Error estimates for multilevel Gaussian quasi-interpolation on the torus

Jeremy Levesley

University of Leicester

with Xingping Sun, University of Southwest Missouri

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Dedicated to Ward Cheney

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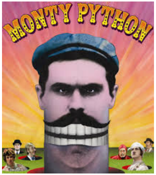
Good

Bad

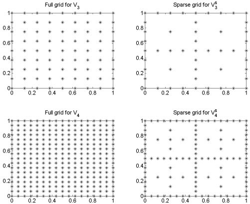


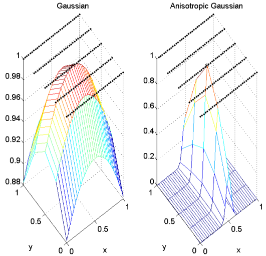
Motivation - sparse kernel approximation

MuSIK (Subhan, Georgoulis, SISC 2014), and QMuSIK Usta 2015.

Use normalised Gaussian

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2).$$





Used in solving PDEs directly using Kansa's method L, Zhang, Zhao, and for solving stochastic collocation problems L, Georgoulis, Dong, Usta.

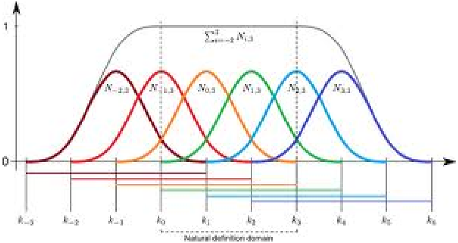
Selected previous research

1. Beatson and Light, approximation with Gaussians on integer grid, changing of shape parameter, 1992.
2. Iske and Floater, multilevel selection of points sets, 1996.
3. Narcowich, Schaback, Ward - multilevel interpolation and approximation, 1999.
4. Hales and L., multilevel interpolation using polyharmonic splines, 2002.
5. Iske and L., multilevel domain decomposition, 2005.
6. Mazya and Schmidt, Approximate approximations, 2007.
7. Wendland, Sloan, Le Gia, use of sampling inequalities for multilevel approximation in Euclidean space and on spheres, 2010-current.
8. Chen and Cao, convergence of quasi interpolation on compact intervals, 2010.
9. Levesley, Convergence of multilevel algorithm, but poor rates (2016).

Some numerical experiments for Stationary Approximation

Using different radial basis functions (RBFs) ϕ compute error in interpolating $f(x) = x^2$ by translates of dilated functions $(\psi(2^n(x - i/2^n)) = \psi(2^n x - i))$

$$s_n(x) = \sum_{i=0}^{2^n} \alpha_i \psi(2^n x - i).$$



Results

$$\phi(r) = \begin{cases} \exp(-r^2), & \text{Gaussian,} \\ r, & \text{norm.} \end{cases}$$

n	Gaussian	norm
2	1.0(-1)	1.5(-2)
3	8.3(-2)	3.9(-3)
4	7.4(-2)	9.7(-4)
5	6.9(-2)	2.4(-4)
6	6.8(-2)	6.1(-5)

Residual correction

Compute error in interpolating $e_{n-1}(x) = f(x) - s_{n-1}(x)$ by

$$t_n(x) = \sum_{i=0}^{2^n} \alpha_i^n \phi(2^n x - i),$$

and then

$$s_n(x) = s_{n-1}(x) + t_n(x).$$

n	$\rho = 1$	norm
2	5.5(-2)	1.5(-2)
3	1.9(-2)	3.9(-3)
4	6.4(-3)	9.7(-4)
5	2.2(-3)	2.4(-4)
6	7.6(-4)	6.1(-5)

Multilevel quasi-interpolation for periodic functions

For

$$f(x) = \sum_{k=0}^{\infty} a_k \cos(2\pi kx),$$

approximate with quasi-interpolant

$$Q_h g(x) = \sum_{z \in \mathbf{Z}} g(hz)\psi(h^{-1}x - z).$$

Then we have errors

$$E_1 f = f - Q_1 f,$$

and

$$E_n f(x) = E_{n-1} f(x) - Q_{2^{1-n}} E_{n-1} f(x), \quad n \geq 2.$$

Simple results I

Let $c_k(x) = \cos(2\pi kx)$, $k = 1, \cdots$.

$n \backslash k$	1	2	4	8	16
1	2.0	2.0	2.0	2.0	?
2	9.9(-1)	2.0	2.0	2.0	?
3	7.0(-1)	9.9(-1)	2.0	2.0	?
4	1.9(-1)	7.0(-1)	9.9(-1)	2.0	?
5	1.4(-2)	1.9(-1)	7.0(-1)	9.9(-1)	?
6	2.6(-4)	1.4(-2)	1.9(-1)	7.0(-1)	?
7	1.2(-6)	2.6(-4)	1.4(-2)	1.9(-1)	?
8	2.8(-9)	1.2(-6)	2.6(-4)	1.4(-2)	?
9	1.0(-10)	2.8(-9)	1.2(-6)	2.6(-4)	?
10	2.9(-11)	1.0(-10)	2.8(-9)	1.2(-6)	?
11	5.6(-14)	2.9(-11)	1.0(-10)	2.8(-9)	?
12	1.0(-15)	5.6(-14)	2.9(-11)	1.0(-10)	?

Simple results II

$n \backslash k$	1	9	11	13	15
1	2.0	2.0	2.0	2.0	2.0
2	9.9(-1)	1.0	1.0	1.0	1.0
3	7.0(-1)	1.3	1.3	1.3	1.3
4	1.9(-1)	1.8	1.1	1.1	1.8
5	1.4(-2)	1.03	1.1	1.5	2.0
6	2.6(-4)	8.0(-1)	9.5(-1)	1.0	1.0
7	1.2(-6)	2.6(-1)	4.0(-1)	5.4(-1)	6.5(-1)
8	2.8(-9)	2.4(-2)	5.4(-2)	9.8(-2)	1.6(-1)
9	1.0(-10)	5.7(-4)	1.9(-3)	4.9(-3)	1.0(-2)
10	2.9(-11)	3.5(-6)	1.8(-5)	6.2(-5)	1.7(-4)
11	5.6(-14)	7.8(-9)	4.6(-8)	2.1(-7)	7.5(-7)
12	1.0(-15)	1.7(-10)	4.5(-10)	8.9(-10)	1.8(-9)

Conclusions

- For $2^{j-1} < k < 2^j$ we can replace the error for c_k with the error for c_{2^j}.
- We can use an estimate for $E_n c_1$ for $E_{n+j} c_{2^j}$ for all j.

Some theory

Lemma

For $m = 1, 2, \cdots$,

$$Q_{1/m}c_0 \quad = \quad c_0 + 2\sum_{j=1}^{\infty} \hat{\psi}(j)c_{mj},$$

and for $k = 1, 2, \cdots$,

$$Q_{1/m}c_k \quad = \quad \sum_{j=1}^{\infty}(\hat{\psi}(k/m - j)c_{k-mj} + \hat{\psi}(k/m + j)c_{k+mj}),$$

where

$$\hat{\psi}(t) = \int_{-\infty}^{\infty} \psi(x)\exp(-2\pi itx)dx = \exp(-2\pi^2 t^2).$$

Consequently

Corollary

$$c_0 - Q_{1/n} c_0 = -2 \sum_{j=1}^{\infty} \hat{\psi}(j) c_{nj} \approx -2\hat{\psi}(1) c_{nj},$$

so no convergence of stationary approximation scheme.

Writing $\|f\| = \sum_{k=0}^{\infty} |a_k|$,

Corollary

$$\|f - Q_{1/n} f\| \leq A\|f\|,$$

where $A = 2 + 4 \times 10^{-7}$.

Example $f = c_1$

First level:

$$Q_1 c_1 = \hat{\psi}(0)c_0 + 2\hat{\psi}(1)c_1 + 2\hat{\psi}(2)c_2 + \cdots$$

$$\begin{aligned} E_1 c_1 &= c_1 - Q_1 c_1 \\ &\approx (1 - 2\hat{\psi}(1))c_1 - c_0 \end{aligned}$$

Second iteration

$$\begin{aligned} Q_{1/2}E_1 c_1 &= (1 - 2\hat{\psi}(1))Q_{1/2}c_1 - Q_{1/2}c_0 \\ &= (1 - 2\hat{\psi}(1))(2\hat{\psi}(1/2)c_1 + 2\hat{\psi}(3/2)c_3) \\ &\qquad -(c_0 + 2\hat{\psi}(1)c_2). \end{aligned}$$

$$\begin{aligned} E_2 c_1 &= E_1 c_1 - Q_{1/2}E_1 c_1 \\ &\approx (1 - 2\hat{\psi}(1))(1 - 2\hat{\psi}(1/2))c_1 + 2\hat{\psi}(1)c_2 - 2\hat{\psi}(3/2)(1 - 2\hat{\psi}(1))c_3. \end{aligned}$$

Bounding $E_n c_1$

Let

$$\begin{aligned}\gamma_1 &= (1-2\hat{\psi}(1)),\\ \gamma_2 &= (1-2\hat{\psi}(1))(1-2\hat{\psi}(1/2)),\\ \gamma_n &= (1-\hat{\psi}(2^{1-n}))\gamma_{n-1}, \quad n\geq 3.\end{aligned}$$

For $n \geq 6$

$$\beta_n = \gamma_n + \gamma_1\hat{\psi}(3/4)\gamma_{n-2} + \gamma_2\hat{\psi}(7/8)\gamma_{n-3} + \gamma_3\hat{\psi}(15/16)\gamma_{n-4} + \gamma_4\hat{\psi}(31/32)\gamma_{n-5}.$$

n	$\|E_n c_1\|_\infty$	β_n
6	2.6(-4)	2.7(-4)
7	1.5(-6)	1.5(-6)
8	2.8(-9)	1.4(-8)
9	1.0(-10)	6.8(-10)
10	2.9(-12)	2.8(-11)
11	5.6(-14)	4.7(-13)
12	1.0(-15)	2.1(-15)

Error estimates

Split error into two parts

$$E_n f \quad = \quad \sum_{k=0}^{2^n} a_k E_n c_k + \sum_{k=2^{n-1}}^{\infty} a_k E_n c_k.$$

Now

$$\left\| \sum_{k=2^{n-1}}^{\infty} a_k E_n c_k \right\| \leq A^n \sum_{k=2^n+1}^{\infty} |a_k|.$$

$$\begin{aligned} \left\| \sum_{k=0}^{2^n} a_k E_n c_k \right\| &= |a_0| \| E_n c_0 \| + \sum_{l=1}^{n} \sum_{j=0}^{2^{l-1}-1} |a_{2^{l-1}+j}| \left\| E_n c_{2^{l-1}+j} \right\| \\ &\leq |a_0| \hat{\psi}(1) \beta_n + \sum_{l=1}^{n} \beta_{n+1-l} \sum_{j=0}^{2^{l-1}-1} |a_{2^{l-1}+j}|. \end{aligned}$$

Example $f(x) = \exp(\cos(2\pi x))$

Level n	$\|E_n f\|_\infty$
1	1.2
2	1.1 (-1)
3	4.4 (-1)
4	9.1 (-2)
5	8.5 (-3)
6	3.1 (-4)
7	3.4 (-6)
8	1.6 (-8)
9	7.3 (-11)
10	2.7 (-12)

For the Torus

Tensor product basis

$$e^{2\pi i \mathbf{k}\cdot\mathbf{x}} = \prod_{j=1}^{d} e^{2\pi i k_j x_j}$$

Tensor product kernel

$$\psi(\|A\mathbf{x}\|) = \prod_{j=1}^{d} \psi(a_j x_j).$$

Then

$$\sum_{\mathbf{k}\in\mathbf{Z^d}} e^{2\pi i \mathbf{k}\cdot\mathbf{h}} \psi(\|h^{-1}\mathbf{x} - \mathbf{k}\|^2) = \prod_{j=1}^{d} e^{2\pi i k_j h} \sum_{k_j\in\mathbf{Z}} \psi(\|h^{-1}x_j - k_j\|^2).$$

to get error on torus.

Conclusions

1. Completed (more or less) convergence of multilevel algorithm on the torus.
2. Now adapt for sparse grids for quasi-interpolation.
3. Look at periodic interpolation case, and quasi-interpolation on compact interval.

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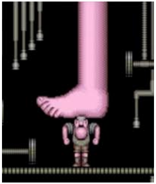

Goodbye our friend