Sparse approximation by modified Prony method

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Outline

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1. Sparse approximation problem for exponential sums
2. The AAK theorem for samples of exponential sums
3. Method for sparse approximation of exponential sums
4. Numerical example

Sparse approximation of exponential sums

Consider a function of the form

$$f(x) = \sum_{j=1}^{N} a_j z_j^x \qquad \text{with } |z_j| < 1,$$

where $a_j, z_j \in \mathbb{C}$.

Goal: Find a function

$$\tilde{f}(x) = \sum_{j=1}^{n} \tilde{a}_j \tilde{z}_j^x \qquad \text{with } |\tilde{z}_j| < 1,$$

such that $n < N$ and

$$\|f - \tilde{f}\| \leq \varepsilon$$

Discrete sparse approximation problem

Consider a sequence of samples $f := (f_k)_{k=0}^{\infty}$ given by

$$f_k := f(k) = \sum_{j=1}^{N} a_j z_j^k \qquad \text{with } |z_j| < 1,$$

where $a_j, z_j \in \mathbb{C}$.

Goal: Find a sequence $\tilde{f} := (\tilde{f}_k)_{k=0}^{\infty}$ of the form

$$\tilde{f}_k = \sum_{j=1}^{n} \tilde{a}_j \tilde{z}_j^k \qquad \text{with } |\tilde{z}_j| < 1,$$

such that $n < N$ and

$$\|f - \tilde{f}\|_{\ell^2} \leq \varepsilon$$

Possible applications

We consider here a *structured low-rank approximation problem* for model reduction.

Problem: Low-rank approximation using the SVD destroys the Hankel structure, [Markovsky, 2008].

Applications

- Approximation of special functions by exponential sums, e.g. Bessel functions, or $x^{-1/2}$ to avoid quadrature methods for Schrödinger equations, [Beylkin & Monzon, 2005], [Hackbusch, 2005].
- Signal compression by sparse representation of the (discrete) Fourier transform.

Our approach

(1) Given a sufficiently large number of samples f_k, reconstruct z_j and a_j such that

$$f_k = \sum_{j=1}^{N} a_j z_j^k \qquad \text{with } |z_j| < 1$$

using a Prony-like method,
[Roy & Kailath, 1989], [Potts & Tasche, 2010].

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using a Prony-like method,
[Roy & Kailath, 1989], [Potts & Tasche, 2010].

(2) Given the representation (1), find $\tilde{z}_j$ and $\tilde{a}_j$ such that for

$$\tilde{f}_k = \sum_{j=1}^{n} \tilde{a}_j \tilde{z}_j^k \qquad \text{with } |\tilde{z}_j| < 1$$

and $n < N$ we have

$$\|f - \tilde{f}\|_{\ell^2} \leq \varepsilon$$

using the AAK Theorem [Adamjan, Arov & Krein, 1971].

Outline

AAK Theorem for Samples of Exponential Sums

Consider the sequence $f := (f_k)_{k=0}^{\infty}$ given by samples

$$f_k = f(k) = \sum_{j=1}^{N} a_j z_j^k \qquad \text{with } 0 < |z_j| < 1$$

and let $\mathbb{D} := \{z \in \mathbb{C} \, : \, 0 < |z| < 1\}$.

We define the infinite Hankel matrix

$$\mathbf{\Gamma}_f := \begin{pmatrix} f_0 & f_1 & f_2 & \cdots \\ f_1 & f_2 & f_3 & \cdots \\ f_2 & f_3 & f_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (f_{k+j})_{k,j=0}^{\infty}$$

with respect to f.

AAK theorem for samples of exponential sums

Then $\mathbf{\Gamma}_f$ has the following properties:

- $\mathbf{\Gamma}_f$ has finite rank N.
- $\mathbf{\Gamma}_f$ defines a compact operator on $\ell^2 = \ell^2(\mathbb{N})$.
- The singular values of $\mathbf{\Gamma}_f$ are of the form

$$\sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_{N-1} > \sigma_N = \ldots = \sigma_\infty = 0.$$

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[Young]	(1988)	An Introduction to Hilbert Space
[Chui & Chen]	(1992)	Discrete H^∞ optimization
[Peller]	(2000)	Hankel Operators and Their Applications
[Beylkin & Monzón]	(2005)	On approximation of functions by exponential sums
[Andersson et al.]	(2011)	Sparse approximation of functions using sums of exponentials and AAK theory

The AAK theorem (Adamjan, Arov, Krein, 1971)

Let $f := (f(k))_{k=0}^{\infty}$ be given as before.
Let (σ_n, u_n) be a fixed singular pair of $\mathbf{\Gamma}_f$ with $\sigma_n \notin \{\sigma_k\}_{k \neq n}$ and $\sigma_n \neq 0$.

- Then

$$P_{u_n}(x) := \sum_{k=0}^{\infty} u_n(k) x^k$$

has exactly n zeros $\tilde{z}_1, \ldots, \tilde{z}_n$ in $\mathbb{D}$, repeated according to multiplicity.

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- If the $\tilde{z}_k$ are pairwise different, then there are $\tilde{a}_1, \ldots, \tilde{a}_n \in \mathbb{C}$ such that for
$$\tilde{f} = (\tilde{f}_j)_{j=0}^{\infty} = \left(\sum_{k=1}^{n} \tilde{a}_k \tilde{z}_k^{\,j} \right)_{j=0}^{\infty}$$
it holds that
$$\|\mathbf{\Gamma}_f - \mathbf{\Gamma}_{\tilde{f}}\|_{\ell^2 \to \ell^2} = \sigma_n.$$

The AAK theorem

Let $\mathbf{\Gamma}_f$ be of rank N and the singular values be of the form

$$\sigma_0 > \sigma_1 > \ldots > \sigma_{N-1} > \sigma_N = \ldots = \sigma_\infty = 0.$$

n	σ_n	zeros of $P_{u_n}(x)$ in $\mathbb{D}$	$\tilde{f}$	$\|\mathbf{\Gamma}_f - \mathbf{\Gamma}_{\tilde{f}}\|$
0	σ_0	$-$	0	σ_0
1	σ_1	$\tilde{z}_1$	$\tilde{f}_j = \tilde{a}\tilde{z}_1^j$	σ_1
2	σ_2	$\tilde{z}_1, \tilde{z}_2$	$\tilde{f}_j = \tilde{a}_1\tilde{z}_1^j + \tilde{a}_2\tilde{z}_2^j$	σ_2
3	σ_3	$\tilde{z}_1, \tilde{z}_2, \tilde{z}_3$	$\tilde{f}_j = \tilde{a}_1\tilde{z}_1^j + \tilde{a}_2\tilde{z}_2^j + \tilde{a}_3\tilde{z}_3^j$	σ_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$N-1$	σ_{N-1}	$\tilde{z}_1, \ldots, \tilde{z}_{N-1}$	$\tilde{f}_j = \sum_{k=1}^{N-1} \tilde{a}_k \tilde{z}_k^j$	σ_{N-1}

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$N-1$	σ_{N-1}	$\tilde{z}_1, \ldots, \tilde{z}_{N-1}$	$\tilde{f}_j = \sum_{k=1}^{N-1} \tilde{a}_k \tilde{z}_k^j$	σ_{N-1}

Original sequence: $f_j = \sum_{k=1}^{N} a_k z_k^j$ 0

Problems with application of AAK theory

Let f be given as before and let (σ_n, u_n) be a fixed singular pair of $\mathbf{\Gamma}_f$ such that $\sigma_n \notin \{\sigma_k\}_{k \neq n}$ and $\sigma_n \neq \sigma_\infty$.

- Then

$$P_{u_n}(x) := \sum_{k=0}^{\infty} u_n(k) x^k$$

  has exactly n zeros $\tilde{z}_1, \ldots, \tilde{z}_n$ in $\mathbb{D}$, repeated according to multiplicity.

- If the $\tilde{z}_k$ are pairwise different, then there are $\tilde{a}_1, \ldots, \tilde{a}_n \in \mathbb{C}$ such that for

$$\tilde{f} = (\tilde{f}_j)_{j=0}^{\infty} = \left(\sum_{k=1}^{n} \tilde{a}_k \tilde{z}_k^j \right)_{j=0}^{\infty}$$

  it holds that

$$\| \mathbf{\Gamma}_f - \mathbf{\Gamma}_{\tilde{f}} \|_{\ell^2 \to \ell^2} = \sigma_n.$$

Outline

Singular values and con-eigenvalues

For a (complex) Hankel matrix $\mathbf{\Gamma}_f$ we call $\sigma \in \mathbb{C}$ a *con-eigenvalue* with the corresponding *con-eigenvector* $v \in \ell^2(\mathbb{N})$ if it satisfies

$$\mathbf{\Gamma}_f \overline{v} = \sigma v.$$

For symmetric matrices like $\mathbf{\Gamma}_f$ we have

- We can always select a nonnegative σ.
- (σ, v) is a singular pair of $\mathbf{\Gamma}_f$ $\xrightleftharpoons{\text{multiplicity is 1}}$ (σ, v) is a con-eigenpair of $\mathbf{\Gamma}_f$

Structure of con-eigenvectors to non-zero con-eigenvalues

Lemma: Let f be given as before, i.e.

$$f_k = \sum_{j=1}^{N} a_j {z_j}^k \qquad \text{with } z_j \in \mathbb{D},$$

and let $\sigma \neq 0$ be a fixed con-eigenvalue of $\mathbf{\Gamma}_f$ with the corresponding con-eigenvector $u := (u_k)_{k=0}^{\infty}$.
Then u can be represented by

$$u_k = \sum_{j=1}^{N} b_j {z_j}^k, \qquad k = 0, 1, \ldots,$$

where $b_j, \ j = 1, \ldots, N$ are some (complex or real) coefficients.

Dimension reduction for the con-eigenvalue problem of $\mathbf{\Gamma}_f$

$$\mathbf{\Gamma}_f \bar{u} = \sigma u \quad \Leftrightarrow \quad \sum_{j=0}^{\infty} f_{j+k} \bar{u}_j = \sigma u_k, \quad \forall k = 0, 1, 2, \ldots$$

$$\Leftrightarrow \quad \sum_{j=0}^{\infty} \left(\sum_{l=1}^{N} a_l z_l^{k+j} \right) \overline{\left(\sum_{s=1}^{N} b_s z_s^{j} \right)} = \sigma \sum_{l=1}^{N} b_l z_l^{k}$$

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$$\Leftrightarrow \quad \sum_{j=0}^{\infty} \left(\sum_{l=1}^{N} a_l z_l^{k+j} \right) \overline{\left(\sum_{s=1}^{N} b_s z_s^j \right)} = \sigma \sum_{l=1}^{N} b_l z_l^k$$

$$\Leftrightarrow \quad \sum_{l=1}^{N} z_l^k \left(a_l \sum_{s=1}^{N} \bar{b}_s \sum_{j=0}^{\infty} (z_l \bar{z}_s)^j \right) = \sigma \sum_{l=1}^{N} b_l z_l^k.$$

$$\Leftrightarrow \quad \sum_{l=1}^{N} z_l^k \left(a_l \sum_{s=1}^{N} \frac{\bar{b}_s}{1 - z_l \bar{z}_s} \right) = \sum_{l=1}^{N} (\sigma b_l)\, z_l^k.$$

$$\Leftrightarrow \quad a_l \sum_{s=1}^{N} \frac{\bar{b}_s}{1 - z_l \bar{z}_s} = \sigma b_l \quad \forall\, l = 1, \ldots, N$$

Dimension reduction for the con-eigenvalue problem of $\mathbf{\Gamma}_f$

The last equation can be seen as the following con-eigenvalue problem of the dimension N

$$A Z \bar{b} = \sigma b,$$

where

$$A := \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_N \end{pmatrix}, \quad Z := \begin{pmatrix} \frac{1}{1-|z_1|^2} & \frac{1}{1-\bar{z}_2 z_1} & \cdots & \frac{1}{1-\bar{z}_N z_1} \\ \frac{1}{1-\bar{z}_1 z_2} & \frac{1}{1-|z_2|^2} & \cdots & \frac{1}{1-\bar{z}_N z_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{1-\bar{z}_1 z_N} & \frac{1}{1-\bar{z}_2 z_N} & \cdots & \frac{1}{1-|z_N|^2} \end{pmatrix}$$

and $b := (b_1, \ldots, b_N)^T$.

Computation of the roots of con-eigenpolynomials of $\mathbf{\Gamma}_f$

Let $P_u(x)$ be the n-th con-eigenpolynomial of $\mathbf{\Gamma}_f$.

Then for $|x| < 1$ we obtain

$$P_u(x) = \sum_{k=0}^{\infty} u_k x^k = \sum_{k=0}^{\infty} \left[\sum_{j=1}^{N} b_j z_j^k \right] x^k$$
$$= \sum_{j=1}^{N} b_j \sum_{k=0}^{\infty} (z_j x)^k = \sum_{j=1}^{N} \frac{b_j}{1 - z_j x}$$

Norm of the Hankel Operator: $\|\mathbf{\Gamma}_f\|$ vs. $\|f\|$

Let $e_1 := (1, 0, 0, \ldots)^T$. Then

$$\|f\|_{\ell^2} = \left(\sum_{j=0}^{\infty} |f_j|^2 \right)^{1/2} = \|\mathbf{\Gamma}_f e_1\|_{\ell^2} \leq \sup_{\|x\|_{\ell^2}=1} \|\mathbf{\Gamma}_f x\|_{\ell^2} = \|\mathbf{\Gamma}_f\|.$$

Therefore we have

$$\|f - \tilde{f}\|_{\ell^2} \leq \|\mathbf{\Gamma}_{f-\tilde{f}}\| = \sigma_n$$

Algorithm for sparse approximation of exponential sums

Input: samples $f_k, \ k = 0, \ldots, L$, for a sufficiently large L,
target approximation error ε.

1. Find the N nodes z_j and the weights a_j of the exponential representation of f using a Prony-like method.
2. Compute a con-eigenvalue $\sigma_n < \varepsilon$ of the matrix AZ and the corresponding con-eigenvector $u = u_n$.
3. Compute the n zeros $\tilde{z}_j$ of the con-eigenpolynomial $P_u(x)$ of $\mathbf{\Gamma}_f$ in $\mathbb{D}$ using the rational function representation.
4. Compute the new coefficients $\tilde{a}_j$ by solving

$$\min_{\tilde{a}_1,\ldots,\tilde{a}_n} \|f - \tilde{f}\|_{\ell^2}^2 = \min_{\tilde{a}_1,\ldots,\tilde{a}_n} \sum_{k=0}^{\infty} |f_k - \sum_{j=1}^{n} \tilde{a}_j (\tilde{z}_j)^k|^2.$$

Output: sequence $\tilde{f}_k = \sum_{j=1}^{n} \tilde{a}_j \tilde{z}_j^k$, such that $\|f - \tilde{f}\|_{\ell^2} \leq \sigma_n < \varepsilon$

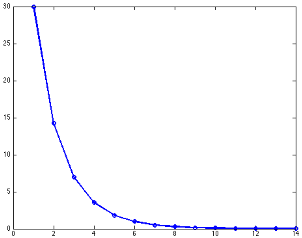
Outline

Numerical example

N=6

$f_k = \sum_{j=1}^{6} a_j z_j^k$

$a_j = 5,\ j = 1, \ldots, 6$



	$n=5$	$n=4$	$n=3$	$n=2$	$n=1$	
$z_1 = 0.3500$	0.3509	0.3550	0.3671	0.3985	0.4889	$\tilde{z}_1$
$z_2 = 0.4000$	0.4103	0.4365	0.4860	0.5684		$\tilde{z}_2$
$z_3 = 0.4500$	0.4802	0.5282	0.5910			$\tilde{z}_3$
$z_4 = 0.5000$	0.5456	0.5981				$\tilde{z}_4$
$z_5 = 0.5500$	0.5998					$\tilde{z}_5$
$z_6 = 0.6000$						
	4.5845e-10	1.6340e-07	3.1318e-05	4.3318e-03	4.8259e-01	σ_n

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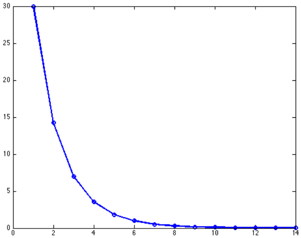

n	σ_n	$\|f - \tilde{f}\|_2$	$\frac{\max_k \|f_k - \tilde{f}_k\|}{\max_k \|f_k\|}$
1	4.8259e-01	4.7095e-01	1.1013e-02
2	4.3318e-03	4.2576e-03	7.6860e-05
3	3.1318e-05	2.8624e-05	5.9415e-07
4	1.6340e-07	1.4449e-07	2.9658e-09
5	4.5845e-10	8.0184e-10	1.1560e-11

Thank You !