Prony's problem and superresolution in several variables: structure and algorithms

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FORWISS

Universität Passau

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The Problem

Exponentials

For an *exponential polynomial*

$$f(x) = \sum_{\omega\in\Omega} f_\omega \, e^{\omega^T x}, \qquad \Omega \subset (\mathbb{R} + i\mathbb{T})^s, \qquad f_\omega \in \mathbb{C},$$

"learn" Ω and f_ω from regular samples of f: $f(\Lambda)$, $\Lambda \subset \mathbb{Z}^s$.

Connections

1. Sparse polynomials: $f(x) = \sum_{\alpha\in A} f_\alpha \, x^\alpha$.
2. Multisource radar: MUSIC & ESPRIT (1D).
3. "Superresolution".
4. $\Re\omega < 0$: damped partials ...

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For a sparse *exponential polynomial*

$$f(x) = \sum_{\omega \in \Omega} f_\omega \, e^{\omega^T x}, \qquad \Omega \subset (\mathbb{R} + i\mathbb{T})^s, \quad 0 \neq f_\omega \in \mathbb{C},$$

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Superresolution

The imaging model [Candes & Fernandez–Granda]

1. Few localized sources.
2. Image acquisition: *low pass*.
3. Point spread function.
4. Localization deteriorates.
5. Remedy: *deconvolution*.
6. For example by minimization.

Mathematical model

$$x = \sum_{\omega \in \Omega} f_\omega \, \delta_\omega$$

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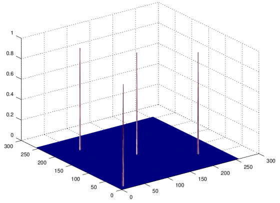

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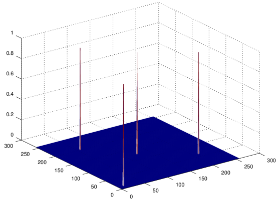

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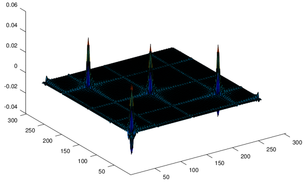

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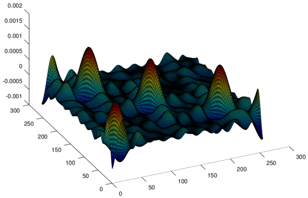

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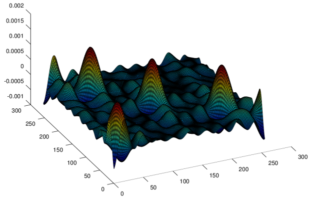

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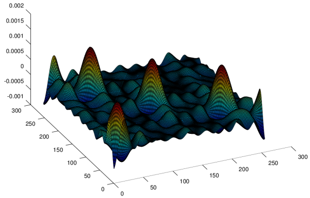

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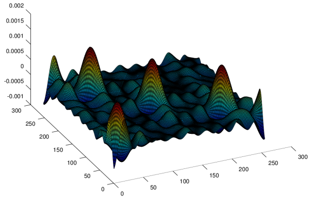

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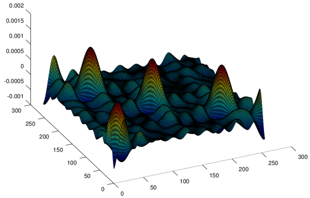

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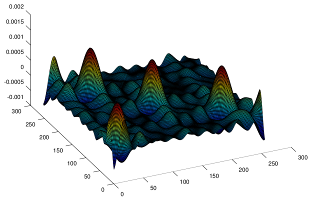

Mathematical model

$$x = \sum_{\omega \in \Omega} f_\omega \, \delta_\omega \qquad \mapsto \qquad \hat{x}(\alpha) = (2\pi)^{-s} \int_{\mathbb{T}^s} x(t) \, e^{-i\alpha^T t} \, dt$$

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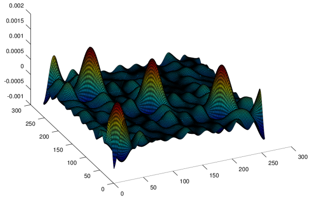

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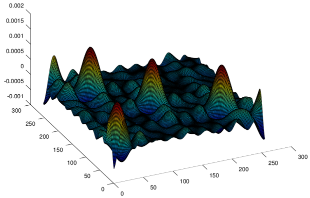

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Moment Problems

Points & moments

1. Point set $X_\Omega = e^\Omega = \{x_\omega := e^\omega : \omega \in \Omega\}$.
2. Discrete measure $\mu = \sum_\omega f_\omega \, \delta_{x_\omega}$.
3. Moments
$$\mu(\alpha) = \int x^\alpha \, d\mu(x)$$

Prony as moment problem

1. Square positive functionals.
2. Flat extensions $\to$ B. Mourrain.

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ESSAI EXPÉRIMENTAL

ET ANALYTIQUE

Sur les lois de la Dilatabilité des fluides élastiques et sur celles de la Force expansive de la vapeur de l'eau et de la vapeur de l'alkool, à différentes températures.

Par R. PRONY.

Remarks on Prony's Problem

Assumptions

1. Problem is *sparse*: $\#\Omega$ small.
2. A priori information: some number $N \geq \#\Omega$.

$$f(x) = \sum_{\omega \in \Omega} f_\omega \, e^{\omega^T x}.$$

Problem structure (Prony)

1. Frequencies: *nonlinear* problem.
2. Coefficients: *linear* problem.
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Notation

Polynomials

1. $\Pi = \mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_s]$.
2. $\Pi_n = \left\{ p(x) = \sum_{|\alpha| \le n} p_\alpha x^\alpha : p_\alpha \in \mathbb{C} \right\}$ of *total degree* $\le n$.
3. *Total degree*: $\deg p = \max \{ |\alpha| : p_\alpha \neq 0 \}$.
4. $\Gamma_n := \{ \alpha \in \mathbb{N}_0^s : |\alpha| \le n \}$.
5. For $A \subset \mathbb{N}_0^s$ define $\Pi_A := \operatorname{span} \{ (\cdot)^\alpha : \alpha \in A \}$, i.e.,

$$\Pi_A \ni p = \sum_{\alpha \in A} p_\alpha (\cdot)^\alpha.$$

Coefficient vectors

$$p \simeq \boldsymbol{p} = [p_\alpha : \alpha \in \dots] \in \mathbb{C}^{\Gamma_n} \text{ or } \mathbb{C}^A.$$

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$$p \simeq \boldsymbol{p} = [p_\alpha : \alpha \in \dots] \in \mathbb{C}^{\Gamma_n} \text{ or } \mathbb{C}^A.$$

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Polynomials

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A Hankel matrix

$$F_n := \left[f(\alpha+\beta) : \begin{array}{c} \alpha \in A \\ \beta \in B \end{array} \right] \in \mathbb{R}^{A \times B}$$

A computation …

For $p \in \Pi_B$ and $\alpha \in A$:

Consequence

1. $p(X_\Omega) = 0$ implies $F_n p = 0$.
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Vandermonde Factorizations

Definition

Vandermonde matrix for $X \subset \mathbb{C}^s$ and $A \subset \Gamma$:

$$\boldsymbol{V}(X,A) := \left[x^\alpha : \begin{array}{c} x \in X \\ \alpha \in A \end{array} \right] \in \mathbb{C}^{X \times A}.$$

Factorizations (known from ESPRIT)

Sampling matrix (Hankel) for $A, B \subset \Gamma$:

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Interpolation Spaces

UNIVERSITÄT PASSAU

Definition

1. $\mathscr{P} \subset \Pi$ *interpolation space* for $X \subset \mathbb{C}^s$:
   for any $q \in \Pi$ there is $p \in \mathscr{P}$ such that $p(X) = q(X)$.
2. *Degree reducing* interpolation space: $\deg p \le \deg q$.
3. $A \subset \mathbb{N}_0^s$ *interpolation set*: Π_A is interpolation space.
4. *Degree reducing* interpolation set:

Interpolation folklore

1. A is interpolation set for X iff $\operatorname{rank} V(X, A) = \# X$.
2. $\# X = \# A$: interpolation iff $V(X, A)$ nonsingular.
3. For fixed A, generic case: nonsingular $V(\cdot, A)$ open & dense.
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Not so much **Interpolation folklore**

1. A is interpolation set for X iff $\operatorname{rank} \boldsymbol{V}(X, A) = \#X$.
2. $\#X = \#A$: *unique* interpolation iff $\boldsymbol{V}(X, A)$ nonsingular.
3. For fixed A, generic case: nonsingular $\boldsymbol{V}(\cdot, A)$ open & dense.
4. Degree reducing: ideal interpolation, Gröber-/H–bases …

Vandermonde Consequences

The Prony ideal

1. $I_\Omega := \{p \in \Pi : p(X_\Omega) = 0\}$.
2. Replacement of univariate *Prony polynomial*.

$$
\begin{aligned}
\boldsymbol{F}_{A,B} &= \boldsymbol{V}(X_\Omega, A)^T \boldsymbol{F}_\Omega \boldsymbol{V}(X_\Omega, B), \\
\boldsymbol{F}_{A,B}\boldsymbol{p} &= \boldsymbol{V}(X_\Omega, A)^T \boldsymbol{F}_\Omega \, p(X_\Omega), \qquad p \in \Pi_B.
\end{aligned}
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Theorem

1. Reconstruct F_Ω from $F_{A,B}$ iff A, B are interpolation sets for X_Ω.
2. If A is interpolation set for X_Ω then
   - $\ker F_{A,B} \simeq I_\Omega \cap \Pi_B$.
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Prony & Interpolation

Needed for solution

1. Find interpolation set A for X_Ω.
2. **But:** X_Ω is unknown.

The generic case

1. Generic interpolation space

$$\Pi_n \qquad \text{where} \qquad \binom{n-1+s}{s} < \#\Omega \leq \binom{n+s}{s}.$$

2. $\ker F_{\Gamma_n,\Gamma_{n+1}}$: nonlinear homogeneous equations.
3. $\leq 2^s \#\Omega$ samples, 2^s best constant.
4. **Linear** in $\#\Omega$, not like $(\#\Omega)^s$.

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Prony & Interpolation

Needed for solution

1. Find interpolation set A for X_Ω.
2. **But:** X_Ω is unknown and life is not generic.

The generic case is of little value

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Minimal Sampling

Two things needed

1. Interpolation space for *all* sets X with $\#X \leq N$.
2. Efficient polynomial solver.

Choices for the set A

1. Interpolation guaranteed for $A = \Gamma_{N-1}$.
2. Minimal for $s = 1$.
3. $\dim \Pi_{N-1} = \binom{N-1+s}{s} \simeq N^s / s!$.
4. F_{Γ_N,Γ_N} too large: e.g. $s = 13$, $N = 200$ yields $10^{20} \times 10^{20}$.

Notabene: naive algorithm

Build $F_n := F_{\Gamma_n,\Gamma_n}$, $n = 0, 1, 2, \dots,$ until $\operatorname{rank} F_n = \operatorname{rank} F_{n+1}$.

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Example: $\Omega = \{\omega, \omega'\}, f_\omega = -f_{\omega'}$

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Universal Interpolation

Definition

$\mathscr{P} \subset \Pi$ *universal* of order N if $\mathscr{P}$ is interpolation space space for any $X \subset \mathbb{C}^s$ with $\# \leq N$.

Classical problem: minimal universal space

Given N, what is the least dimensional subspace of Π that allows for interpolation at any $X \subset \mathbb{C}^s$ with $\#X \leq N$?

Prony version, monomial

Given N, what is the smallest set $\Upsilon_N \subset \Gamma$ with:

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Definition (poor man's Haar space)

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$\mathscr{P} \subset \Pi$ *universal* of order N if $\mathscr{P}$ is interpolation space space for any $X \subset \mathbb{C}^s$ with $\# \le N$.

Classical problem: minimal universal space

Given N, what is the least dimensional subspace of Π that allows for interpolation at any $X \subset \mathbb{C}^s$ with $\#X \le N$?

Prony version, monomial

Given N, what is the smallest set $\Upsilon_N \subset \Gamma$ with:

- for any $X \subset \mathbb{C}^s$, $\#X \le N$,
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Universal Interpolation and Lower Sets

Definition

1. $A \subset \Gamma$ is called *lower set* if $\alpha \in A \quad \Rightarrow \quad \{\beta : \beta \le \alpha\} \subseteq A$.
2. $\mathscr{L}_j :=$ all lower sets of cardinality j.

Theorem

1. If Π_Θ is degree reducing monomially universal then
$$\Theta \supseteq \bigcup_{j=1}^{N} \bigcup_{A \in \mathscr{L}_j} B. \tag{1}$$
2. The set on the right hand side of (1) is universal …
3. $\alpha \in \Upsilon$ iff $\prod_{j=1}^{s} (1+\alpha_j) \le N$.

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3. $\alpha \in \Upsilon$ iff $\prod_{j=1}^{s} (1 + \alpha_j) \leq N$. Positive part of *hyperbolic cross*.

Summary & Algorithm

Theorem

The hyperbolic cross $\Upsilon_N \subset \mathbb{N}_0^s$ of order N

1. is the unique minimal universal monomial degree reducing interpolation space.
2. has cardinality $\le N \log^{s-1} N$.
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Algorithm (Prony ideal & interpolation space)

For increasing sets $\{0\} = A_0 \subset A_1 \subset \cdots \subset \mathbb{N}_0^s$

1. build $F_j = F_{\Upsilon_N, A_j} = \left[F_{\Upsilon_N, A_{j-1}} \mid * \right]$,
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- T. Sauer, *Prony's method in several variables*, Numer. Math., to appear

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For increasing sets $\{0\} = A_0 \subset A_1 \subset \cdots \subset \mathbb{N}_0^s$ filling $\Gamma_0, \Gamma_1, \ldots$

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Numerically or Symbolically?

1. Symbolic:

   Sparse Monomial Interpolation with Least Elements .

2. Symbolic/Numeric:

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Theorem (Small sample sets)

SMILE computes Gröbner basis and interpolation space from at most $s\,N^2 \log^{s-1} N$ samples of f on Γ.

Remarks

1. N^2 cannot be improved.
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Augmenting Matrices

Rank detection

1. Compute SVD of *block augmented* $\boldsymbol{F}_j = [\boldsymbol{F}_{j-1} \,|\, \boldsymbol{A}_j]$.
2. $\tau \geq \sigma_{k+1} \geq \cdots \geq \sigma_n$ yields

$$\|\boldsymbol{F}_j \boldsymbol{x}_j\|_2 \leq \tau \|\boldsymbol{x}\|_2, \qquad \boldsymbol{x} \in \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-k}\}.$$

3. $\boldsymbol{F}_j = \boldsymbol{U}\Sigma\boldsymbol{V}$: $\boldsymbol{V}$ gives bases for $\ker \boldsymbol{F}_j$ and $(\ker \boldsymbol{F}_j)^c$.

SVD update [with J. M. Peña]

1. Use SVD of $\boldsymbol{F}_{j-1}$ for SVD of $\boldsymbol{F}_j$.
2. Strategy for given threshold τ.
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2. Strategy for given threshold τ.
3. Uses QR and SVD of smaller matrices.
4. Guarantees $\operatorname{rank} \boldsymbol{F}_j \ge \operatorname{rank} \boldsymbol{F}_{j-1}$.

Augmenting Matrices

Rank detection

1. Compute SVD of *block augmented* $\boldsymbol{F}_j = [\boldsymbol{F}_{j-1} \,|\, \boldsymbol{A}_j]$.
2. $\tau \ge \sigma_{k+1} \ge \cdots \ge \sigma_n$ yields

$$\|\boldsymbol{F}_j \boldsymbol{x}_j\|_2 \le \tau \|\boldsymbol{x}\|_2, \qquad \boldsymbol{x} \in \operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-k}\}.$$

3. $\boldsymbol{F}_j = \boldsymbol{U}\Sigma\boldsymbol{V}$: $\boldsymbol{V}$ gives bases for $\ker \boldsymbol{F}_j$ and $(\ker \boldsymbol{F}_j)^c$.

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Augmenting Matrices

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3. Uses QR and SVD of smaller matrices.
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Completing the Job: Companion Matrices

Multiplication tables for ideal projectors

1. Multiplication: $\mathscr{P} \ni p$
2. Linear operation on $\mathscr{P} \to$ matrix M_j for a basis P of $\mathscr{P}$.
3. Implicitly computable: *reduction*.
4. Monomial basis: *Frobenius companion matrix*.

Theorem (Stetter, Sticklberger, ...)

The *eigenvalues* of the M_j are $(x_\omega)_j$ and the *eigenvectors* ℓ_ω, $\omega \in \Omega$.

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$$L_{\mathscr{P}} p = \sum_{\omega \in \Omega} p(x_\omega)\, \ell_\omega, \qquad \ell_\omega(\omega') = \delta_{\omega,\omega'}$$

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$$L_{\mathscr{P}}\left((\cdot)_j \ell_\omega\right)$$

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$$L_{\mathscr{P}}\left((\cdot)_j \ell_\omega\right) = \sum_{\omega' \in \Omega} \left((x_{\omega'})_j \, \ell_\omega(x_{\omega'})\right) \, \ell_{\omega'}$$

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Some Results

Random frequencies & coefficients, real, 100 tests

parameters			average error		max error	
s	# freq.	n	coeff	freq	coeff	freq
2	5	3	1.3688e-11	1.8332e-09	3.5131e-09	2.4165e-07
2	10	5	4.9366e-08	2.6388e-06	7.3010e-05	5.3330e-04
2	15	8	7.0614e-07	2.9725e-04	1.4659e-04	4.4493e-02
2	20	9	Inf	Inf	NaN	NaN
3	20	6	1.5874e-08	1.4165e-06	4.7337e-05	8.9382e-04
4	20	5	8.4712e-12	4.6565e-11	9.0309e-09	3.7456e-09
5	20	5	1.6879e-12	5.9416e-11	1.9510e-09	1.3243e-08
5	50	5	1.1079e-10	6.6070e-10	3.1709e-07	6.6913e-08
5	100	6	2.9307e-09	1.9431e-08	1.0034e-05	1.3912e-06
5	150	8	1.3142e-08	8.4199e-08	5.7281e-06	4.3975e-06

Some Results

Random frequencies, purely imaginary, 100 tests

parameters			average error		max error	
s	# freq.	n	coeff	freq	coeff	freq
2	10	5	1.3476e-14	3.4744e-13	6.0290e-12	1.3724e-10
2	20	7	2.5148e-14	1.2420e-12	3.2103e-11	7.8847e-10
2	50	11	5.9357e-14	3.9721e-12	1.1845e-10	5.5214e-09
2	100	15	9.0480e-13	5.7684e-11	8.8308e-09	2.0468e-07
5	100	6	2.3796e-15	4.3794e-15	3.1431e-11	3.2918e-14
5	150	8	2.3954e-15	4.7773e-15	1.1702e-11	6.9726e-14

Observations

1. Performs very well:

2. be combined with hyperbolic cross.

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1. Performs very well:
   $\#\Omega = 200$, $s = 13$: 2180 quartic equations in 154.26s, accuracy $\sim 10^{-14}$
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Observations

1. Performs very well:
   $\#\Omega = 200$, $s = 13$: 2180 quartic equations in 154.26s, accuracy $\sim 10^{-14}$
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Some Results

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1. Performs very well:
   $\#\Omega = 200$, $s = 13$: 2180 quartic equations in 154.26s, accuracy $\sim 10^{-14}$
2. **Must** be combined with hyperbolic cross.

Some Results

Perturbation of imaginary input data, 100 tests, 100 frequencies

parameters			average error		max error	
s	ε	fail	coeff	freq	coeff	freq
5	10^{-5}	0	3.7885e-08	1.1462e-06	2.0235e-06	1.3860e-05
5	10^{-7}	0	3.7916e-10	1.1133e-08	2.1059e-08	7.8396e-08
5	10^{-10}	0	3.7221e-13	1.1200e-11	1.5896e-11	1.6209e-10
3	10^{-4}	27	0.0023822	0.0791873	5.3002	148.6645
4	10^{-4}	1	1.2563e-04	5.9020e+01	1.1638e+00	2.9213e+05
5	10^{-4}	2	3.7969e-07	1.1228e-05	6.8484e-06	7.1493e-05
10	10^{-4}	0	1.2672e-07	3.7955e-06	7.7848e-07	1.4004e-05

Explanation

1. SVD threshold tolerance adapted to error.
2. Compensates errors smaller than “conditioning”.

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Prony's problem in several variables ...

1. ... is interpolation & ideal theory.
2. ... motivation for universal interpolation.
3. ... efficiently solvable.

To do

1. Quantitative analysis, error estimates.
2. Good implementation

It's on the `arXiv`!

- T. Sauer, *Prony's method in several variables*. `arXiv:1602.02352`
- T. Sauer, *Prony's method in several variables: symbolic solutions by universal interpolation*. `arXiv:1603.03944`

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Summary

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Thank you for your attention!

Passau

Where three rivers meet . . .

Passau

...lies the "Bavarian Venice" ...

Passau

. . . lies the "Bavarian Venice" . . .

Passau

. . . lies the "Bavarian Venice" . . .

Passau

. . . with a "University at the beach" . . .

Passau

…with a "University at the beach" …

Passau

. . . with a "University at the beach" . . .

Passau

UNIVERSITÄT PASSAU

. . . and great students

Passau

…and great students

Start