# Univariate Non-linear Approximation Scheme for Piecewise Smooth functions

(In progress)

Jungho Yoon
Ewha W. University, Seoul, Korea

(Joint work with Hyoseon Yang)

MAIA 2016, CIRM, Luminy, Sep. 23, 2016

# Outline


Motivations

Linear Scheme: Quasi-Interpolation
- Quasi-Interpolation based on B-splines
- Modifed Quasi-Interpolation based on B-splines

Construction of a Non-Linear Scheme
- Troubled-Cell Detector
- Smoothness Indicators
- Error Analysis

Experimental Results

# Outline

Motivations

Linear Scheme: Quasi-Interpolation
- Quasi-Interpolation based on B-splines
- Modifed Quasi-Interpolation based on B-splines

Construction of a Non-Linear Scheme
- Troubled-Cell Detector
- Smoothness Indicators
- Error Analysis

Experimental Results

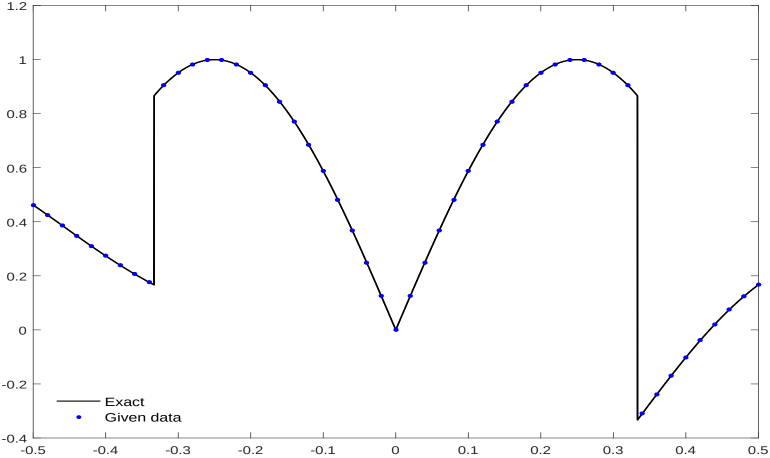
# Approximation to Piecewise Smooth Functions

The issue of approximation to piecewise-smooth functions arises in many scientific areas:

## Applications:

- Numerical Solution of Hyperbolic Conservation Laws
- Computer graphics
- (Medical) Image Processing (zooming, registeration, motion correction, deformation)

# An example of piecewise-smooth function



# Linear Schemes

## Advantages

- Flexibility (e.g., Polynomial/RBF interpolation, Quasi-interpolation, Subdivision, MLS, ...)
- Simplicity and Fast Computation
- High Accuracy to Smooth Data

## Limitation

- Introduce artifacts (e.g., Gibbs(-like) Phenomenon near discontinuities and and Staircase/ringing effects in image interpolation)

# Linear Schemes

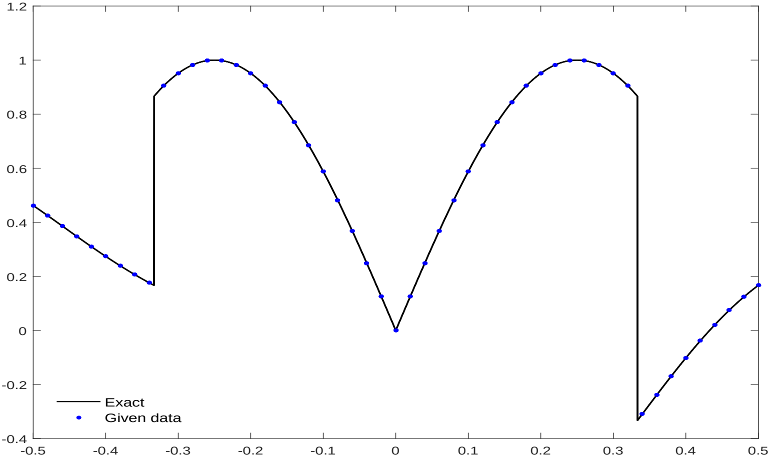
## Advantages

- Flexibility (e.g., Polynomial/RBF interpolation, Quasi-interpolation, Subdivision, MLS, ...)
- Simplicity and Fast Computation
- High Accuracy to Smooth Data

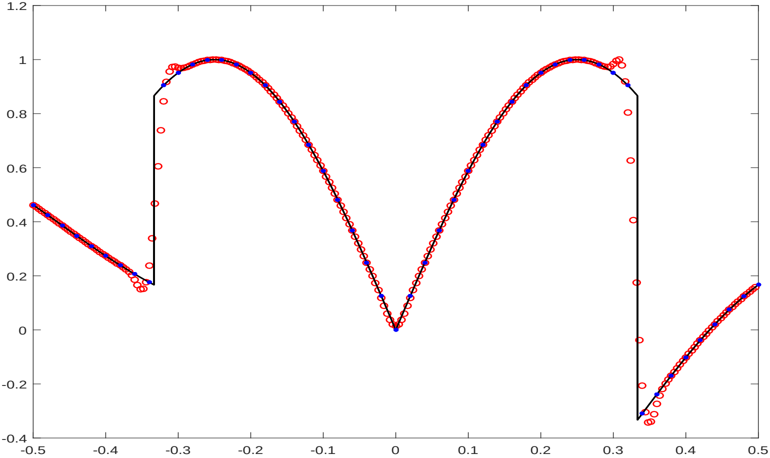
## Limitation

- Introduce artifacts (e.g., Gibbs(-like) Phenomenon near discontinuities and and Staircase/ringing effects in image interpolation)

# Quasi-interpolation based on cubic B-spline



# Quasi-interpolation based on cubic B-spline



# Previous Works of Non-linear Methods:

- ENO(Essentially Non-Oscillatory)/ENO-SR Schemes
- Weighted ENO (WENO) Schemes
- ETC

# Goal:

Construct a WENO-type non-linear scheme capable of

- maintaining high-order accuracy in smooth regions
- capturing steep gradients without creating spurious oscillations.

# Outline

Motivations

Linear Scheme: Quasi-Interpolation
- Quasi-Interpolation based on B-splines
- Modifed Quasi-Interpolation based on B-splines

Construction of a Non-Linear Scheme
- Troubled-Cell Detector
- Smoothness Indicators
- Error Analysis

Experimental Results

# Quasi-interpolation

Assume that a function $f : \mathbb{R} \to \mathbb{R}$ is known only at discrete values on the uniform grid $h\mathbb{Z}$. The quasi-interpolations based on the B-spline of degree $k$ is defined by

$$L_h f(x) = \sum_{n \in \mathbb{Z}^d} B_{n,h}(x) Q f(hn)$$

with a suitable operator $Q$ such that $L_h$ reproduces polynomials up to degree $k$, i.e., $L_h p = p$ for all $p \in \Pi_k$, where

$$B_{n,h}(x) = B_k(x/h - n)$$

and $\Pi_k$ is the space of polynomials up to degree $k$.

For the case of cubic B-spline, $Qf$ may be definded as

$$Qf = f - \frac{1}{6} f''$$

Note that

$$Qf(hn) \approx f(x)$$

## Notation

For a positive odd integer $k = 2\bar{k} + 1$ with $\bar{k} \in \mathbb{N}$, let

$$\mathbb{Z}_k := \{0, \pm n : n = 1, \ldots, \bar{k}\}$$

Define $\ell_n$ be the Lagrange polynomial on $\mathbb{Z}_k$ such that

$$\ell_n(j) = \delta_{n,j}, \quad j, n \in \mathbb{Z}_k$$

where $\delta_{n,j}$ is the Kronecker delta.

# Modified Quasi-interpolation

Define define an operator $\overline{Q}_{n,h}$ for each $n \in \mathbb{Z}$ by

$$\overline{Q}_{n,h}f(x) := \sum_{j\in\mathbb{Z}_k} \ell_j(h^{-1}x - n)f(h(n+j)), \tag{1}$$

which is in fact the polynomial interpolation on $\mathbb{Z}_k$.
The quasi-interpolations based on the B-spline of degree $k$ is defined by

$$L_h f(x) := \sum_{n\in\mathbb{Z}} B_{n,h}(x)\overline{Q}_{n,h}f(x)$$

Note here that $\overline{Q}_{n,h}f - f(x) = O(h^k)$ if $f$ is $C^k$ around $x$.

# Modified Quasi-interpolation

## Lemma

*For a given $h > 0$, we have $L_h p = p$ for all $p \in \Pi_k$.*

## Theorem

*Suppose that $f \in C^{k+1}$ and $f^{(k+1)}$ is uniformly bounded. For a given $h > 0$, we have*

$$|L_h f(x) - f(x)| \leq c\|f^{(k+1)}\|_\infty h^{k+1} \tag{2}$$

*with a constant $c > 0$ depending on $f$, but independent of $h > 0$*

# Outline

Motivations

Linear Scheme: Quasi-Interpolation
- Quasi-Interpolation based on B-splines
- Modifed Quasi-Interpolation based on B-splines

Construction of a Non-Linear Scheme
- Troubled-Cell Detector
- Smoothness Indicators
- Error Analysis

Experimental Results

# Non-linear scheme

In this talk, we particularly consider the scheme $L_h f$ based on the cubic B-spline. As a non-linear counterpart of $L_h f$, we wish to define a non-linear map as follows:

$$N_h f(x) = \sum_{n \in \mathbb{Z}} K_{n,h}(x) \overline{Q}_{n,h} f(x) \tag{3}$$

where

$$\overline{Q}_{n,h} f(x) := \sum_{j=-1}^{1} \ell_j (h^{-1}x - n) f(h(n+j)), \tag{4}$$

# Non-linear Scheme

The local solutions $\overline{Q}_{n,h}f(x)$ are determinded by the follwing stencils:

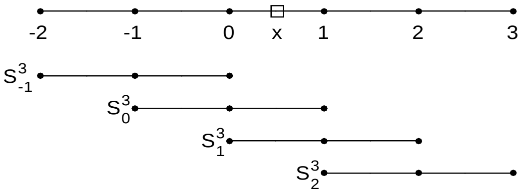



Figure: Stencils corresponding to $\overline{Q}_{n,h}f$ with $n = -1, \ldots, 2$.

# Modified Quasi-interpolation

The non-linear kernels $K_{n,h}$ are designed to reflect the local feature of the data around $x$. They are especially required to satisfy the following properties:

- It prevents spurious oscillations around discontinuities.
- The approximation $N_h f$ mimics the linear quasi-interpolation $L_h f$ in smooth regions

In this talk, we are particularly interested in detecting jump discontinuity in the first derivative and analyzing error bound near discontinuity.

Suppose that $f$ is continuous and $f \in C^4(\mathbb{R} \setminus \{s\})$ and $f^{(\alpha)}$, $1 \leq \alpha \leq 4$ is uniformly bounded on $\mathbb{R} \setminus \{s\}$. For convenience, assume that the evaluation point $x$ is in $I_0 := [0, h)$ and a singular point $s$ is near $x$, that is $s \in (-2h, 3h)$.

Note that $N_h f(x)$ is constructd by envolving the values $f(hn)$, $n = -2, \ldots, 3$.

## Key Ingriedients for $N_h f(x)$:

- Troubled Cell Detector
- Smoothness Indicators

# Divided Difference

Let $S := \{-1, 0, 1, 2\}$ be the $4$-point stencil. The generalized undivided difference of order $\nu$ is defined by the following two steps. First, for each $\nu = 2, 3$, we find the coefficient vector $\boldsymbol{\alpha}^{[\nu]} := (\alpha_i^{[\nu]} : i = -1, 0, 1, 2)^T$ by solving the linear system

$$\mathbf{V} \cdot \boldsymbol{\alpha}^{[\nu]} = \boldsymbol{\delta}^{[\nu]} \tag{5}$$

with

$$\mathbf{V} := ((i - 1/2)^\ell / \ell! : i = -1, \dots, 2, \quad \ell = 0, \dots, 3),$$
$$\boldsymbol{\delta}^{[\nu]} := (\delta_{\nu,\ell} : \ell = 0, \dots, 3)^T.$$

These coefficients are then utilized in defining the generalized undivided differences of order $\nu$ on the stencil $S$:

$$D_j^\nu f := \sum_{n=-1}^{2} \alpha_n^{[\nu]} f(h(j + n)). \tag{6}$$

# Troubled Cell Detector

Define

$$W_n f := |D_n^2 f| + \frac{1}{h}|D_n^3 f|$$

Consider the measurement $\theta$ defined by

$$\theta := \theta(f,s) := \frac{\min(W_{-1}f,\ W_1 f) + h^2}{\max(W_{-2}f, W_2 f) + h^2}, \tag{7}$$

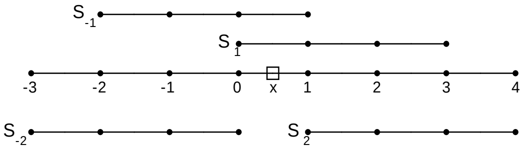



**Figure :** Stencils for troubled-cell detector

# Properties of $\theta$

- If $f$ is smooth, then $\theta$ converges to $1$ as $h \to 0$.
- If $s$ is outisde of $I_0$, then

$$\theta \leq 1 + c_f h$$

  for some constant $c_f$ depending on $f$.
- Let $A > 1$. Then there exists $c_A > 0$ such that

$$\theta > A$$

  for $s \in (\delta, h - \delta)$ with $\delta = c_A h^3$.

## Notation

$$I_\delta := (\delta, h - \delta).$$

# The case $\theta < A$

In this case, $s$ is not in $I_\delta$. Then the smoothness indicators $\mathrm{SI}_n$ for $\overline{Q}_{n,h}f$ are defined by

$$\begin{aligned} \mathrm{SI}_{-1} &:= W_{-1}f, \\ \mathrm{SI}_n &:= W_n f \quad (n = 0, 1) \\ \mathrm{SI}_2 &:= W_2 f. \end{aligned} \tag{8}$$

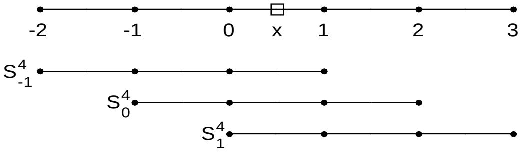



Figure: Stencils for the case $\theta < A$.

# Case $\theta < A$

In particular, let

$$\mathrm{SI}_\mu = \min\{\mathrm{SI}_n : n = -1, \dots, 2\}.$$

Then, we put

$$\begin{aligned} \alpha_n &:= \frac{B_{n,h}(x)}{(\mathrm{SI}_n + \epsilon_h)^2}, \qquad \text{if} \quad \frac{\mathrm{SI}_\mu}{\mathrm{SI}_n} \geq (1 - Ch) \\ \alpha_n &:= \frac{B_{n,h}(x)}{\mathrm{SI}_n + \epsilon_h}, \qquad \text{otherwise}, \end{aligned}$$

where $0 < \varepsilon_h < 1$ is introduced to prevent the denominator becoming zero.

# Case $\theta \geq A$

Choose a smooth solution $\overline{Q}_{-1,h}f$ or $\overline{Q}_{2,h}f$ in terms of the evaluation point $x$, say $\overline{Q}_{\mu,h}f$. Then we put

$$\alpha_\mu := \frac{B_{\mu,h}(\frac{h}{2})}{(\mathrm{SI}_n + \epsilon_h)^2},$$
$$\alpha_n := \frac{B_{n,h}(x)}{\mathrm{SI}_n + \epsilon_h}, \quad \text{if } n \neq \mu,$$

where $0 < \varepsilon < 1$ is introduced to prevent the denominator becoming zero.

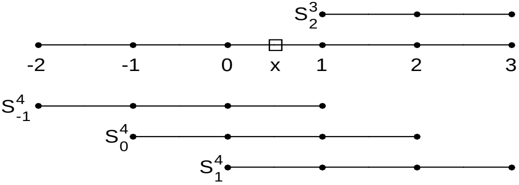
# Smoothness Indicators for $\theta \geq A$

The smoothness indicators on a stencil $S$ (with $\#S = 3$ or $4$) is defined by

$$\begin{aligned} \mathrm{SI}_\mu &:= |\Delta_\mu^2 f|, && \text{if } \#S = 3, \\ \mathrm{SI}_n &:= W_n f && \text{if } \#S = 4, \end{aligned} \tag{9}$$

where $\Delta_n^2 f$ is the second order difference, that is,

$$\Delta_n^2 f := \Delta_{n,h}^2 f := f(hn) - 2f(h(n+1)) + f(h(n+2))$$



# Final Approximation

Then, we define nonlinear weights $K_{n,h}(x)$ corresponding to $B_{n,h}(x)$ by

$$K_{n,h}(x) := \frac{\alpha_n(x)}{\sum_\ell \alpha_\ell(x)}. \tag{10}$$

Accordingly, the final non-linear approximation $N_h f$ is given as

$$N_h f(x) = \sum_n K_{n,h}(x) \overline{Q}_{n,h} f(x).$$

# Error Analysis (in Smooth Region)

## Theorem 3.1

Let the non-linear approximation scheme $N_h$, $h > 0$, be defined as above. If $f \in C^4(\Omega)$ and $x$ is inside of $\Omega$, then,

$$|f(x) - N_h f(x)| \leq c_f h^4$$

with a constant $c_f > 0$ depending on $f$.

Proof is based on the relation:

$$\begin{aligned} f(x) - N_h f(x) &= f(x) - L_h f(x) + L_h f(x) - N_h f(x) \\ &= f(x) - L_h f(x) \\ &\quad + \sum_{n \in \mathbb{Z}} \big(B_{n,h}(x) - K_{n,h}(x)\big(f(x) - \overline{Q}_{n,h}(x)\big) \end{aligned}$$

Also, it can be shown by a direct calculation that in smooth region

$$|B_{n,h}(x) - K_{n,h}(x)| = O(h^2), \quad h \to 0. \tag{11}$$

# Error Analysis (in Non-Smooth Region)

Let $S = \{nh : n = -1, \ldots, 2\}$ and assume that $s \in (0, h)$. Note that

$$\begin{aligned} D^3 f &= \sum_{n \in S} c_n f(hn) \\ &= \tau_1(s) h [f'](s) + \tau_2(s) h^2 [f''](s) + O(h^3) \end{aligned}$$

where $S^+ = \{hn \in S : hn > s\} = \{h, 2h\}$

$$\begin{aligned} \tau_1(s) &= \sum_{n \in S^+} c_n (n - s/h) \\ \tau_2(s) &= \sum_{n \in S^+} c_n (n - s/h)^2 \end{aligned}$$

# Error Analysis (in Non-Smooth Region)

## Theorem 3.1

Let

$$h_0 := \max\left(\frac{\left|[f'](s)\right|}{2\|f''\|_\infty^*},\ \frac{1}{2\|f'''\|_\infty^*}\left|\tau_1(s)\frac{[f'](s)}{h} + \tau_2(s)[f''](s)\right|\right)$$

where

$$\|f\|_\infty^* := \|f\|_{L^\infty(\mathbb{R}\setminus\{s\})}.$$

If $f \in C^3(\Omega \setminus \{s\})$ and $f^{(\alpha)}$, $0 \le \alpha \le 3$, is uniformly bounded on $\Omega \setminus \{s\}$ for $\alpha \le 3$, then,

$$|f(x) - N_h f(x)| \le c_f h^3, \quad \text{for} \quad h \le h_0$$

with a constant $c > 0$ depending on $f$.

# Error Analysis (in Non-Smooth Region)

Proof is based on the relation:

$$\begin{aligned} f(x) - N_h f(x) &= f(x) - \sum_{n\in\mathbb{Z}} K_{n,h}(x)\overline{Q}_{n,h}(x) \\ &= \sum_{n\in\mathbb{Z}} K_{n,h}(x)\big(f(x) - \overline{Q}_{n,h}(x)\big) \end{aligned}$$

# Outline

Motivations

Linear Scheme: Quasi-Interpolation
- Quasi-Interpolation based on B-splines
- Modifed Quasi-Interpolation based on B-splines

Construction of a Non-Linear Scheme
- Troubled-Cell Detector
- Smoothness Indicators
- Error Analysis

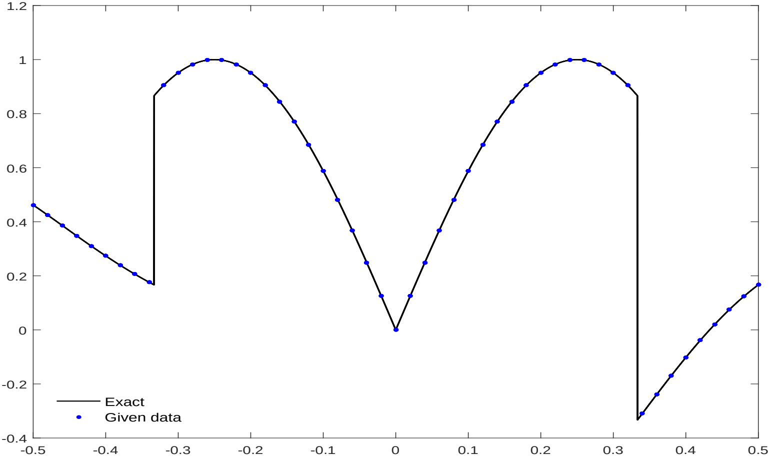
Experimental Results

# Experimental Results

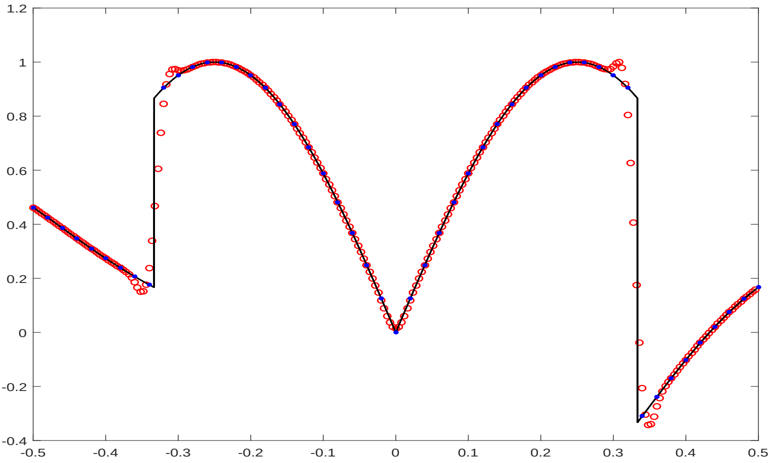
## Approximation Performance

$$f_5(x) := \begin{cases} -x\sin\left(\frac{3\pi}{2}x^2\right), & \text{if } -1 \le x \le -\frac{1}{3}, \\ |\sin(2\pi x)|, & \text{if } -\frac{1}{3} < x \le \frac{1}{3}, \\ 2x - 1 - \frac{1}{6}\sin(3\pi x), & \text{if } \frac{1}{3} < x \le 1. \end{cases}$$

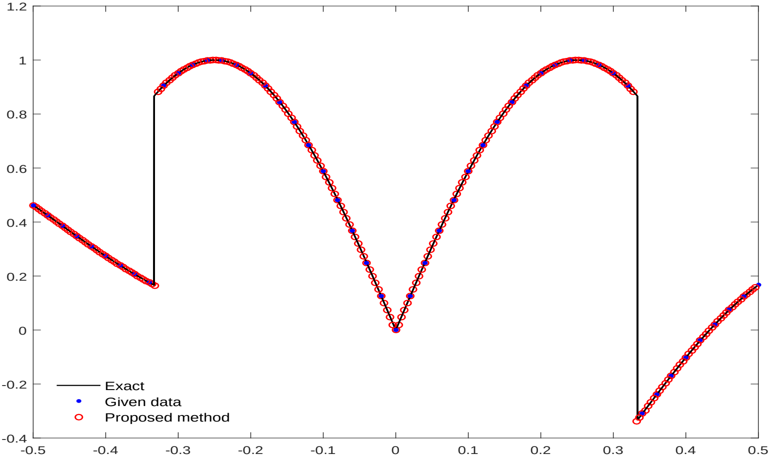
# An example of piecewise-smooth function



# Approximation by QI based on cubic B-spline



# Approximation by the proposed scheme



# Experimental Results

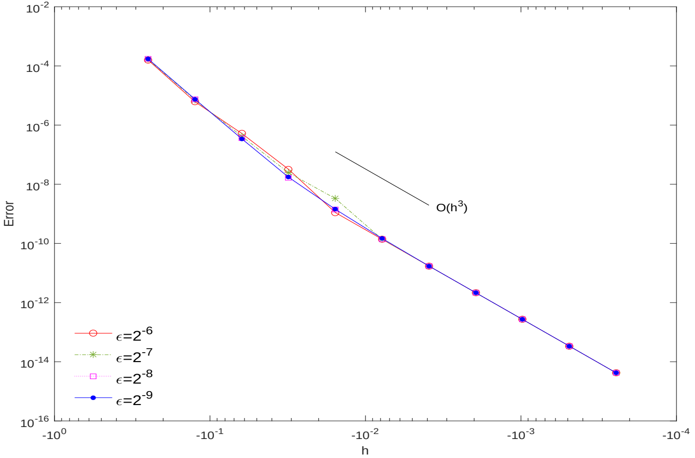
## Convergence Order

Consider the following function [ACDD, SINUM 2005]:

$$f_\epsilon(x) := \begin{cases} (x-\pi/6)(x-\pi/6-\epsilon)+\sin(\pi x/8)/8, & \text{if } x \leq \pi/6, \\ \sin(\pi x/8)/8, & \text{otherwise,} \end{cases}$$

We generate uniformly spaced $(2^k+1)$ points on $[0,1]$ for each $k$-th step.

# Experimental Results



# Future Works:

- High Order approximation Scheme
- Jump discontinuities in the high order derivatives.
- n-Dimension $(n = 2, 3)$
- Noisy data

Thank you for your attention !!