Applications of Variably Scaled Kernels

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- discuss their potentialities and show that
  - they work quite satisfactorily in cases spoiled by excessive instability of the standard method
  - they can significantly improve the recovery quality by preserving shape properties and particular features as gradient discontinuities.

Background

A symmetric *kernel*

$$K \ : \ \Omega \times \Omega \to \mathbf{R}$$

is very useful for a variety of purposes going from interpolation or approximation to PDE solving, if certain *centers*

$$X =: \{x_1, \ldots, x_N\} \subset \mathbf{R}^d$$

are used to define *kernel translates*

$$K(\cdot, x_j)$$

as *trial functions*.

If the kernel is positive definite, i.e. the *kernel matrices*

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- there is a *native* Hilbert space $\mathcal{H}$ in the background in which the kernel is reproducing, i.e.

$$g(x) = (g, K(\cdot, x))_{\mathcal{H}} \quad \forall x \in \Omega, \quad g \in \mathcal{H}.$$

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- the coefficient vector $a \in \mathbf{R}^N$ allows the interpolant function to be written as

$$s_{X,f}(x) := \sum_{j=1}^{N} a_j K(x, x_j).$$

- If the kernel K is *translation–invariant* on $\mathbf{R}^d$, it is of the form

$$K(x,y) = \Phi(x-y) \quad \text{forall} \quad x,y \in \mathbf{R}^d.$$

- If the kernel is *radial*, i.e. of the form

$$K(x,y) = \phi(\|x-y\|_2)$$

for a scalar function

$$\phi \ : \ [0,\infty) \to \mathbf{R},$$

the function ϕ is called a *radial basis function (RBF)*.

Scale parameter

Kernels on $\mathbf{R}^d$ can be *scaled* by a positive factor δ

$$K(x,y;\delta) := K(x/\delta, y/\delta) \quad \forall x,y \in \mathbf{R}^d.$$

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- Large δ increase the condition of kernel matrices,
- small δ let the translates turn into sharp peaks which approximate functions badly, if separated too far from each other.

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Within kernel–based interpolation and its many applications, how to chose the scale parameter δ (or $\epsilon = 1/\delta$) is a well–documented but still an unsolved problem.

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- special case of scaling the *flat limit* $\delta \to \infty$ [Driscoll, Fornberg 02], [Fornberg, Wright, Larsson 04], [Larsson, Fornberg 05], [Lee, Yoon, Yoon 07], [Schaback 05], [Schaback 08]...

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- The scale of a kernel translate varies with the translation. E.g.

$$\phi(\|x - x_j\|_2/\delta_j), \; 1 \le j \le N$$

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In these cases, it is easy to come up with examples that let interpolation fails for certain nonuniform choices of scale.

- In [Bozzini, Lenarduzzi,R. Schaback 04] sufficient conditions for the unique solvability of such interpolation processes are given.

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This allows us

- varying scales in a continuous way
- without leaving the firm grounds of kernel–based interpolation.

This approach can be fully understood as the standard fixed–scale method applied to a certain sub-manifold of $\mathbf{R}^{d+1}$.

Definition

Let K be a kernel on $\mathbf{R}^{d+1}$. If a *scale function*

$$c \ : \ \mathbf{R}^d \to (0, \infty)$$

is given, we define a *variably scaled* kernel on $\mathbf{R}^d$ by

$$K_c(x, y) := K((x, c(x)), (y, c(y))) \ \ \forall x, y \in \mathbf{R}^d.$$

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Theorem

If K is positive definite on $\mathbf{R}^{d+1}$, so is K_c on $\mathbf{R}^d$.

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proceeds as usual via solving a linear system

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- if we project points $(x, c(x)) \in \mathbf{R}^{d+1}$ back to $x \in \mathbf{R}^d$, the projection of the kernel turns into the variably scaled kernel

$$K_c(x, y)$$

  on $\mathbf{R}^d$ whenever $c(x)$ is not constant.

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$$h(X,\Omega) := \sup_{y\in\Omega} \min_{x\in X} \|x-y\|_2 \quad q(X) := \min_{X \ni x \neq y \in X} \|x-y\|_2.$$

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4. but fill distance may also blow up, increasing the usual error bounds.

Applications

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- improve stability
- to reproduce features like discontinuities in the gradient of the underlying function

Stability: Chebyshev Point

For Chebyshev points

$$x_j = -\cos(\pi(j-1)/(N-1)),\ 1 \leq j \leq N$$

the fill distance behaves like $1/N$, while the separation distance behaves like $1/N^2$, leading to a very large condition in the kernel matrices.

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If we map the interval $\Omega = [-1, +1] \subset \mathbf{R}$ to the semi–circle $C(\Omega) \subset \mathbf{R}^2$ via

$$C(x) = (x, \sqrt{1-x^2})$$

the separation distance behaves like $1/N$.

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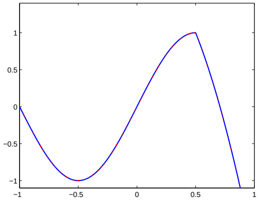
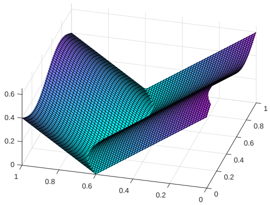
Points and scaling	Condition	no noise	0.001 noise
Chebyshev, single scale	$1 \cdot 10^{16}$	$1.1 \cdot 10^{-5}$	1.4294
Chebyshev, variable scale	$8 \cdot 10^{5}$	$1.3 \cdot 10^{-4}$	0.0012

Table: Interpolation of Runge function by Gaussians

Interpolating functions with discontinuities

Functions with discontinuities appear in many scientific applications including signal and image processing and geophysics.





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Idea

To use a scale function $c(x)$ that reproduces the shape of the discontinuity. In this way we use translates of basis functions that *change their smoothness locally* according to the position of the discontinuity.

Case $d = 1$

Let the data sites X be scattered in $[a, b]$ and let us assume that the derivative of the underlying function is discontinuous at $x^* \in [a, b]$.

We fix the scale function to be

$$c(x) = \begin{cases} 1 - 3/2|x - x^*|/R + 1/2|x - x^*|^3/R^3, & |x - x^*| < R; \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

$2R$ is the support of $c(x)$ which goes to zero smoothly. Generally $R \leq (b - a)/2$.

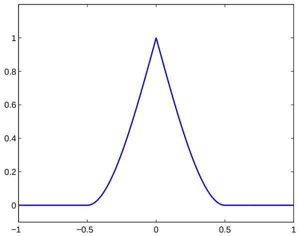

Figure: $c(x)$, $x^* = 0$

The shifts of the variably scaled kernel change their shape: they are no more radial functions and, as expected, exhibit, if the center is "next to" x^*, a discontinuity in the first derivative.

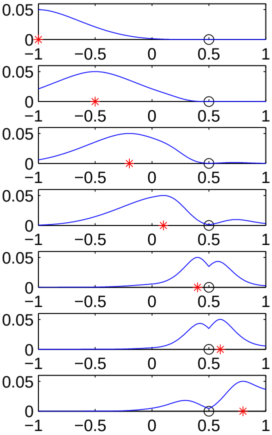

Figure: Wendland VS-basis for some centers

In the examples we have considered, the C^2-Wendland function with, $\delta = 2$, $N = 61$ scattered points in $[-1, 1]$, and $R = 0.5$.

Example 1

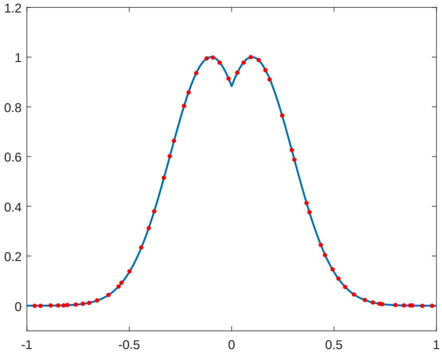

Figure: Test function and data points

Results

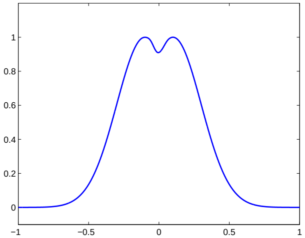

Figure: Standard interpolant

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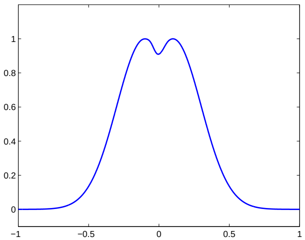

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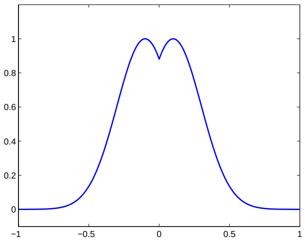

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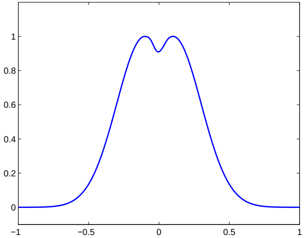

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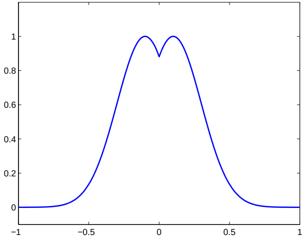

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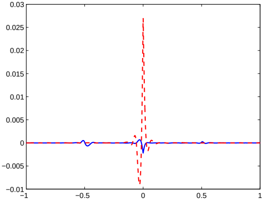
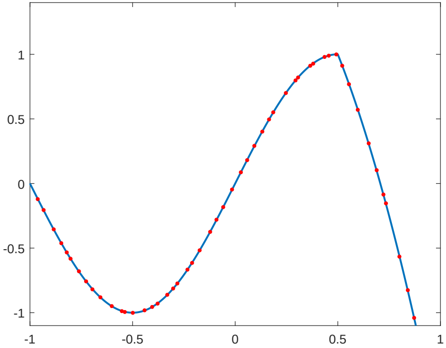


Example 1	e_∞-error
Standard interpolant	2.692631e-02
VKS-interpolant	2.193475e-03

Table: Errors for Example 1

Example 2



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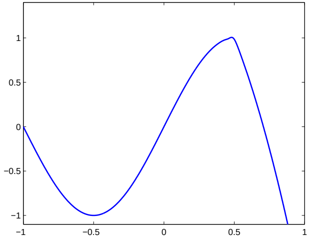

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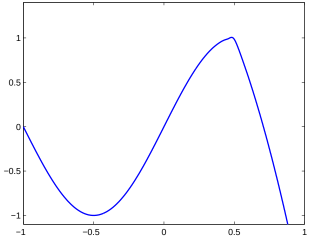

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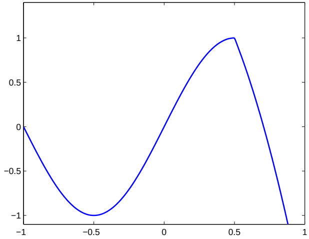

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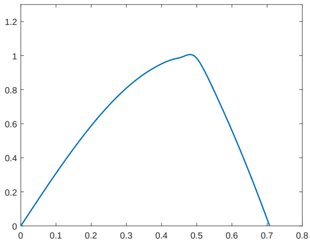

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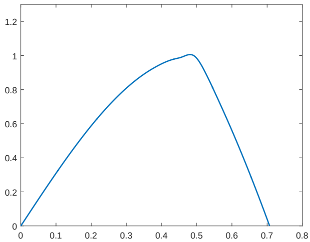


Figure: Standard Interpolant

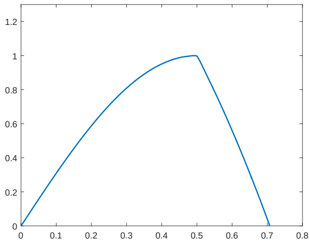
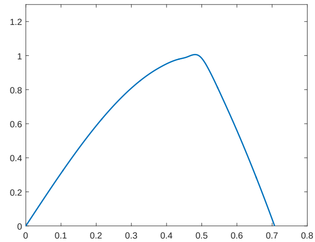


Figure: VSK-interpolant

Results



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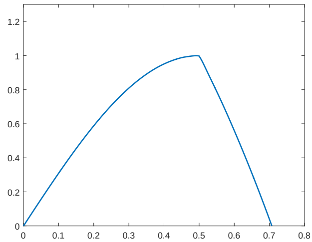

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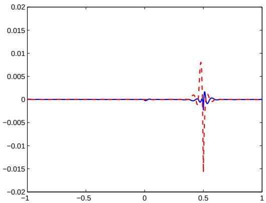

Example 2	e_∞-error
Standard interpolant	1.573100e-002
VKS-interpolant	2.250510e-003

Table: Errors for Example 2

Case $d = 2$

$$f : \Omega \subset \mathbf{R}^2 \to \mathbf{R}, \quad \Omega = [0,1]^2.$$

The gradient $\nabla f(x,y)$ of f is discontinuous across a curve Γ of Ω and smooth in any neighborhood $\mathcal{U}$ of Ω which does not intersect Γ. Let us assume to know the explicit equation of Γ, for instance $y = \Gamma(x)$.

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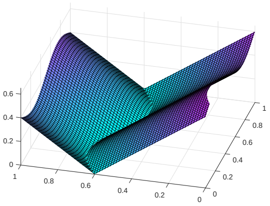
Scale function

$$c(x,y) = \begin{cases} 1 - 3/2|y-\Gamma(x)|/R + 1/2|y-\Gamma(x)|^3/R^3, & |y-\Gamma(x)| < R; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Again the scale function goes to zero smoothly.

In the following examples, we have considered $N = 256$ scattered points in the unitary square, fixed $\delta = 1$, and $R = 0.3$.

Example 3



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Example 3

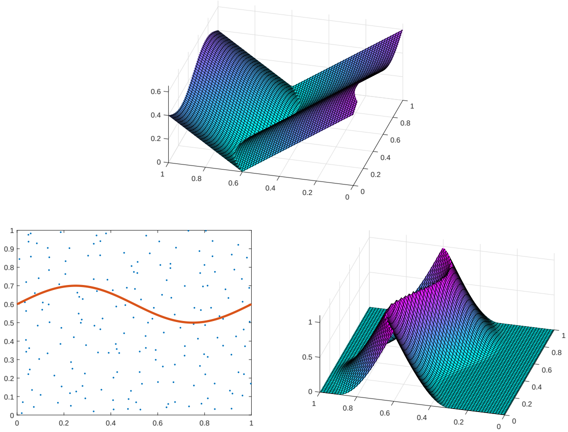


Figure: Data sites, discontinuity line and scale function

Results

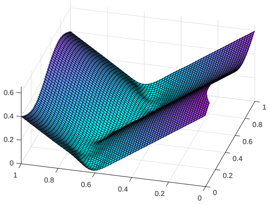


Figure: Standard interpolant

Results

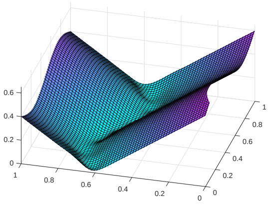


Figure: Standard interpolant

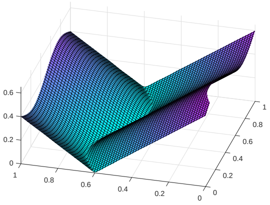
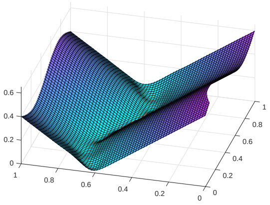


Figure: VSK-interpolant

Results



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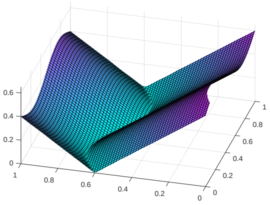
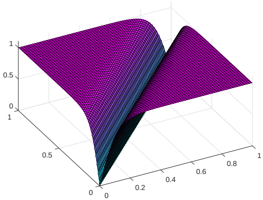


Figure: VSK-interpolant

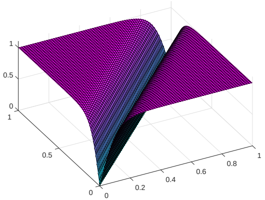
Example 4	RMS-error	e_∞-error
Standard interpolant	2.184125e-03	2.648879e-02
VKS-interpolant	5.101804e-04	4.605954e-03

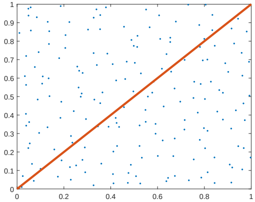
Table: Errors for Example 3

Example 4



Example 4





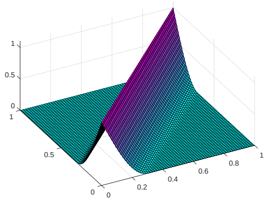


Figure: Data sites, discontinuity line and scale function

Results

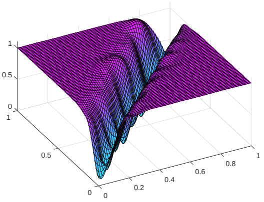


Figure: Standard interpolant

Results

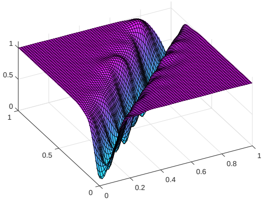


Figure: Standard interpolant

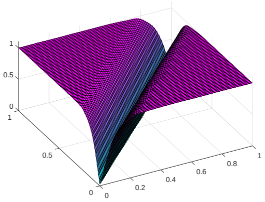


Figure: VSK-interpolant

Results

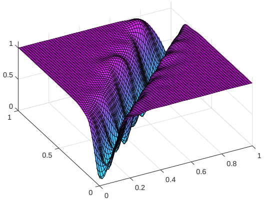


Figure: Standard interpolant

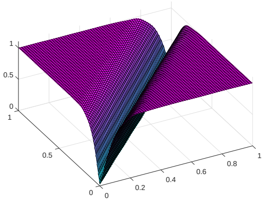


Figure: VSK-interpolant

Example 5	RMS-error	e_∞-error
Standard interpolant	3.531240e-02	2.811598e-01
VKS-interpolant	3.456239e-03	3.804824e-02

Table: Errors for Example 4

Concusions

The examples show how the proper use of the variable scaled kernel K_c can lead

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....THANK YOU!......

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