Supra-Spherical Splines

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Standard Spline Schemes

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For

$$\mathbf{u} := \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in \mathbb{R}^3 \quad \text{and}$$

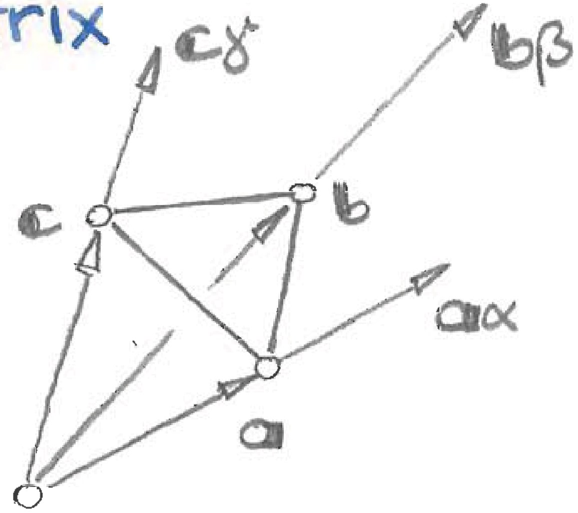
$$\mathbf{i} := \begin{bmatrix} i \\ j \\ k \end{bmatrix} \in \mathbb{N}_0^3, \quad |\mathbf{i}| := i+j+k = n,$$

$$B_{\mathbf{i}}(\mathbf{u}) := \binom{n}{\mathbf{i}} \mathbf{u}^{\mathbf{i}} = \frac{n!}{i!\,j!\,k!} u^i v^j w^k$$

is a homogenous Bernstein polynomial.

$\mathbf{A} := [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ a reg. 3×3 matrix

$\mathbf{x} := \mathbf{A}\mathbf{u}$



Then

$$p(\mathbf{x}) := \sum_{\mathbf{i}} b_{\mathbf{i}} B_{\mathbf{i}}(\mathbf{u})$$

is the hom. B-form w.r.t. A of $p(\mathbf{x})$.

Changing A to $A\begin{bmatrix}\alpha & & \\ & \beta & \\ & & \gamma\end{bmatrix}$,

changes $b_{\mathbf{i}}$ to $b_{\mathbf{i}}\,\alpha^i\beta^j\gamma^k$.

For

$$\mathbb{u} := \begin{bmatrix} \mathbf{u} \\ 1-|\mathbf{u}| \end{bmatrix} \in \mathbb{R}^4$$

$$\mathbb{i} := \begin{bmatrix} \mathbf{i} \\ n-|\mathbf{i}| \end{bmatrix} \in \mathbb{N}_0^4, \quad |\mathbf{i}| \leq n,$$

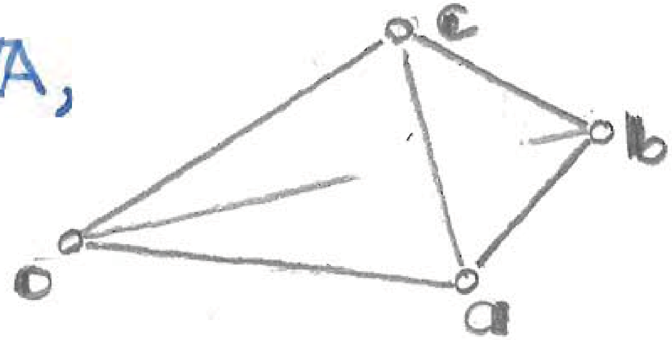
$$B_{\mathbb{i}}(\mathbb{u}) := \binom{n}{\mathbb{i}} \mathbb{u}^{\mathbb{i}}$$

is the standard Bernstein polynomial, where for $|\mathbf{i}| = n$

$$B_{\mathbb{i}}(\mathbb{u}) = \frac{n!}{i!\,j!\,k!\,o!} u^i v^j w^k (1-u-v-w)^o = B_{\mathbf{i}}(\mathbf{u}).$$

Thus $$\mathbf{x} = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}\ \mathbf{o}]\begin{bmatrix} u \\ v \\ w \\ 1-u-v-w \end{bmatrix} =: \mathbb{A}\mathbf{u},$$

i.e., $\mathbf{u}$ = barycentric coord. vector w.r.t. the tetrahedron $\mathbb{A}$,

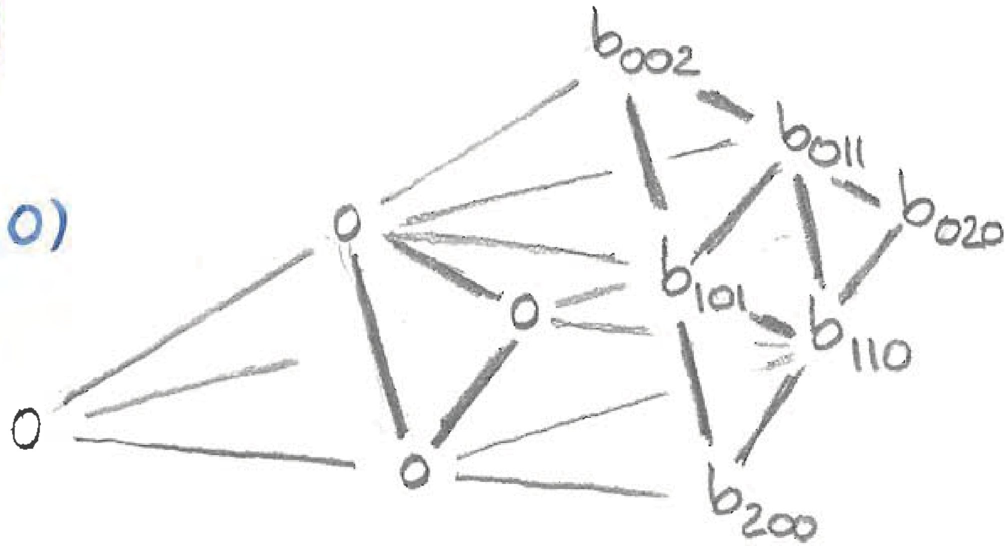


and

$$p(\mathbf{x}) = \sum_{\mathbf{i}} b_{\mathring{\mathbf{i}}} B_{\mathring{\mathbf{i}}}(\mathbf{u})$$

with

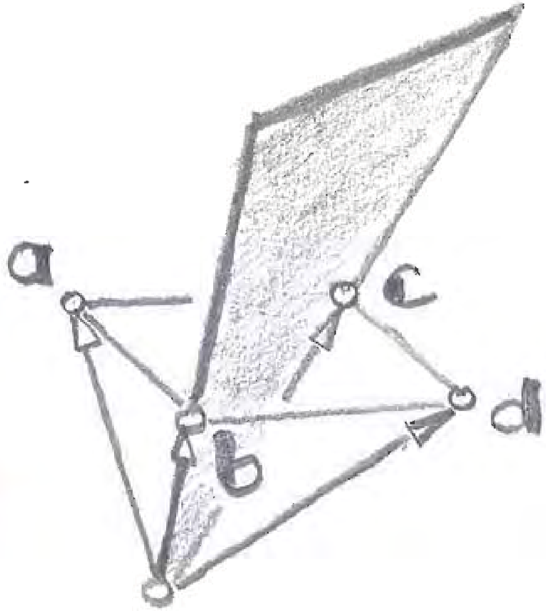
$$b_{\mathring{\mathbf{i}}} := \begin{cases} b_{\mathbf{i}}, & \mathring{\mathbf{i}} = (\mathbf{i}, 0) \\ 0, & \text{else} \end{cases}$$



homog. polynomials

$p := \sum b_{\mathbf{i}} B_{\mathbf{i}}$ w.r.t. **abc**

$q := \sum c_{\mathbf{i}} B_{\mathbf{i}}$ w.r.t. **bcd**



have C^k contact over the plane **bc**

$\Longleftrightarrow$

p and q restr. to the plane **abcd**

or the sphere S^2

have it.

$p(\mathbf{x})$ a hom. polyn.

Alfeld et al. [1996] call

$p|_{S^2}$ a spherical polyn. and

$\mathbf{x}\ p(\mathbf{x})|_{S^2}$ a spherical BB patch.

pw sph. polyn. splines	ext./ restr.
pw hom. polyn. splines	
bivar. splines over $\mathbb{R}^2$	

B-form looses its geom. meaning
for spherical polyn. & patches.

Way out (for odd degr.)

$$\mathbf{x}\, p(\mathbf{x}) \quad \hat{=} \quad \begin{bmatrix} \mathbf{x}\, p(\mathbf{x}) \\ 1 \end{bmatrix} \quad \hat{=} \quad \begin{bmatrix} \mathbf{x}\, p(\mathbf{x}) \\ |\mathbf{x}|^{d+1} \end{bmatrix} \quad \hat{=} \quad \frac{\mathbf{x}\, p(\mathbf{x})}{\mathbf{x}^{d+1}}$$

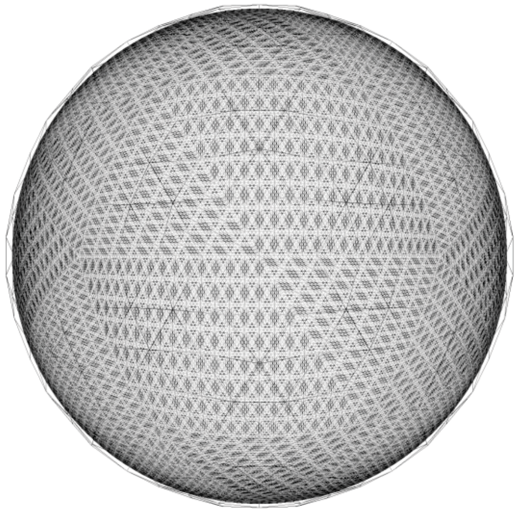
sph. patch $\mathbf{x} \in S^2$

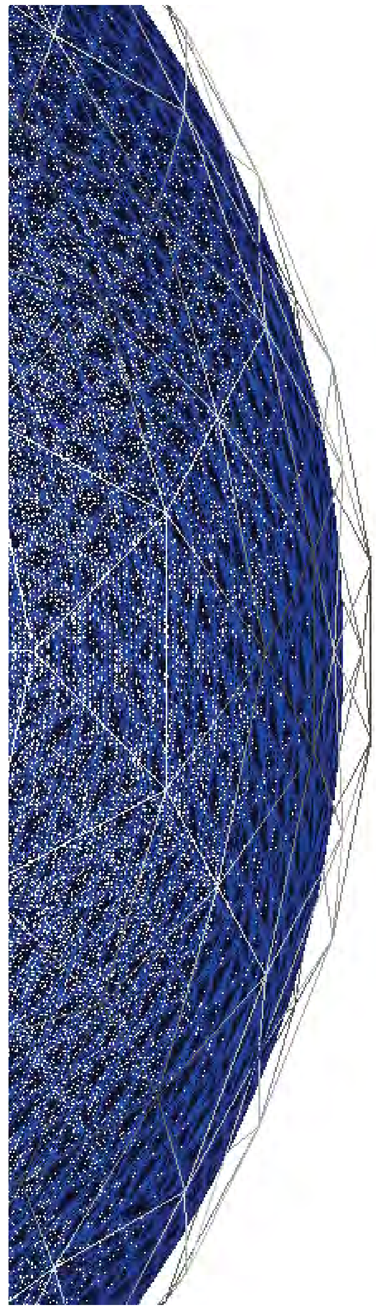
ext. coord.

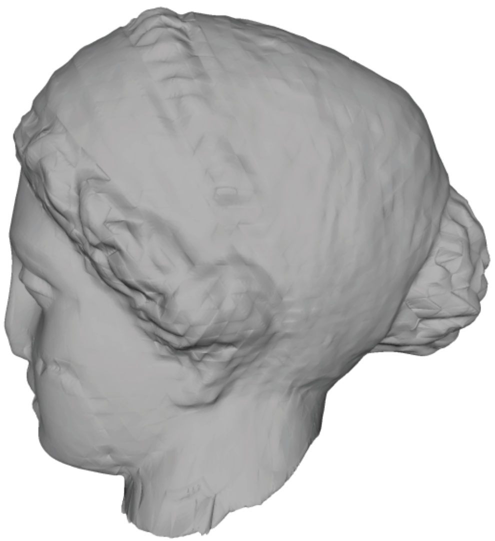
hom. coord. $\mathbf{x} \in \mathbb{R}^3$

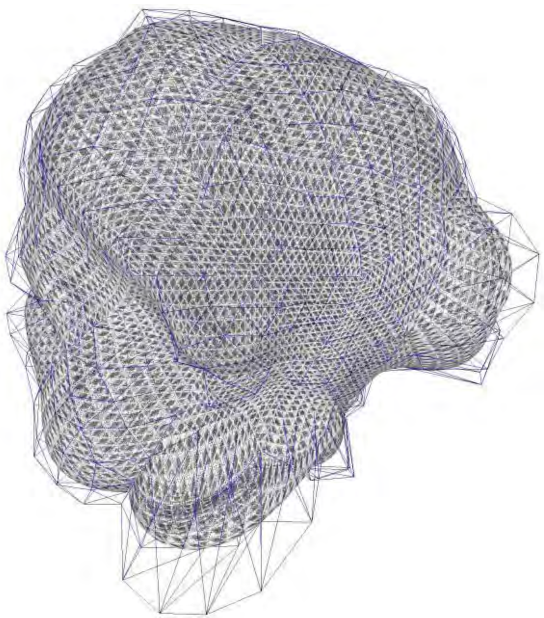
ratl. polyn. $\mathbf{x} \in S^2$
elliptic plane
proj. plane

2 p.w. cubic C^0 approximations and their B-nets

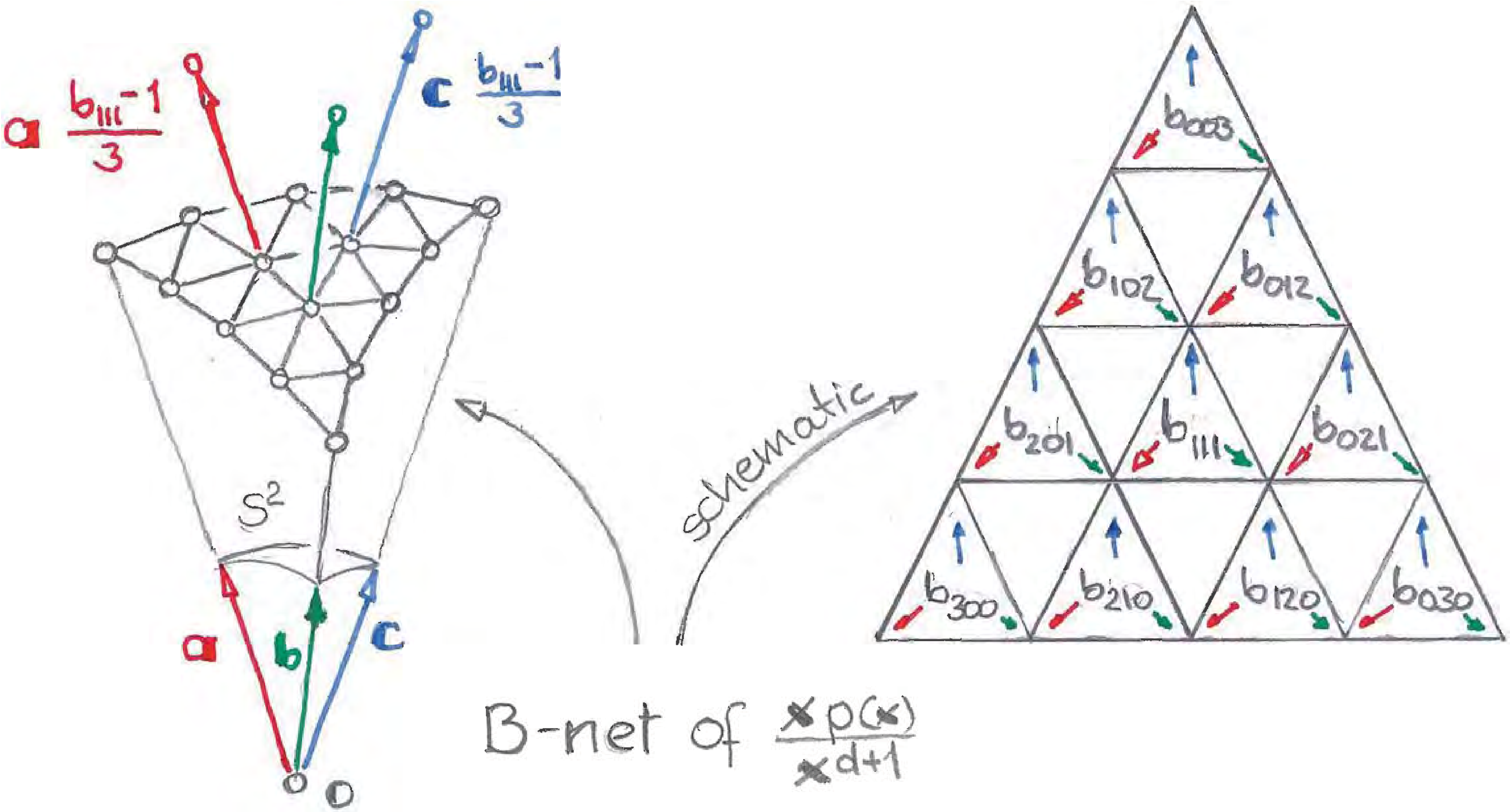








For $p = \sum b_i B_i$



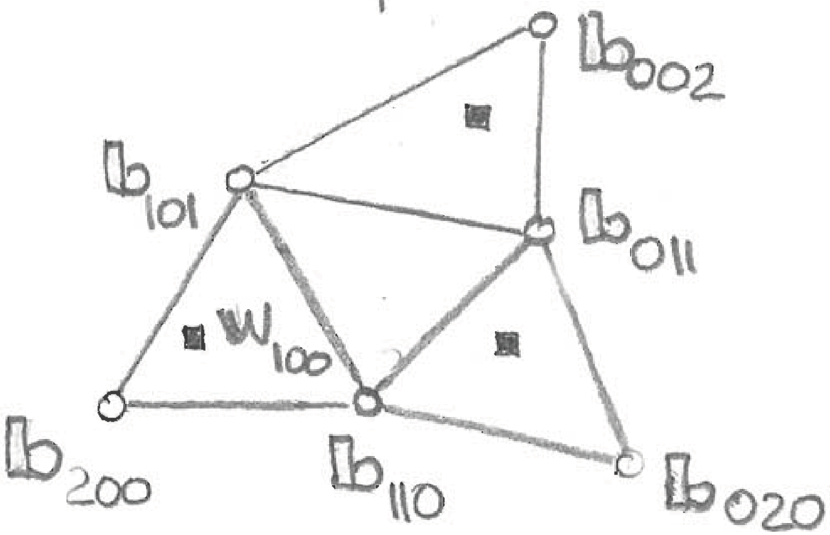
B-net of $\frac{\cancel{x}\, p(x)}{\cancel{x}^{d+1}}$

homog. polyn.

$$\mathbb{r} := \begin{bmatrix} \mathbb{p} \\ q \end{bmatrix} := \sum \mathbb{b}_{\mathbf{i}} B_{\mathbf{i}} \quad : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$$

repr. ratl. polyn.

$$\mathbb{r} := \frac{\mathbb{p}}{q} \; : \; \mathcal{P}^2 \text{ or } S^2 \longrightarrow \mathbb{R}^3$$



weight points

$$w_{\mathbf{j}} := \sum_{\ell=1}^{3} \mathbb{b}_{\mathbf{j}+\mathbf{e}_\ell}$$

$$|\mathbf{j}| = |\mathbf{i}| - 1$$

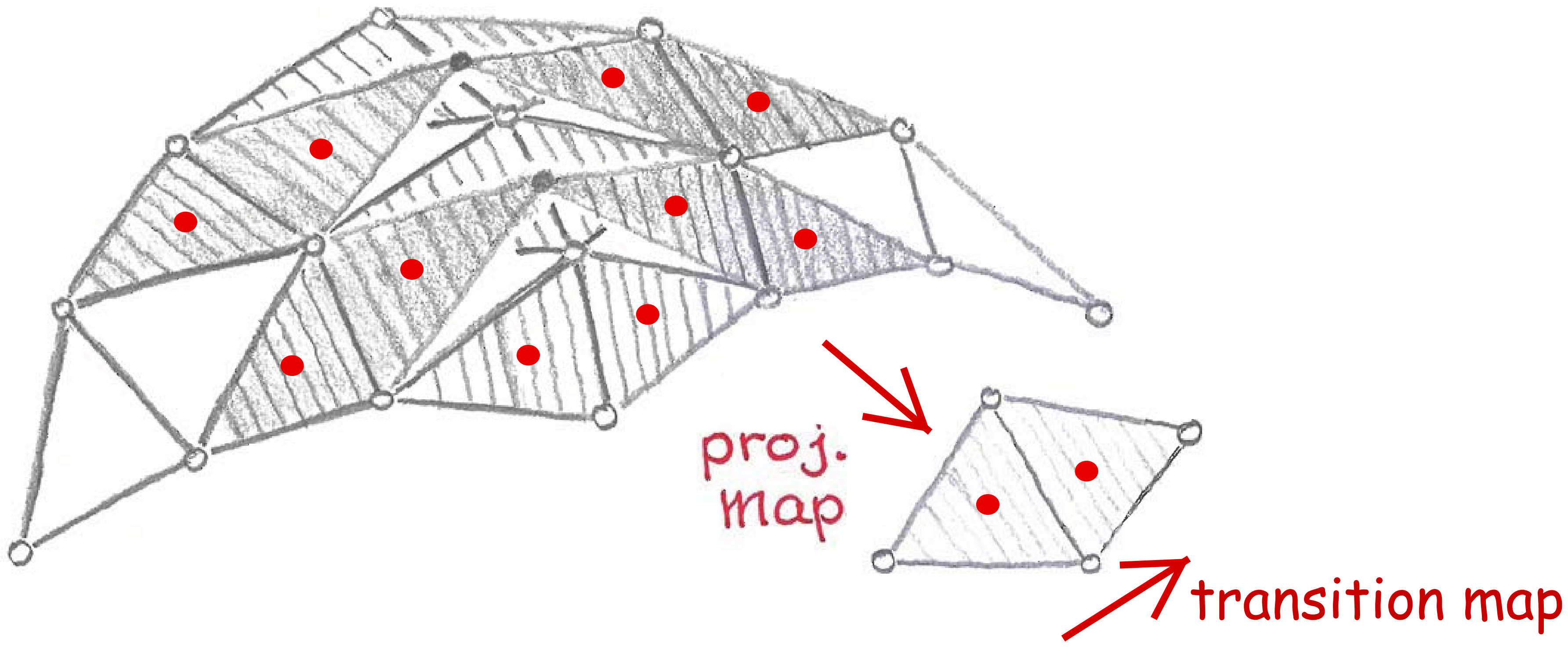
Changing the domain triangle

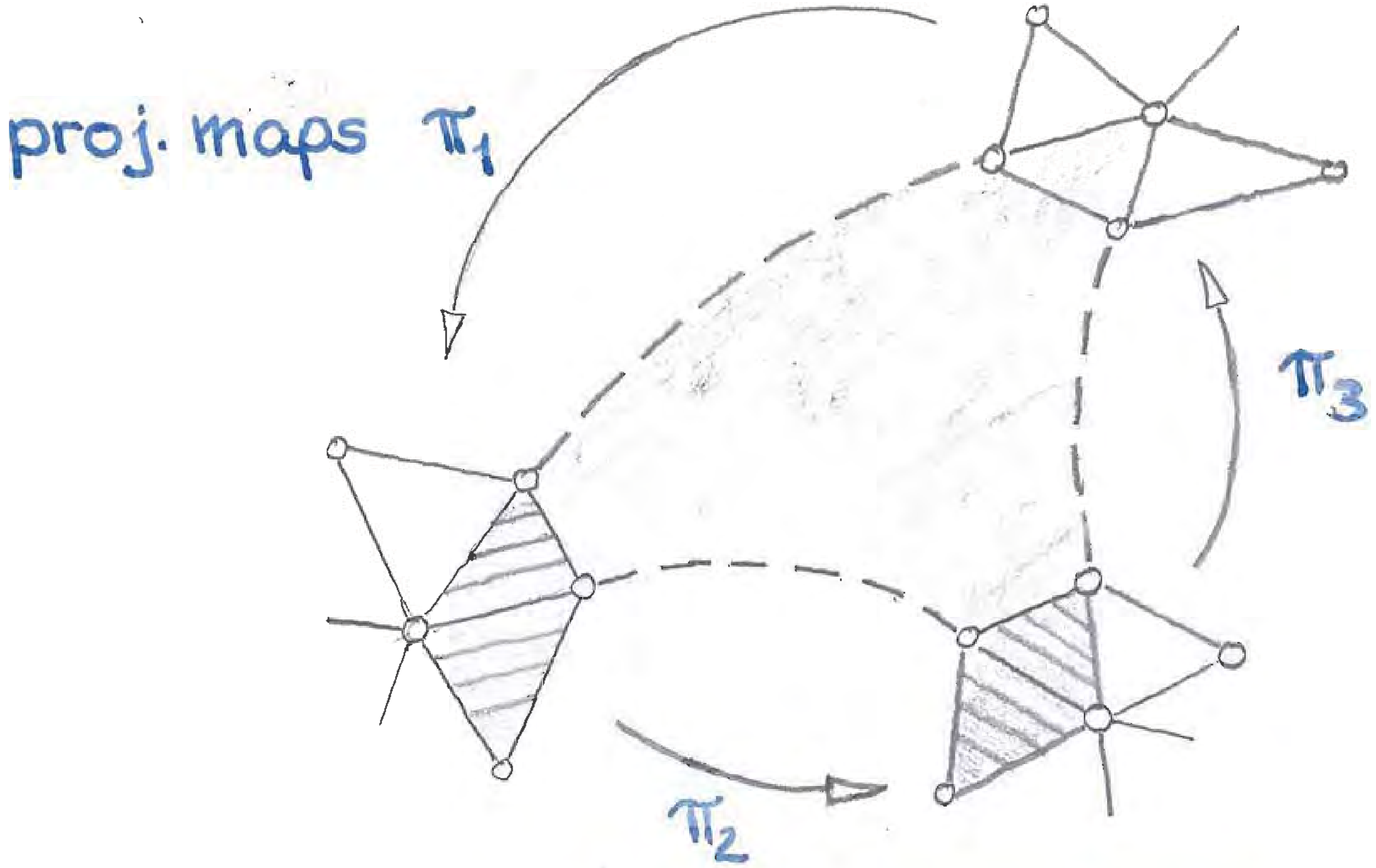
$$A \text{ to } A\begin{bmatrix}\alpha & & \\ & \beta & \\ & & \gamma\end{bmatrix},$$

changes the $\mathbf{w}_j$, not the $\mathbf{b}_i$.

($\triangleq$ reparam. by a proj. map mapping $\mathbf{abc} \mapsto \mathbf{abc}$.)

Ratl. polynomials have a C^k joint ~~in $\mathbb{R}^4$~~
if
their C-quads are ~~affine~~ projective
maps of their domain triangles,
proj. version of [Farin 1979].





compatibility condition

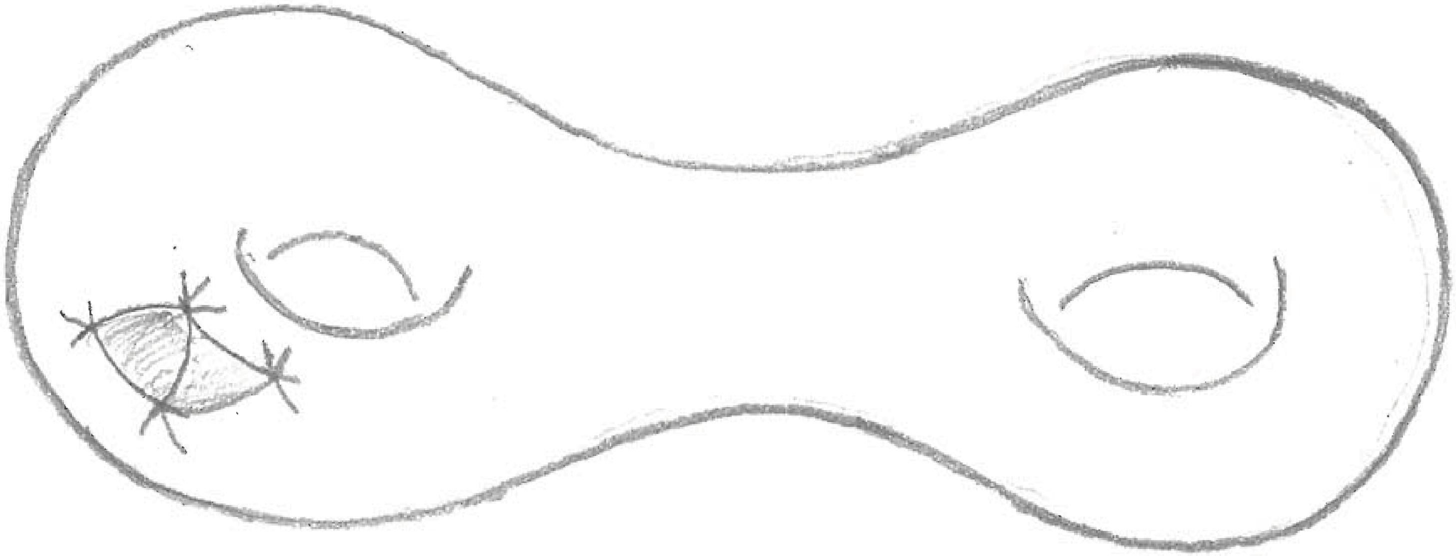
$\forall$ patches: $\pi_3 \circ \pi_2 \circ \pi_1 = id$

Constructing spherical splines

- choose patch layout
- determine a B-net satisfying all compatibility constraints
  - choosing one domain triangle, all adjacent triangles are determined by the C-quads.

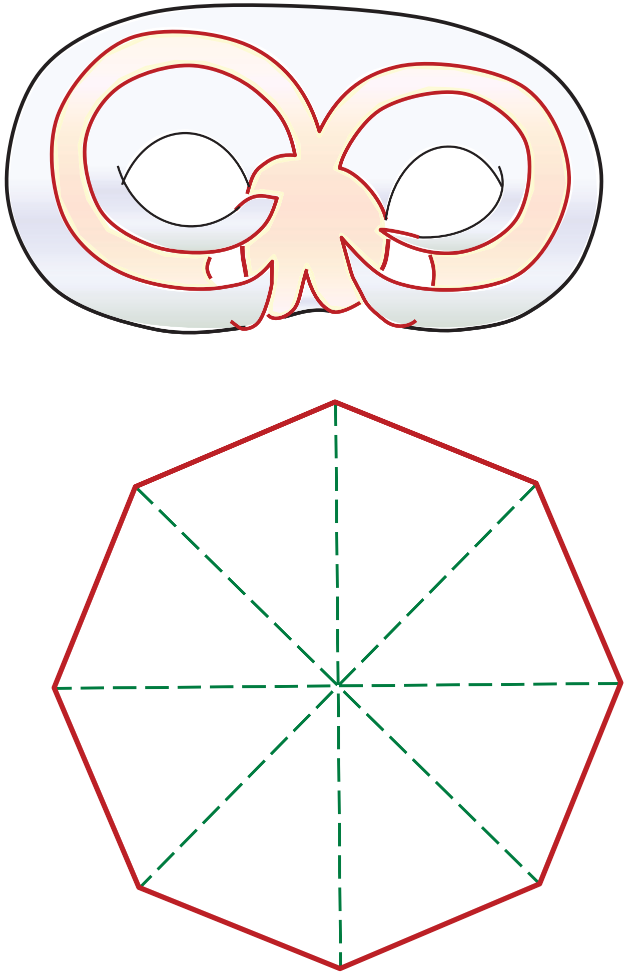
The compatibility conditions

- can not be satisfied by affine maps only,
- but can be satisfied for surfaces of any genus γ

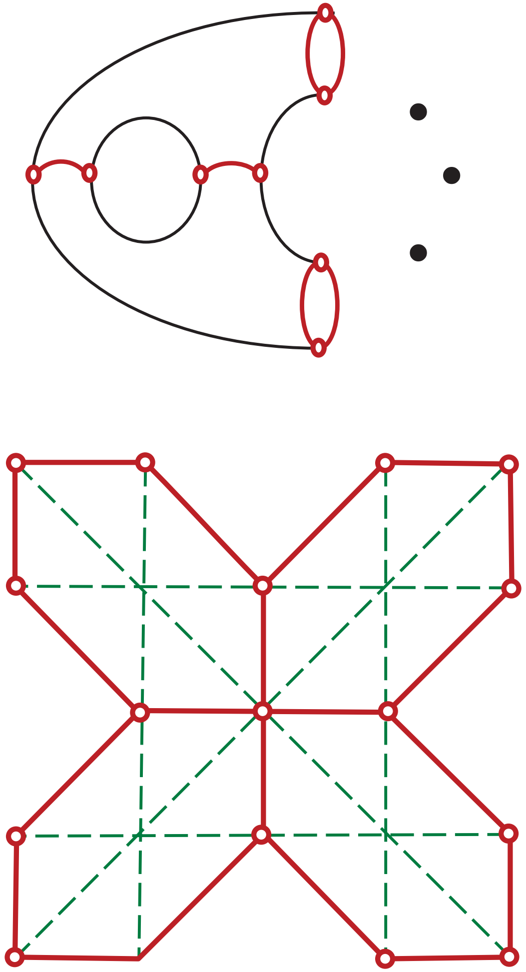


special triangulations

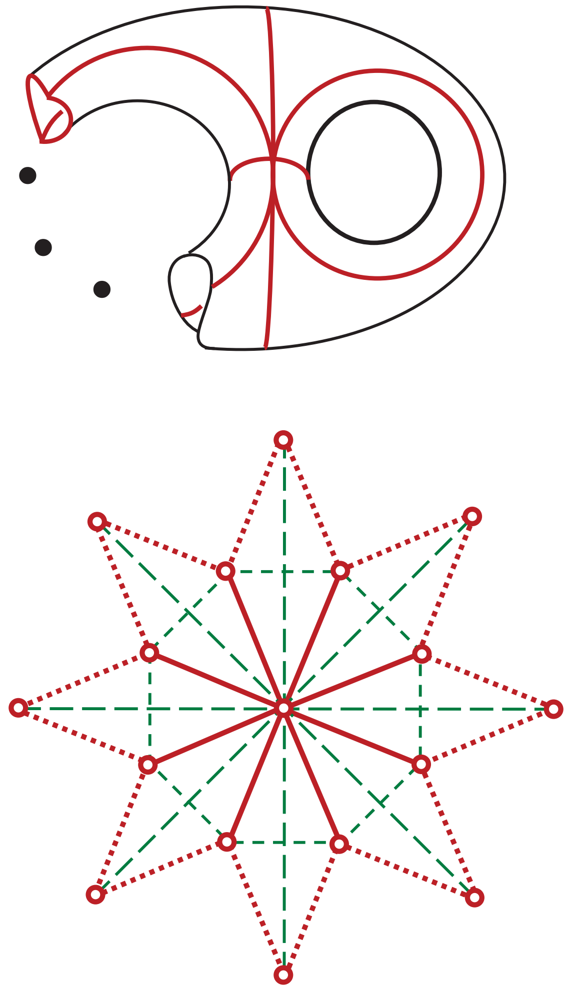
4γ-gon



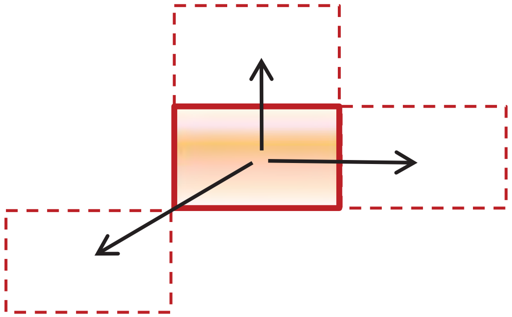
hexagons

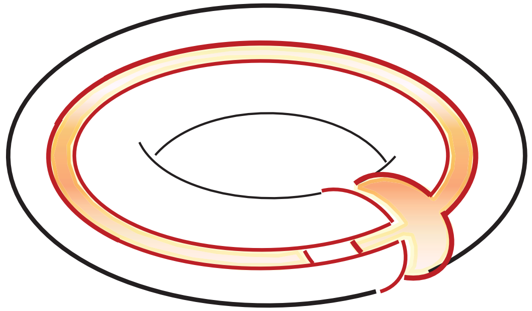


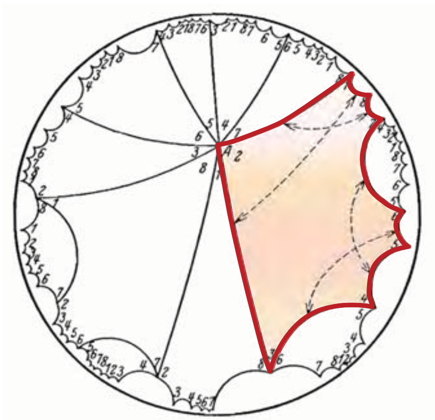
quads



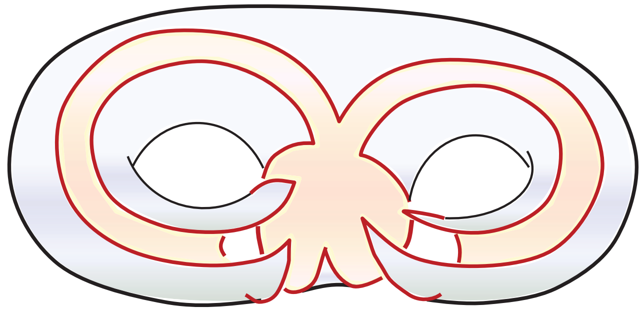
Parameter triangles form a fundamental domain in the hyperbolic plane H^2,







Hilbert & Cohn-Vossen 1932



which can be triangulated arbitrarily.

Transition maps between adjacent triangles for a mesh of n-gons with knot valencies ν:

$$A := \begin{bmatrix} 1 & 2c & 0 \\ 0 & 2-2c & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 1 & b & 0 \\ 0 & b & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

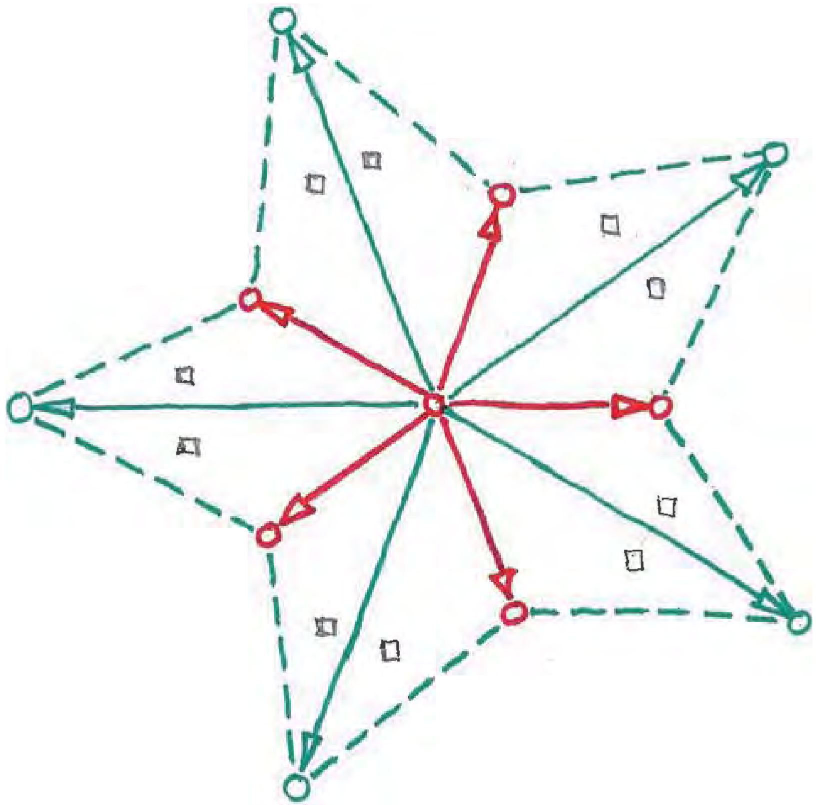
where

$$c := \cos\frac{2\pi}{n} \quad \text{and} \quad b := \frac{2\cos^2\frac{\pi}{\nu}}{1-\cos\frac{2\pi}{n}}.$$

For $n = 4$, $\nu \leq 24$ [Peters, Fan 2010].

Proof for all n, ν:

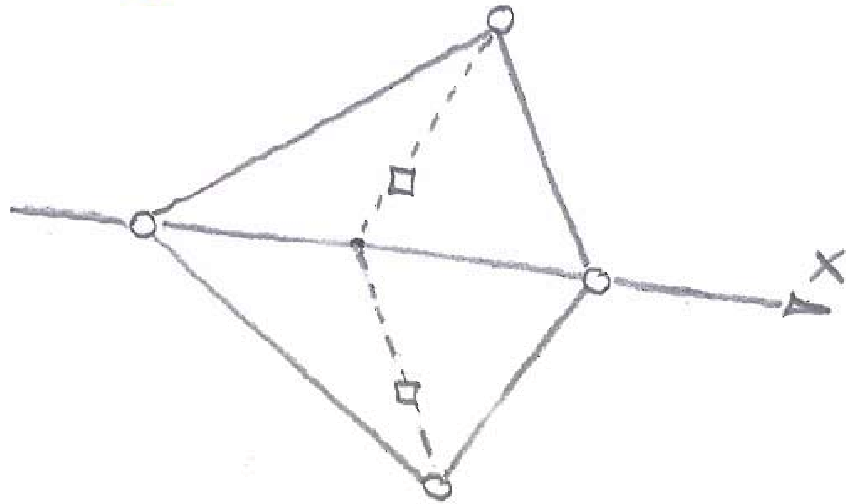
Check Poincarés identity $(AB)^\nu = \mathrm{Id}$.



Projective structures, i.e., triangulated fundamental domains have been used in CAGD to build rational spline manifolds by:

Ferguson & Rockwood	1993
Wallner (& Pottmann)	1995 (1997)
Beccari, Gonsor, Neamtu	2014
Beccari & Neamtu	2016

transition map $\big|_{\text{common edge}} = \mathrm{id}$



$\Rightarrow$ edge parallels are mapped to edge parallels

$\Rightarrow$ transition maps are of the form

$$\mathbf{t}(x,y) = \begin{bmatrix} x \\ \pi(y) \end{bmatrix}, \quad \pi \text{ a proj. map,}$$

with integral cross bdy. derivatives along the edges.

Conclusion

Integral spline manifolds over proj. structures exist, cf.

Peters & Fan		2010
"	& Jianhua	2014
Sarov & Peters		2016

for special proj. structures.

Final destination reached,

the

Supra-Spherical Spline Stuff Story Stop