c} (N+1) \frac{K_6}{a^4}$$

- $K_4 a^{-6}$ Repulsive term from ACS
- $N \equiv$ Number of gravitons
- $\lim_{a\to 0} a^3 E_N(a^{3/2}) = 0$

$\Longrightarrow$ repulsive term dominant



Three trajectories in the half-plane (a, H)

QC : Bianchi IX (beyond BO)$^{(a)}$

Vibronic approximation : $|\Psi_t\rangle = |q_t, p_t\rangle \otimes |e(t)\rangle, \quad |e(t)\rangle = \sum_N c_N(t)|e_N(q_t)\rangle$

- **Semi-Classical Lagrangian-Hamiltonian dynamics (Klauder)** :

$$\mathsf{L}(\Psi, \dot{\Psi}, \mathcal{N}) = \langle \Psi_t \left| \left(\left(i\hbar \frac{\partial}{\partial t} - \mathcal{N}\mathcal{C} \right) \right| \Psi_t \right\rangle$$

- **Equations from** $\mathsf{L}(\Psi, \dot{\Psi}, \mathcal{N})$:

$$\dot{q} = \mathcal{N} \frac{\partial \langle \Psi | A_{\mathcal{C}} | \Psi \rangle}{\partial p}, \ \dot{p} = -\mathcal{N} \frac{\partial \langle \Psi | A_{\mathcal{C}} | \Psi \rangle}{\partial q}, \ \frac{\hbar}{i} \frac{\partial}{\partial t} |e\rangle = \mathcal{N} \langle q, p | A_{h^{\mathsf{anis}}}(q) | q, p \rangle |e\rangle$$

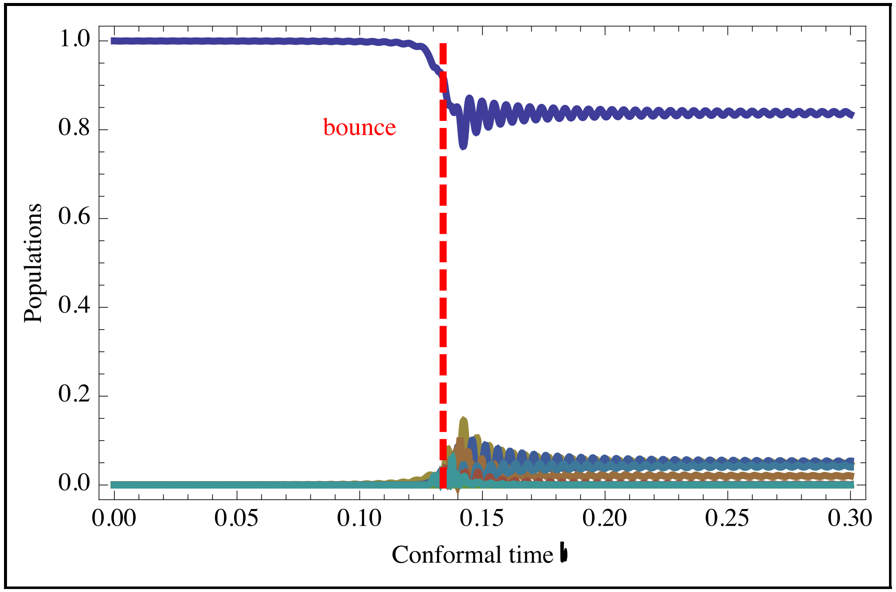
- **Consistency of the semi-classical constraint** :

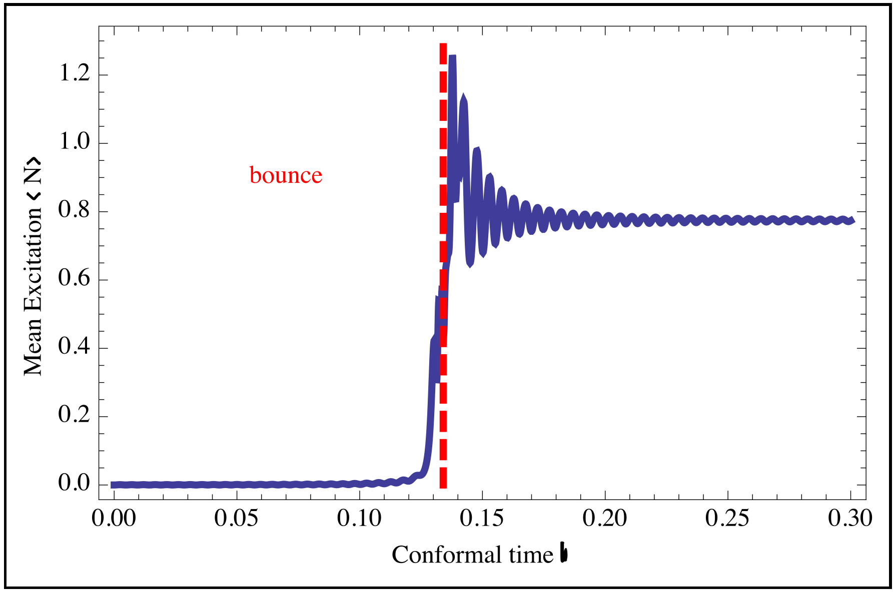
$$\mathsf{L}(\Psi, \dot{\Psi}, \mathcal{N}) \implies \frac{\partial \mathsf{L}}{\partial \mathcal{N}} = \langle \Psi | A_{\mathcal{C}} | \Psi \rangle = 0$$

(a) *"H. Bergeron, E. Czuchry, J.P. Gazeau, P. Malkiewicz, Phys. Rev. D* ***93****, 064080 (2016), arXiv :1512.00304v1 [gr-qc]*

Bianchi IX (beyond BO) continued

First Example $|e(t=0)\rangle = |N=0\rangle$

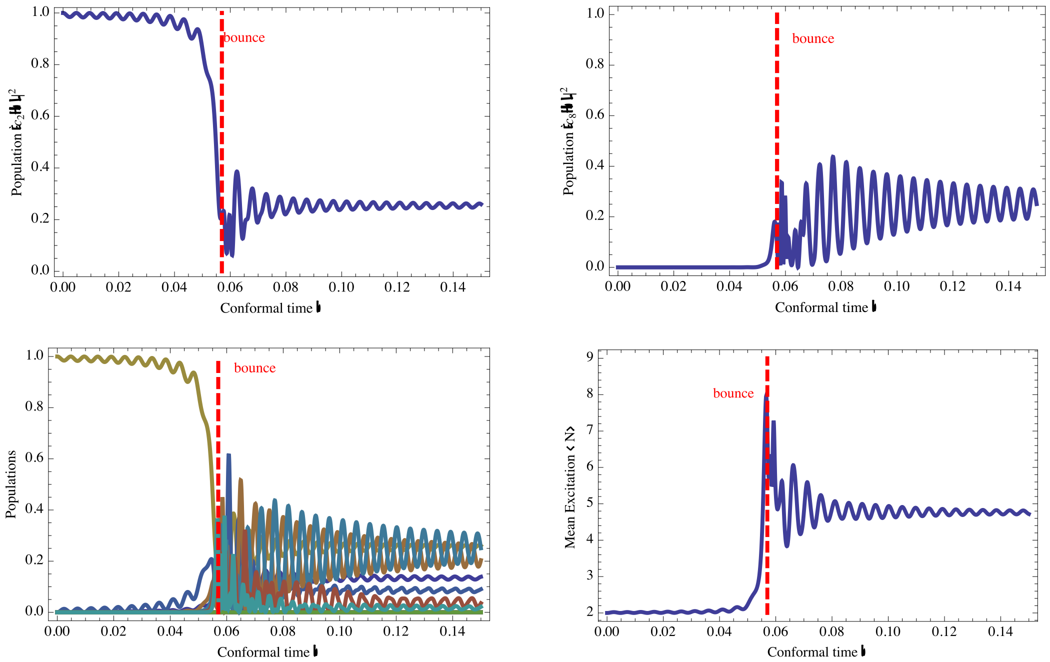




Evolution with **conformal time** η when $a(\eta=0)=5$ and $|e(\eta=0)\rangle = |N=0\rangle$.
Left panel : the evolution of the populations $|c_N(\eta)|^2$ for $N = 0, 1, \ldots, 12$.
$|c_0(\eta)|^2$ corresponds to the curve on the top.
Right panel : the mean excitation $< \hat{\mathsf{N}} > (\eta)$.

Bianchi IX (beyond BO) continued

Second Example $|e(t=0)\rangle = |N=2\rangle$



Evolution of the quantum state when $a(\eta = 0) = 5$ and $|e(\eta = 0)\rangle = |N = 2\rangle$. **Top left panel** : the decay of the initial energy level $N = 2$. **Top right panel** : the excitation of the energy level $N = 8$. **Bottom left panel** : the evolution of the populations $|c_N(\eta)|^2$ for $N = 0, 1, \ldots, 12$. **Bottom right panel** : the mean excitation $< \hat{\mathrm{N}} > (\eta)$.

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Summary

- (Covariant) Integral Quantization provides a very nice framework to quantize non-standard phase spaces.
- In the case of Quantum Cosmology :
  - ACS resolve the singularity (all situations),
  - ACS provide a very convenient semi-classical framework,
  - ACS combined with methods coming from molecular physics (Born-Oppenheimerg, vibronic approximations) provide a detailed description of quantized oscillatory singularities.

THANK YOU

Recent Papers

1. “*Are the canonical and coherent state descriptions physically equivalent ?*", H. Bergeron, J.P. Gazeau, A. Youssef, Phys. Lett. A, **377** 598 (2013)
2. “*Integral quantizations with two basic examples*", H. Bergeron and J.P Gazeau, Annals of Physics (NY), **344** 43-68 (2014)
3. “*Smooth big bounce from affine quantization*", H. Bergeron, A. Dapor, J.P. Gazeau and P. Malkiewicz, Phys. Rev. D **89**, 083522 (2014)
4. “Smooth Bounce in Affine Quantization of Bianchi I", H.Bergeron, A.Dapor, J.P.Gazeau, P. Malkiewicz, Phys. Rev. D, **91**, 124002 (2015), [arXiv :1501.07718]
5. “*Smooth quantum dynamics of the mixmaster universe*", H.Bergeron, E. Czuchry, J.P.Gazeau, P. Malkiewicz and W. Piechocki, Phys. Rev. D, **92**, 061302(R) (2015)
6. “Singularity avoidance in a quantum model of the Mixmaster universe", H.Bergeron, E.Czuchry, J.P.Gazeau, P.Malkiewicz, W.Piechocki, Phys.Rev. D, **92**, 124018 (2015) [arXiv :1501.07871]
7. “*Vibronic framework for quantum mixmaster universe*", H.Bergeron, E.Czuchry, J.P. Gazeau, P. Malkiewicz, Phys. Rev. D **93**, 064080 (2016)
8. “ Nonadiabatic bounce and an inflationary phase in the quantum mixmaster universe", H.Bergeron, E.Czuchry, J.P.Gazeau, P.Malkiewicz, Phys. Rev. D, **93**, 124053 (2016)