PARTIALLY NESTED HIERARCHICAL B-SPLINES

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INTRODUCTION

Adaptive spline refinement

T-splines (Sederberg et al. 2003):
tensor-product B-splines defined on a mesh with T-junctions

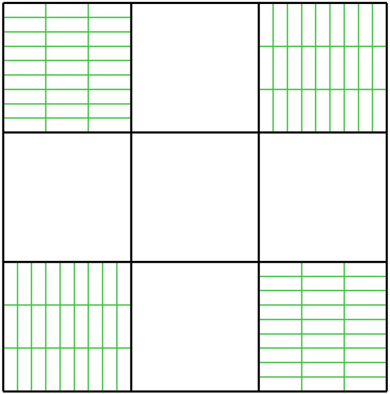
PHT splines (Falai Chen, Jiansong Deng 2008):
algebraically complete basis for splines on a mesh with T-junctions

HB-splines, THB-Splines (Kraft 1997, Giannelli et al. 2012):
obtained by selecting B-splines from different levels in a hierarchy

LR Splines (Dokken et al. 2010):
constructed by repeatedly splitting tensor-product B-splines

This talk focuses on ***HB-splines***.

Motivation



Independent refinement strategies
$\rightsquigarrow$ cannot be achieved with HB-splines.

Motivation

State of the art: Hierarchical B-splines that use sequences of nested spline spaces, $V^0 \subseteq V^1 \subseteq \ldots \subseteq V^N$.

Limits: Independent refinement strategies are not possible.

Possible application of independent refinement strategies:

- Modeling: designing objects with creases or similar features.
- IGA: using different refinement techniques (e.g., h- and p-refinement) in different parts of the domain.

Goal: Generalization of the selection mechanism for hierarchical B-splines to obtain sequences of ***partially nested hierarchical spline spaces***.

Preliminaries

We consider:

- A finite sequence of bivariate tensor-product spline spaces:

$$V^\ell = \text{span}\, B^\ell, \quad \ell = 1, \ldots, N.$$

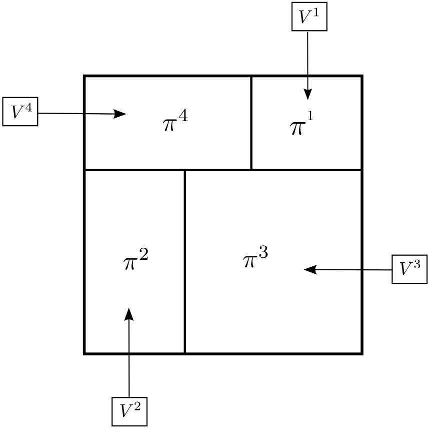
  - The spline bases B^ℓ consist of tensor-product B-splines.
  - The index ℓ will be called ***level***.
  - Note: V^ℓ not necessarily subspace of $V^{\ell+1}$

- An associated sequence of open sets

$$\pi^\ell \subseteq (0,1)^2, \quad \ell = 1, \ldots, N.$$

  - The sets are called ***patches***.
  - We assume that they are mutually disjoint, i.e., $\pi^\ell \cap \pi^k \neq \emptyset \;\Rightarrow\; \ell = k$.

Preliminaries



Patches and associated spline spaces.

The PNH spline space

Collecting all patches results in the ***domain*** Ω, i.e.,

$$\Omega = \text{int}\left(\bigcup_{\ell=1}^{N} \overline{\pi^{\ell}} \right) \subseteq (0,1)^2.$$

Now we define the ***partially nested hierarchical spline space*** (PNH spline space):

$$H = \{ f \in \mathcal{C}^{\mathbf{s}}(\Omega) : f|_{\pi^{\ell}} \in V^{\ell}|_{\pi^{\ell}} \ \forall \ell = 1, \ldots, N \},$$

with the order of smoothness being

$$\mathbf{s} = \mathbf{p} - 1.$$

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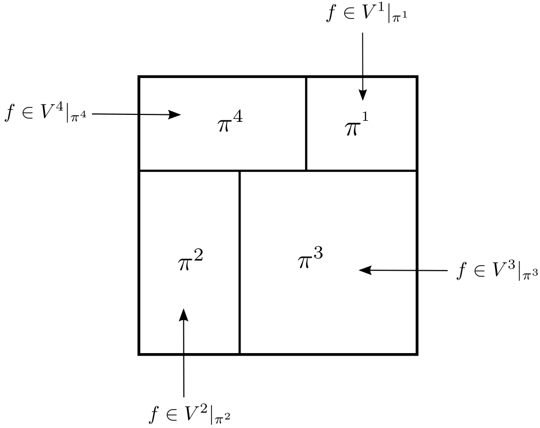
The PNH spline space

$$H = \{f \in \mathcal{C}^{\mathbf{s}}(\Omega) : f|_{\pi^\ell} \in V^\ell|_{\pi^\ell} \ \forall \ell = 1, \ldots, N\},$$



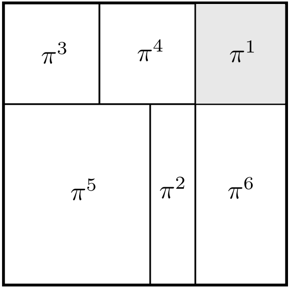
The partially nested hierarchical spline space.

Constraining boundaries

For each patch there exists a ***constraining boundary***

$$\Gamma^\ell = \bigcup_{k=1}^{\ell-1} \overline{\pi^k} \cap \overline{\pi^\ell},$$

i.e., that part of the boundary shared with patches of a lower level.



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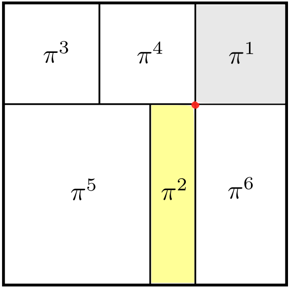

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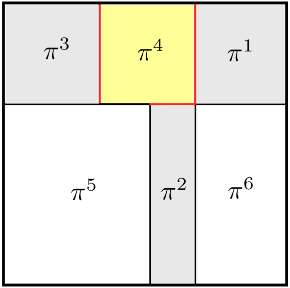

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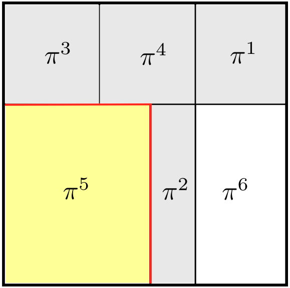

Constraining boundaries

BASIS FUNCTIONS

Selection mechanism

Generalizing Kraft's selection mechanism leads to a sequence of B-splines from all levels ℓ:

$$K^\ell = \{\beta^\ell \in B^\ell : \beta^\ell|_{\pi^\ell} \neq 0 \quad \text{and} \quad \beta^\ell|_{\Gamma^\ell} = 0\}.$$

Definition: The ***partially nested hierarchical B-splines*** (PNHB-splines) are obtained by forming the union over all levels,

$$K = \bigcup_{\ell=1}^{N} K^\ell.$$

Selection mechanism

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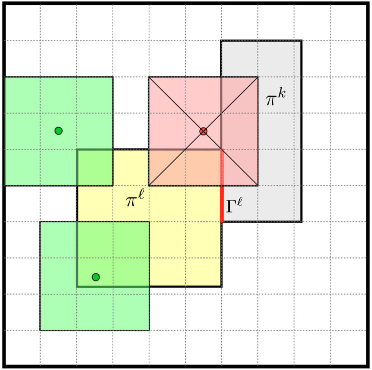
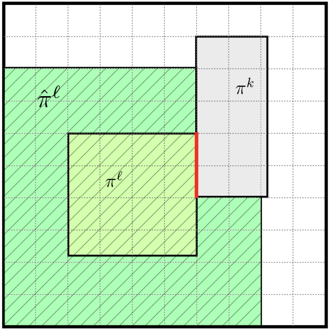


The selection mechanism for PNHB-splines ($k < \ell$).

Shadow

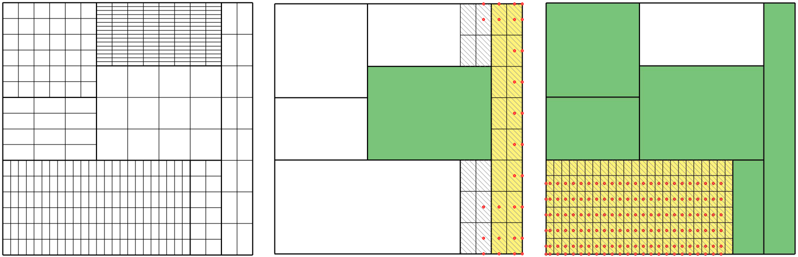
We define the ***shadow*** of a patch π^ℓ as the union of all supports of the selected basis functions,

$$\hat{\pi}^\ell = \text{supp}\, K^\ell = \bigcup_{\beta^\ell \in K^\ell} \text{supp}\, \beta^\ell.$$



Example

The knot lines of the spline space V^ℓ define a ***mesh*** M^ℓ of level ℓ.



A partially nested hierarchical mesh (left), the shadows and selected basis functions of two different patches (middle and right). Basis functions are represented by Greville points.

THE SPLINE SPACE

Assumptions

Assumption	Acronym	Used for
Shadow Ordering Assumption	SOA	proving linear independence
Shadow Compatibility Assumption	SCA	characterizing the space span K
Constrained Boundary Alignment	CBA	characterizing the space span K
Full Boundary Alignment	FBA	proving algebraic completeness
Support Intersection Condition	SIC	proving algebraic completeness

Assumptions and results.

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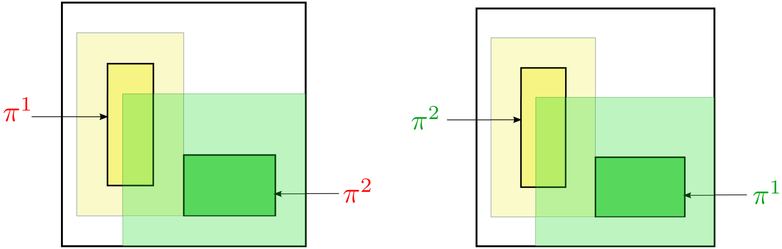
Shadow Ordering Assumption (SOA)

Assumption If the shadow $\hat{\pi}^\ell$ of the patch of level ℓ intersects another patch π^k of level k, then the first level is lower than the second one,

$$\hat{\pi}^\ell \cap \pi^k \neq \emptyset \quad \Rightarrow \quad \ell \leq k.$$



SOA not satisfied. SOA satisfied.

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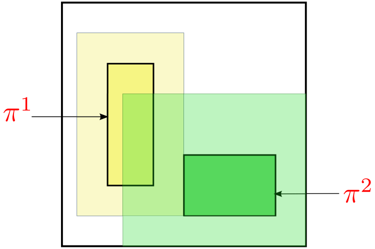
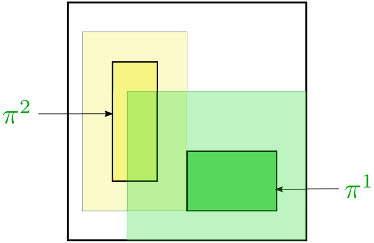


SOA not satisfied.



SOA satisfied.

Theorem: The PNHB-splines are ***linearly independent*** on Ω if SOA holds.

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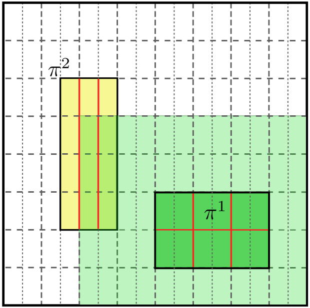
Assumptions and results.

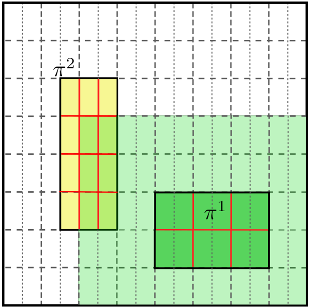
Shadow Compatibility Assumption (SCA)

Assumption If the shadow $\hat{\pi}^\ell$ of the patch of level ℓ intersects another patch π^k of a different level k, then the first level precedes * the second one,

$$\hat{\pi}^\ell \cap \pi^k \neq \emptyset \quad \Rightarrow \quad \ell \leq k \quad \text{and} \quad V^\ell \subseteq V^k.$$



SCA not satisfied.

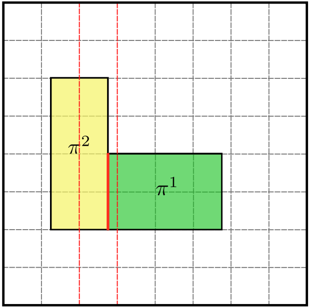


SCA satisfied.

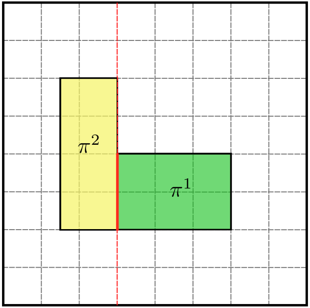
$^\star \ell < k$ and $V^\ell \subseteq V^k$

Constrained Boundary Alignment (CBA)

Assumption For each level ℓ, the constraining boundary Γ^ℓ of the patch π^ℓ is aligned with the knot lines of the spline space V^ℓ.



CBA not satisfied.



CBA satisfied.

Space characterization

Theorem The PNHB-splines span the partially nested hierarchical spline space H if both SCA and CBA are satisfied.

Thus, we have ***two different characterizations*** of the PNH spline space:

$$H = \{f \in \mathcal{C}^{\mathbf{s}}(\Omega) : f|_{\pi^\ell} \in V^\ell|_{\pi^\ell} \ \forall \ell = 1, \dots, N\},$$

(***"implicit"*** definition: space defined by properties of functions)

$$H = \text{span} \bigcup_{\ell=1}^{N} \{\beta^\ell \in B^\ell : \beta^\ell|_{\pi^\ell} \neq 0 \quad \text{and} \quad \beta^\ell|_{\Gamma^\ell} = 0\}$$

(***"constructive"*** definition: space defined as linear hull of basis functions)

Completeness question: Is $H = F$?

Recall: The PNH spline space is defined as

$$H = \{f \in \mathcal{C}^{\mathbf{s}}(\Omega) : f|_{\pi^\ell} \in V^\ell|_{\pi^\ell} \ \forall \ell = 1, \ldots, N\}.$$

Question: Do PNHB-splines span the ***full spline space***

$$F = \{f \in \mathcal{C}^{\mathbf{s}}(\Omega) : f|_c \in \Pi_{\mathbf{p}} \ \forall c \in C^\ell \ \forall \ell = 1, \ldots, N\},$$

of C^s smooth piecewise polynomial functions on the mesh?

It is clear that $H \subseteq F$. How about $H \supseteq F$?

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Assumptions and results.

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Completeness

Assumption (Full Boundary Alignment) All boundaries of the patches π^ℓ are aligned with the mesh of level ℓ.

Assumption (Support Intersection Condition) The support intersections of the basis functions of level ℓ with the associated patches π^ℓ are all connected,

$$\operatorname{supp} \beta^\ell \cap \pi^\ell \quad \text{is connected} \quad \forall \beta^\ell \in B^\ell,\ \ell = 1, \ldots, N.$$

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$$\text{supp}\, \beta^\ell \cap \pi^\ell \quad \text{is connected} \quad \forall \beta^\ell \in B^\ell,\ \ell = 1, \dots, N.$$

Theorem: The PNHB-splines span the full spline space if SCA, FBA and SIC are satisfied.

TRUNCATION

Restoring partition of unity

Truncation mechanism

Kraft 1997 $\rightarrow$ Giannelli et al. 2012:

Hierarchical B-splines $\rightarrow$ Truncated Hierarchical B-splines (**THB-splines**)

The recipe

Truncated function: "original function ***minus*** contribution of selected basis functions from higher levels"

Is there a generalization to truncated PNHB-splines?

Restoring partition of unity

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The recipe

Truncated function: "original function ***minus*** contribution of selected basis functions from higher levels"

Is there a generalization to truncated PNHB-splines? **Yes!**

For the truncated PNHB-splines we can show that

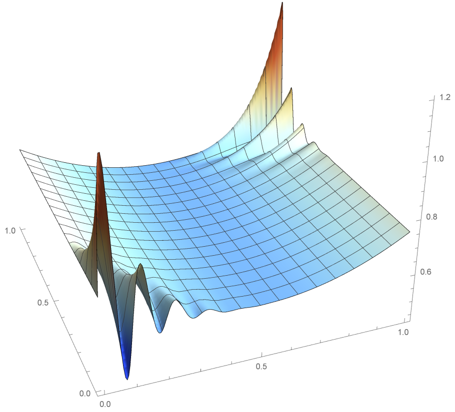
- they are linearly independent,
- they form a **partition of unity** and
- they span the partially nested spline space.

We could not prove non-negativity so far (but did not find negative TPNHB-splines either).

PNHB SPLINES IN SURFACE APPROXIMATION

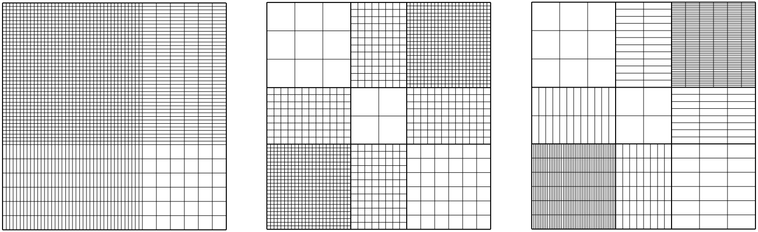
Example

We want to approximate the following function:



Function for approximation.

Manual mesh generation



The meshes used for defining the approximating spline functions.

	no. of dof	% of dof	max. error	average error
tensor-product B-splines	2304	100%	3.39e-3	3.81e-5
HB-splines	1633	71 %	3.08e-3	4.37e-5
PNHB-splines	769	33 %	8.12e-4	1.89e-5

Numerical results of the least-squares approximation.

Automatic mesh refinement

Determining the refinement direction with a ***local fitting-based*** method:

- Perform local fitting on patches π^ℓ.
- Try different refinement strategies, e.g., uniform knot refinement in x- vs. in y-direction.
- The strategy that performs better, i.e., produces less error, determines the refinement direction.

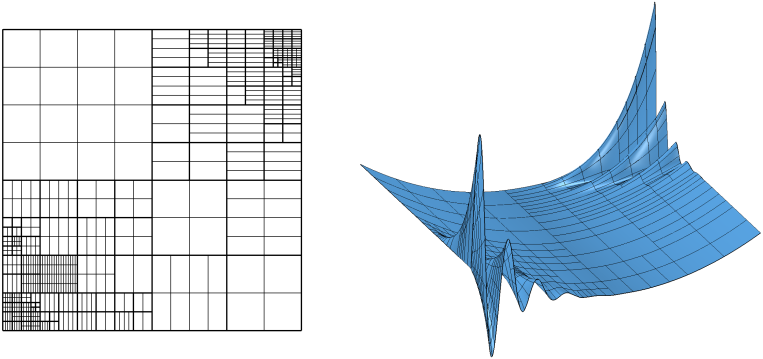
Advantages:

- No assumptions on data,
- simple.

Disadvantages:

- Could become slow if too many strategies are tested.

Automatic mesh refinement - results



PNH spline mesh after 6 steps of adaptive refinement and resulting surface.

	no. of dof	% of dof	max. error	average error
HB-splines	1260	100 %	1.05e-3	6.14e-5
PNHB-splines	576	46 %	8.86e-4	6.79e-5

Numerical results of the least-squares approximation.

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SUMMARY AND OUTLOOK

Summary

- Generalization of a selection mechanism from hierarchical B-splines to PNHB-splines
- Identification of certain assumptions for defining a hierarchical spline basis
- Introduction of a truncation mechanism
- Derivation of a completeness result
- Application of PNHB-splines in surface approximation
- Presentation of a first automatic refinement algorithm for PNHB-spline meshes
- PNHB splines need fewer degrees of freedom than HB splines

Current work and outlook

- Generalization of the PNHB-splines to
  - arbitrary dimension d,
  - degrees $\mathbf{p}^\ell = (p_1^\ell, \ldots, p_d^\ell)$ and
  - multiplicities $m_i^\ell(x_i) \geq 1$.
- Truncation: alternative construction
  - Goal: proving non-negativity
- Implementation of the truncation mechanism
- Development of further automatic mesh refinement strategies
- Application in industry (e.g., fitting of structural components).