Smoothing of vector and Hermite subdivision schemes

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Joint work with Nira Dyn

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Overview

- Subdivision and smoothing of subdivision schemes

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- Hermite subdivision

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- Smoothing procedure for Hermite schemes

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- Examples

Subdivision schemes

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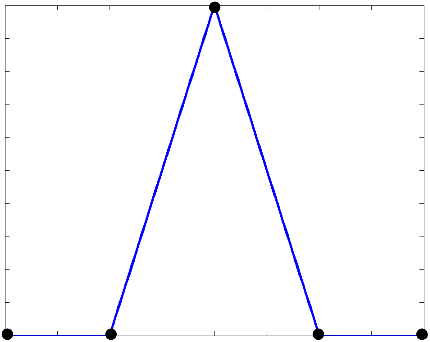

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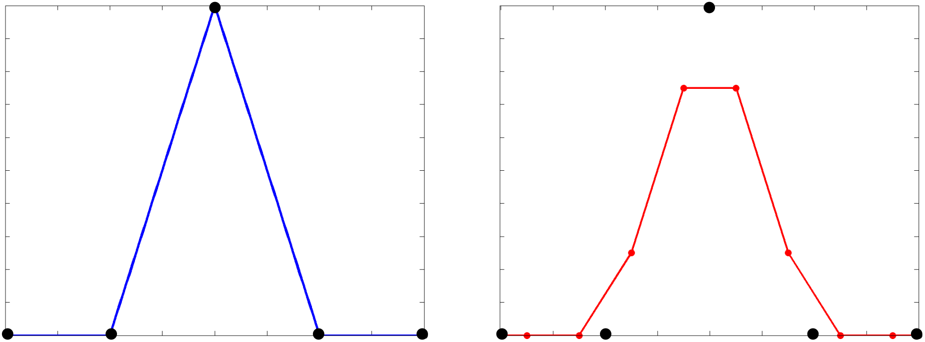

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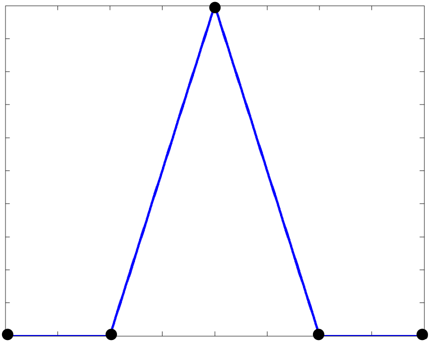

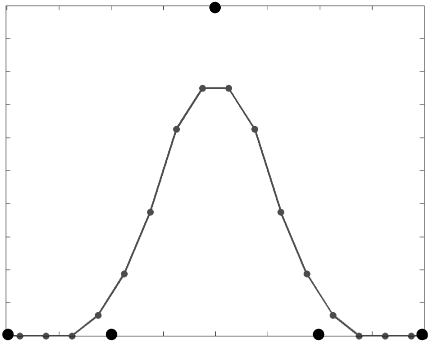

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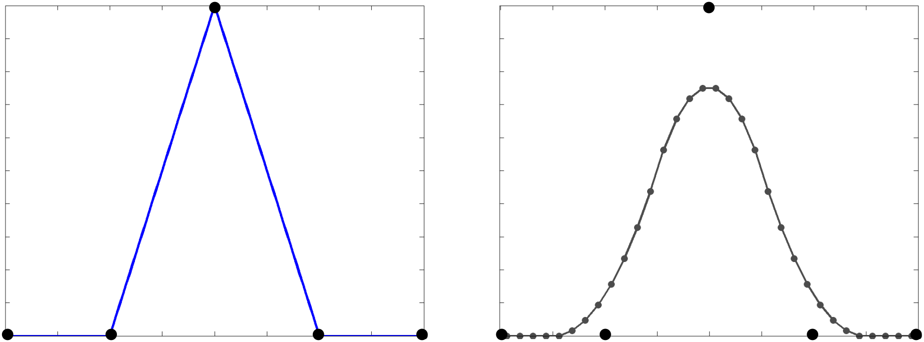

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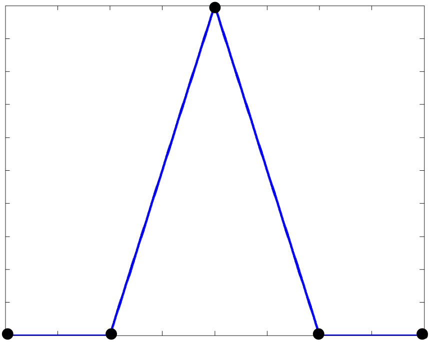

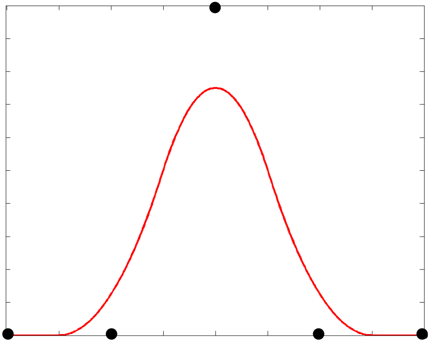

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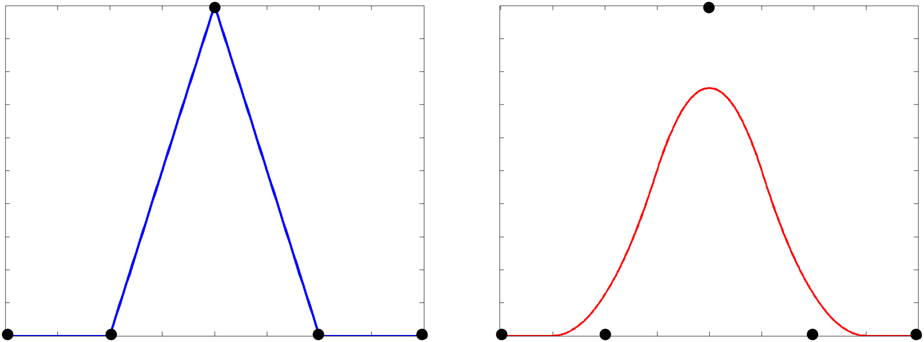

The limit of this subdivision process is a degree 2 spline.

Subdivision schemes

Chaikin's algorithm in more detail:

$$(S\delta)_{2i} = \frac{3}{4}\delta_i + \frac{1}{4}\delta_{i+1}$$

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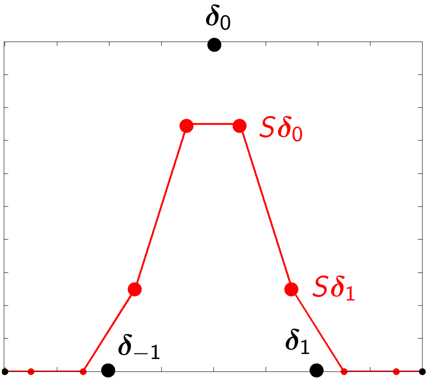

Input: δ

First step: $S\delta$

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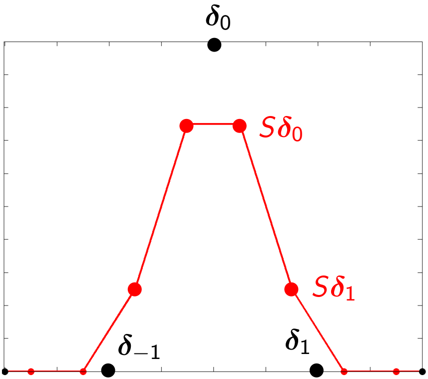

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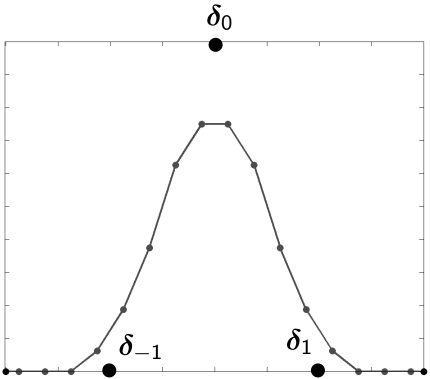

Input: δ

Second step: $S^2\delta$

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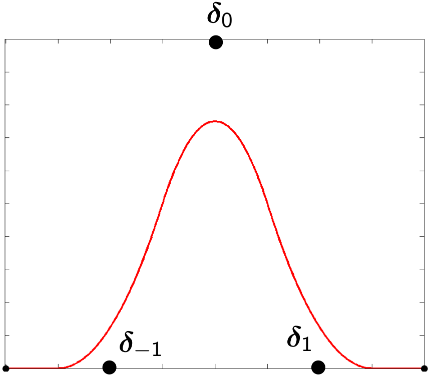

Input: δ

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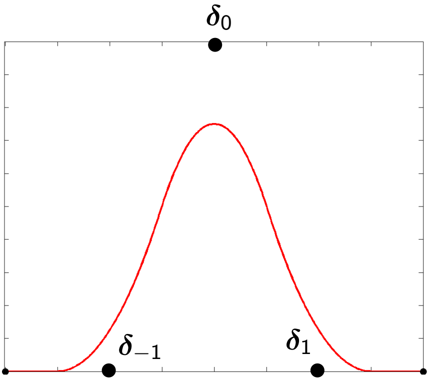
- In this example the limit is C^1



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Subdivision schemes

Start from input data p, a subdivision operator can be defined by two rules:

$$(Sp)_{2i} = \sum_{j \in \mathbb{Z}} a_{-2j} p_{i+j},$$
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For example, Chaikin's algorithm:

$$a(z) = \frac{1}{4} z^{-2} + \frac{3}{4} z^{-1} + \frac{3}{4} + \frac{1}{4} z.$$

Smoothing of subdivision schemes

Necessary condition for convergence:

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Smoothing of subdivision schemes:

$$S_*,\ C^\ell \quad \xrightarrow{\times \frac{z+1}{2z}} \quad S,\ C^{\ell+1}$$

Smoothing of subdivision schemes

The mask of the Lane-Riesenfeld algorithm for degree k B-Splines:

$$a_k(z) = \frac{(z+1)^{k+1}}{(2z)^k}$$

Apply subdivision operator S_k to 2D input data $\boldsymbol{\delta} = (i, \delta_{i,0})_{i \in \mathbb{Z}}$:

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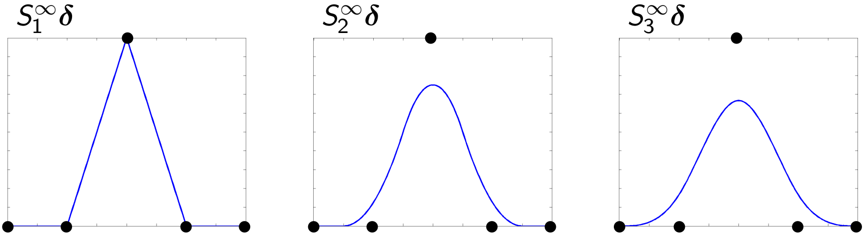

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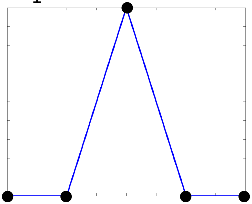
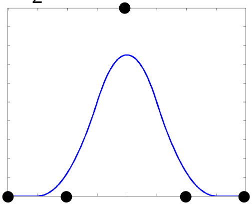


$a_1(z) = \frac{(z+1)^2}{2z}$

C^0

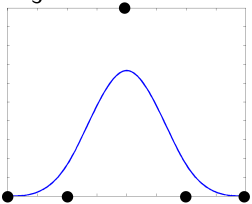
$S_2^\infty \boldsymbol{\delta}$



$a_2(z) = \frac{z+1}{2z} a_1(z)$

C^1

$S_3^\infty \boldsymbol{\delta}$



$a_3(z) = \frac{z+1}{2z} a_2(z)$

C^2

Hermite subdivision

Successive refinement of point-vector data for generating a function and its derivative

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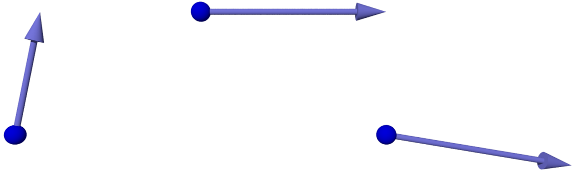

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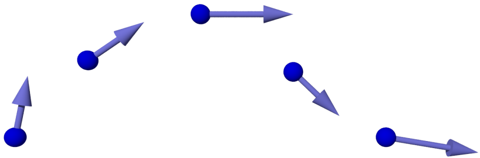

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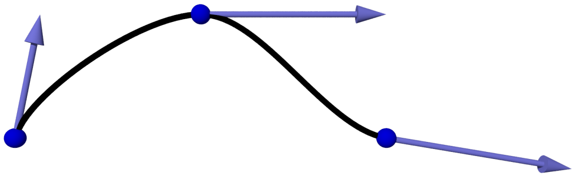

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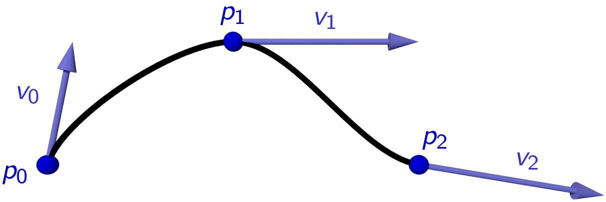

Subdivision operator: $S\binom{p}{v}_i = \sum_{j\in\mathbb{Z}} \begin{pmatrix} a_{i-2j} & b_{i-2j} \\ c_{i-2j} & d_{i-2j} \end{pmatrix} \binom{p_j}{v_j}$.

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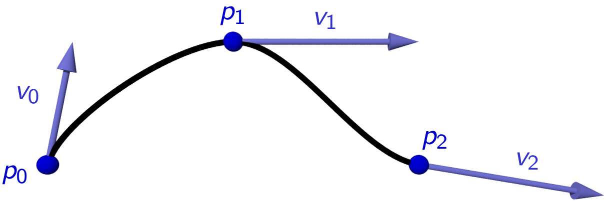

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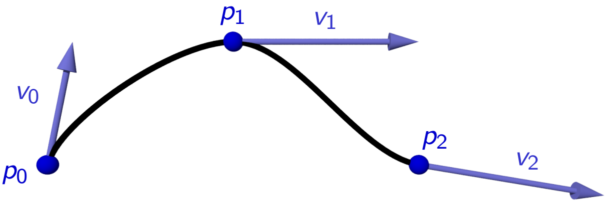

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- The iterates $S^n\binom{p}{v}$ describe the refined point-vector data
- $S^n\binom{p}{v}$ converges to function and its derivative (after appropriate scaling)

Smoothing of Hermite schemes

The spectral condition implies the existence of the derived scheme S_* with respect to the Taylor operator:

$$\begin{pmatrix} \Delta & -1 \\ 0 & 1 \end{pmatrix} S = \frac{1}{2} S_* \begin{pmatrix} \Delta & -1 \\ 0 & 1 \end{pmatrix}$$

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Theorem (Merrien and Sauer 2012; Conti, Merrien and Romani 2014)

If the vector scheme S_* is C^ℓ, $\ell \geq 0$, then the Hermite scheme S is $C^{\ell+1}$

Smoothing of Hermite schemes

$$
\begin{tikzcd}
\text{Hermite } S,\ C^{\ell} \arrow[rr, "\text{Taylor factorisation}"] \arrow[d, dashed, "\text{smoothing}"] & & \text{vector } S_*,\ C^{\ell-1} \arrow[d, "\text{vector smoothing}"] \\
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Theorem (Dyn, M. 2016)

Any* Hermite scheme S which is $C^{\ell}, \ell \geq 1$, can be transformed to a new Hermite scheme S of regularity $C^{\ell+1}$.

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${\color{red}S}$ is constructed from S by manipulating generating functions.

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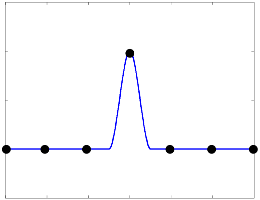
Disadvantage: Makes support larger by a maximum of 5.

Examples

We smoothen an interpolatory C^1 Hermite scheme by J.-L. Merrien.

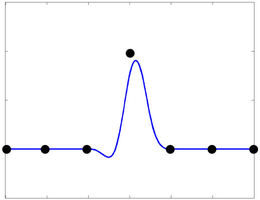
C^1 Hermite limit

$S^\infty \delta$



C^2 Hermite limit

$S^\infty \delta$



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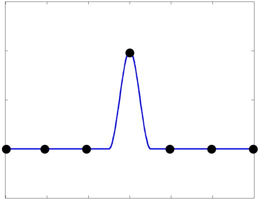
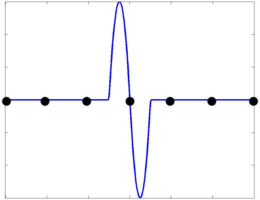
C^1 Hermite limit

$S^\infty \delta$

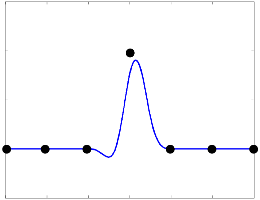


$S^{\infty\prime} \delta$

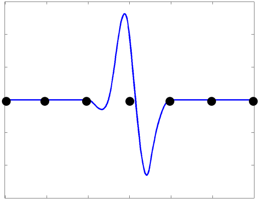


C^2 Hermite limit

$S^\infty \delta$



$S^{\infty\prime} \delta$

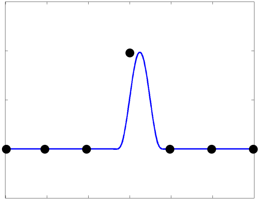


Examples

We smoothen a C^2 Hermite scheme constructed by a de Rham transform.

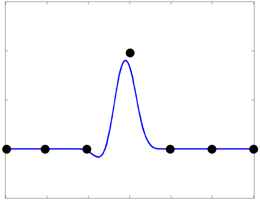
C^2 Hermite limit

$S^\infty\delta$



C^3 Hermite limit

$S^\infty\delta$

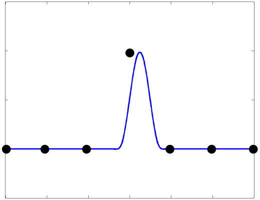


Examples

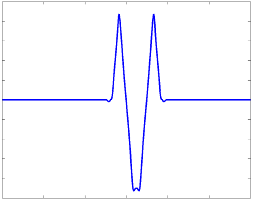
We smoothen a C^2 Hermite scheme constructed by a de Rham transform.

C^2 Hermite limit

$S^\infty\delta$

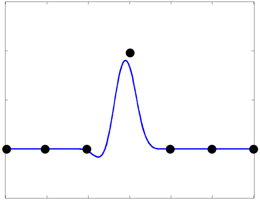


$S^{\infty\prime\prime}\delta$

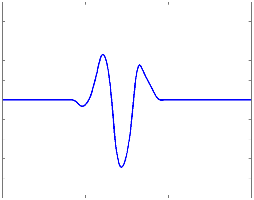


C^3 Hermite limit

$S^\infty\delta$



$S^{\infty\prime\prime}\delta$



Conclusion

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Thank you!

FWF Der Wissenschaftsfonds.

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Smoothing of Hermite schemes

For example, if S has the mask $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $b(1) = 0$, then the mask of S is given by

$$\begin{aligned}
a(z) &= \tfrac{(z+1)}{2z}\Big((z^{-2}-2)b(z) + a(z)\Big), \\
b(z) &= \tfrac{1}{2}\frac{zb(z)}{(1-z)}, \\
c(z) &= \tfrac{1}{2}(z^{-2}-1)\Big(c(z) - a(z)(z^{-1}-2) \\
&\quad + d(z)(z^{-2}-2) - b(z)(z^{-1}-2)(z^{-2}-2)\Big), \\
d(z) &= \tfrac{1}{2}(d(z) - (z^{-1}-2)b(z)).
\end{aligned}$$