RAtional Geometric Splines:

construction and applications in the representation of smooth surfaces

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Multivariate Approximation and Interpolation with Applications

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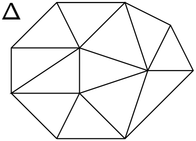
Bivariate splines on planar triangulations



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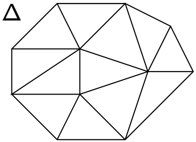

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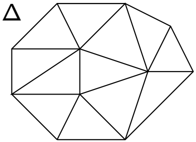

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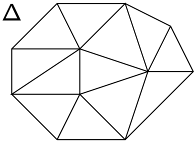

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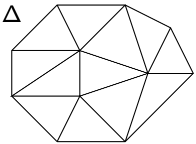

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- a plethora of available methods based on such splines for data fitting, geometric modeling, finite elements, etc.
- our objective is to define a generalization of $S_n^m(\Delta)$ for 3D triangulations Δ

– Spline spaces $S_n^m(\Delta)$ are effective in the problem of interpolating/approximating

functional data (x_i, y_i, z_i), where $(x_i, y_i) \in \Omega$, $\Omega \subset \mathbb{R}^2$, i.e. in finding a spline $s \in S_n^m(\Delta)$ such that $s(x_i, y_i) = z_i \in \mathbb{R}$

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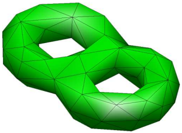
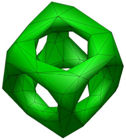
- What if the sought-for surface cannot be parameterized by a planar domain?





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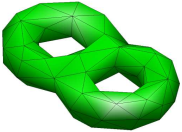

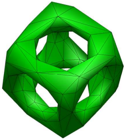

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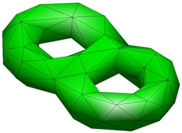
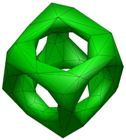

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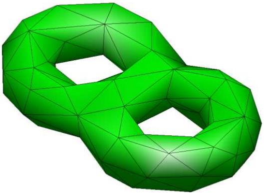
- Alternative methods such as implicit surfaces, manifold splines, subdivision surfaces, T-splines, ambient B-splines, . . .

Splines on 3D triangulations of arbitrary genus

Δ



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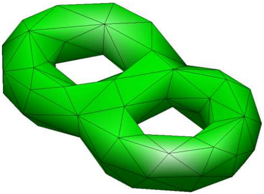

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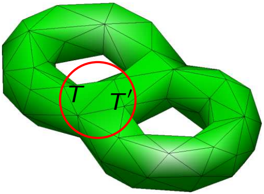

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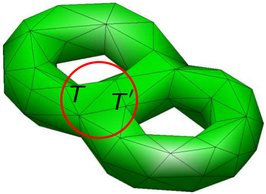

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- define the space of functions $C^m(\Delta)$ as a space of geometrically continuous functions $f : \Delta \to \mathbb{R}$ (or $\mathbb{R}^d, d > 1$), such that for all such pairs T, T', $f|_{T'}$ and $f|_T \circ \phi_{T',T}$ join with ordinary C^m continuity along common edge [DeRose '85, Haas '89, Peters 2002, etc.]

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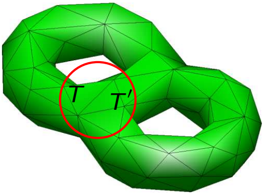

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- for fixed transition maps, $C^m(\Delta)$ is a linear space, hence so is $S_n^m(\Delta)$, assuming ρ_n^T are chosen as linear spaces [DeRose '85, Höllig-Mogerle '90, Goodman '91, Prautzsch '97, Reif '98, Peters '02, papers on manifold splines, etc.]

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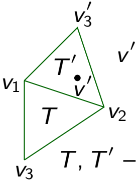

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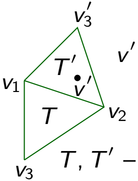

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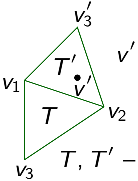

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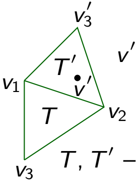

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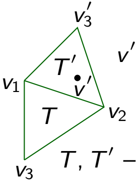

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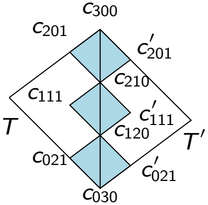

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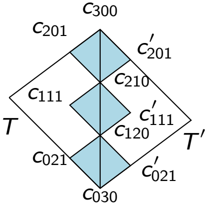

- standard interpolation/approximation methods can be extended to 3D triangulations (no need to invent brand new methods)

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2. Δ_H gives rise to linear rational transition maps $\phi_{T',T}$, corresponding to parameters $\lambda_1, \lambda_2, \lambda_3$, given by [B. & Neamtu '16]

$$\lambda_1 = \frac{\sin(\alpha_2+\alpha_2')\sin\alpha_1\sin\alpha_1'}{\sin\alpha_3'\sin\alpha_1\sin\alpha_2}$$

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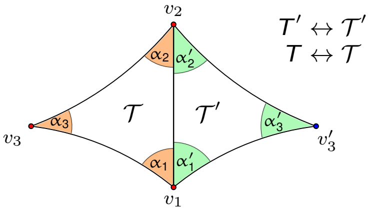

Construction of RAGS via homogeneous geometry

1. Construct a triangulation Δ_H in one of the three homogeneous geometries (spherical, affine, hyperbolic), which is combinatorially equivalent to Δ (the type of geometry will depend on the genus of Δ)

2. Δ_H gives rise to linear rational transition maps $\phi_{T',T}$, corresponding to parameters $\lambda_1, \lambda_2, \lambda_3$, given by [B. & Neamtu '16]

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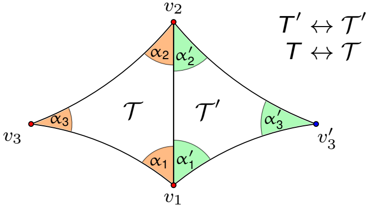

3. Define spaces of homogeneous splines:

$$S_n^m(\Delta_H) := \{s \in C^m(\Delta_H) : s|_{\mathcal{T}} \in \pi_n|_{\mathcal{T}}, \mathcal{T} \in \Delta_H\}$$

$\pi_n|_{\mathcal{T}}$ – trivariate homogeneous polynomials of total degree $\leq n$, restricted to triangle $\mathcal{T} \in \Delta_H$.

Construction of RAGS via homogeneous geometry

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4 function $p \in \pi_n|_{\mathcal{T}}$ can be written in terms of (homogeneous) barycentric coordinates w.r.t. $\mathcal{T}$:

$$p(u) = \sum_{i+j+k=n} c_{ijk} B_{ijk}^n(u), \quad u \in \mathcal{T}, \quad B_{ijk}^n(u) = \frac{n!}{i!j!k!} b_1^i(u) b_2^j(u) b_3^k(u),$$

where $b = (b_1(u), b_2(u), b_3(u))$ are the homogeneous barycentric coordinates of u w.r.t. vertices of $\mathcal{T}$.

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5 Parametrize $S_n^m(\Delta_H)$ by Δ to obtain $S_n^m(\Delta)$. For $T \in \Delta \leftrightarrow \mathcal{T} \in \Delta_H$, $v \in T \leftrightarrow u \in \mathcal{T}$:

$$r_H^{\mathcal{T}}(b(u)) = \frac{\sum c_{ijk}^T B_{ijk}^{n,\mathcal{T}}(u)}{\sum w_{ijk}^T B_{ijk}^{n,\mathcal{T}}(u)} = \frac{\sum c_{ijk}^T (b_1 + b_2 + b_3)^n B_{ijk}^{n,T}(v)}{\sum w_{ijk}^T (b_1 + b_2 + b_3)^n B_{ijk}^{n,T}(v)}$$

$$= \frac{\sum c_{ijk}^T B_{ijk}^{n,T}(v)}{\sum w_{ijk}^T B_{ijk}^{n,T}(v)} =: r^T(\bar{b}(v)),$$

Parametrization - determining Δ_H

Since elements of Δ_H are equivalence classes of congruent triangles, we can represent Δ_H as a collection of triples of angles:

$$\Delta_H = \{\langle \alpha_1^T, \alpha_2^T, \alpha_3^T \rangle, T \in \Delta\}, \quad \alpha_1^T, \alpha_3^T, \alpha_3^T \in (0, \pi)$$

Any triple $\alpha_1^T, \alpha_2^T, \alpha_3^T \in (0, \pi)$ gives rise to a unique triangle (up to congruence) in one of the three homogeneous geometries:

$\alpha_1^T + \alpha_2^T + \alpha_3^T < \pi$ (hyperbolic)
$\alpha_1^T + \alpha_2^T + \alpha_3^T = \pi$ (affine)
$\alpha_1^T + \alpha_2^T + \alpha_3^T > \pi$ (spherical)

Necessary and sufficient conditions for a consistent triangulation Δ_H:

- the sum of angles around each vertex is 2π;
- common sides of adjacent triangles have equal lengths:

$$\frac{\cos \alpha_1^T \cos \alpha_2^T + \cos \alpha_3^T}{\sin \alpha_1^T \sin \alpha_2^T} = \frac{\cos \alpha_1^{T'} \cos \alpha_2^{T'} + \cos \alpha_3^{T'}}{\sin \alpha_1^{T'} \sin \alpha_2^{T'}}$$

Weights and control points

- Instances of RAGS have been obtained before:
  [Liu, Schumaker '96, Wallner '96, Pottmann, Wallner '97, He, Gu, Qin '06]

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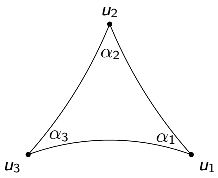
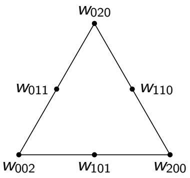
  - for a quadratic polynomial

$$w = \sum_{i+j+k=2} w_{ijk} B^2_{ijk} \equiv 1$$

yields:

$$w_{200} = w_{020} = w_{002} = 1$$





$$w_{011} = \frac{\cos\alpha_2 \cos\alpha_3 + \cos\alpha_1}{\sin\alpha_2 \sin\alpha_3}, \quad w_{101} = \frac{\cos\alpha_1 \cos\alpha_3 + \cos\alpha_2}{\sin\alpha_1 \sin\alpha_3}, \quad w_{110} = \frac{\cos\alpha_1 \cos\alpha_2 + \cos\alpha_3}{\sin\alpha_1 \sin\alpha_2}$$

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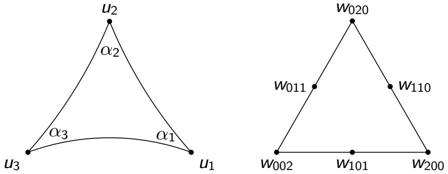

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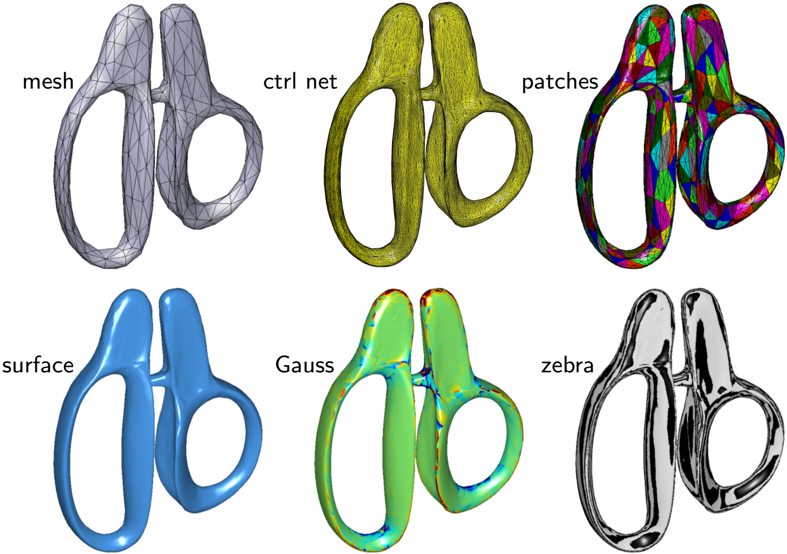
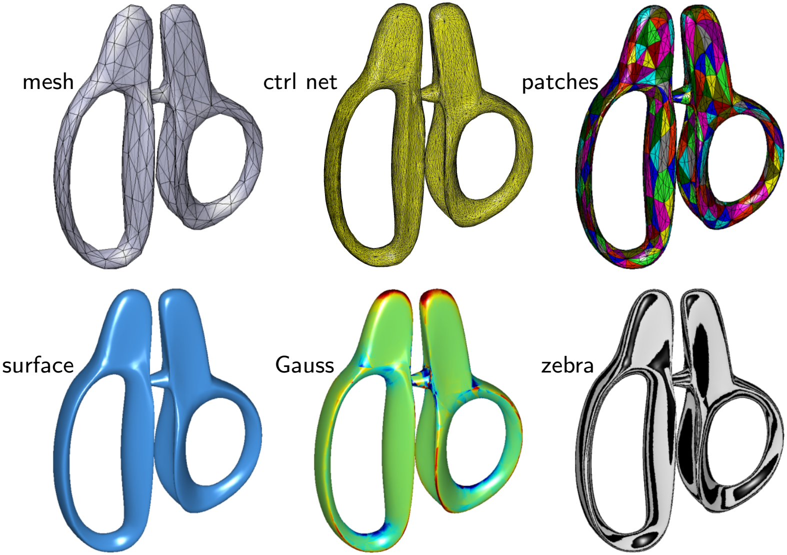
- Specific interpolation methods can be implemented as a direct extension of bivariate methods. Examples are:
  - Powell-Sabin macro-element methods (local and global)
  - energy-minimizing interpolating splines

Example: $S_2^1(\Delta_{PS})$ local interpolation

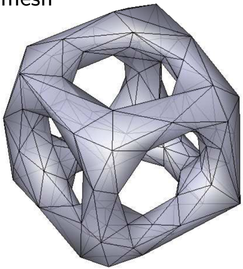


Example: $S_2^1(\Delta_{PS})$ interpolation & energy min.

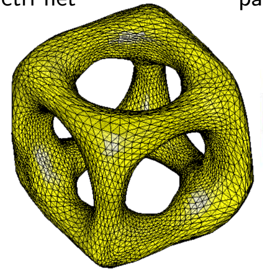


Example: $S_6^2(\Delta)$ interpolation & energy min.

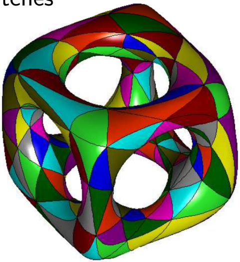
mesh



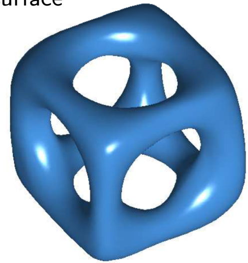
ctrl net



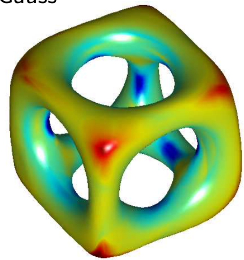
patches



surface

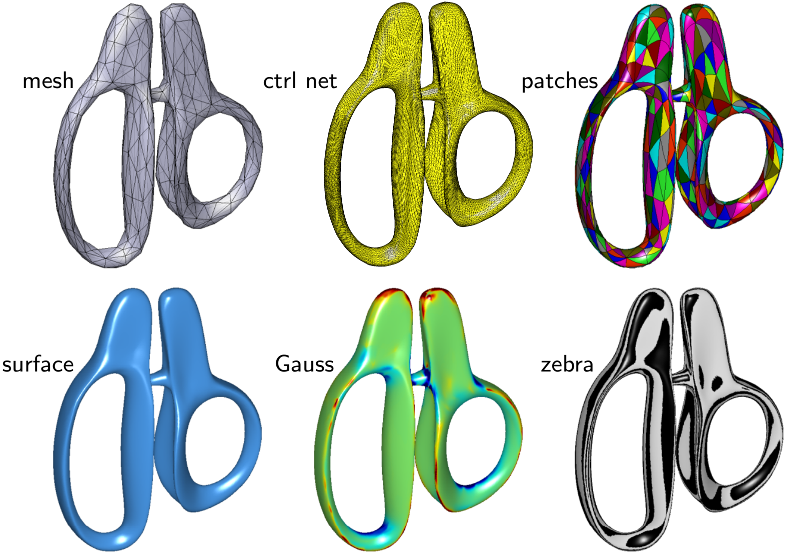


Gauss

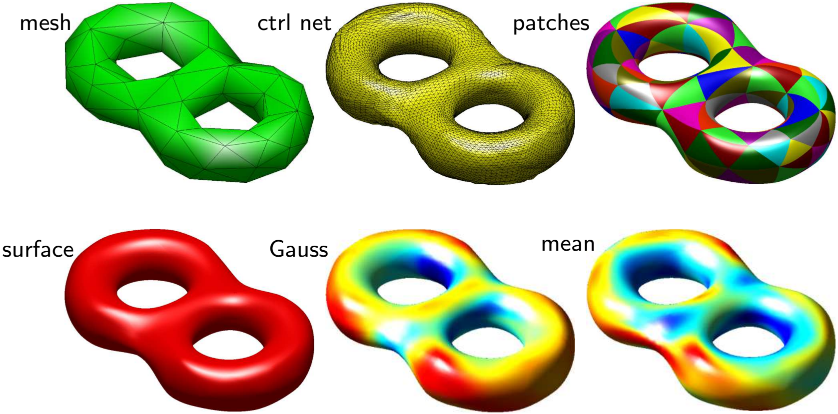


zebra

Example: $S_6^2(\Delta)$ interpolation & energy min.



Example: $S^4_{10}(\Delta)$ interpolation & energy min.



Thank you!