Hydrodynamic spectrum and dynamical phase transition in one-dimensional bulk-driven particle gases

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CIRM, Marseille, January 2016

Introduction

Statistical physics: bridging the gaps.

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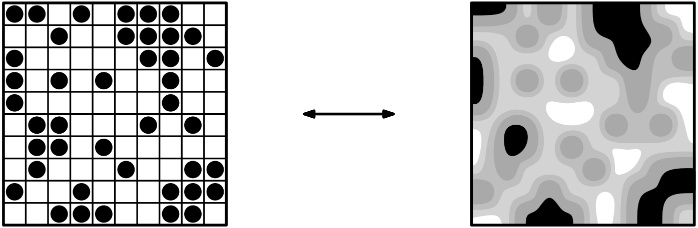

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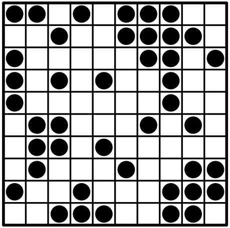
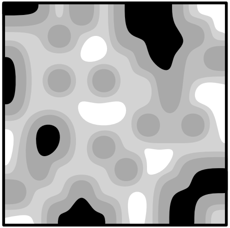




$$\mathrm{d}_t \mathrm{P}(C) =$$

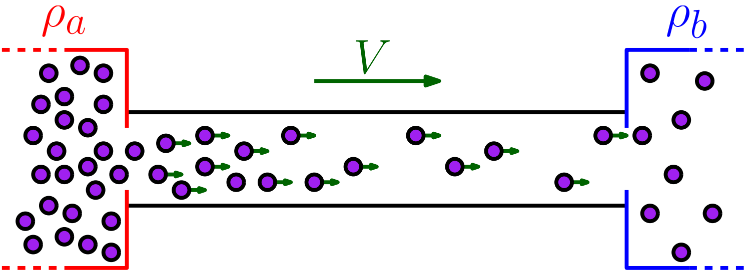
$$\sum_{C' \sim C} k(C, C')\mathrm{P}(C') - k(C', C)\mathrm{P}(C)$$

$$\mathrm{d}_t \rho(x) =$$

$$-\nabla[F\sigma(\rho(x)) - D\nabla\rho(x) + \xi]$$

Introduction

Simple situation: one-dimensional conduit, two reservoirs ρ_a and ρ_b, a driving field V in the bulk, and interactions between the particles.



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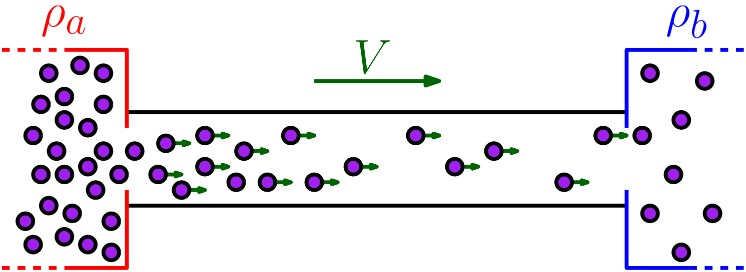


Field and/or reservoir imbalance $\Rightarrow$ macroscopic current of particles (related to microscopic production of entropy)

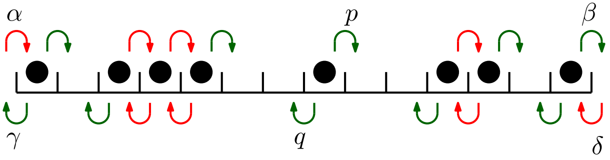
Layout

- Introduction

- I – Model and formalism: current fluctuations in the open ASEP
- II – Macroscopic Fluctuation Theory and hydrodynamic states
- III – Generic dynamical phase transition

- Conclusion

I - The open Asymmetric Simple Exclusion Process



The open Asymmetric Simple Exclusion Process (ASEP):

- one-dimensional lattice of size L
- entry at the left with rate α and at the right with rate δ
- exit at the right with rate β and at the left with rate γ
- jumps in the bulk with rate p to the right and $q < p$ to the left (if the target site is free)

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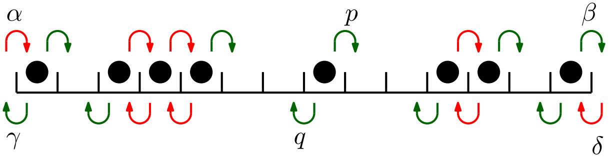

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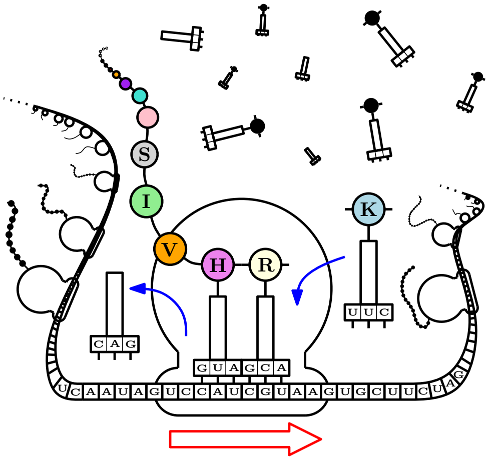
Totally Asymmetric case (TASEP): $q = \gamma = \delta = 0$

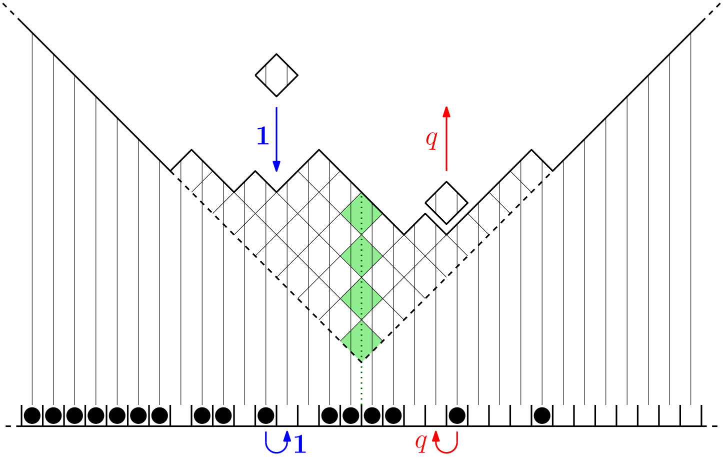
I - The open Asymmetric Simple Exclusion Process

First invented to describe biological transport.



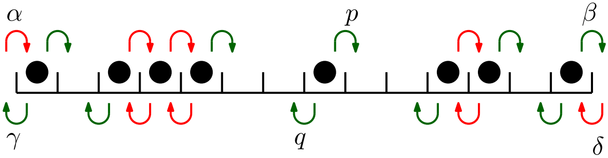
[C. T. MacDonald, J. H. Gibbs, A. C. Pipkin, **Biopolymers**, 1968]

Can be related to other statistical models, such as surface growth.

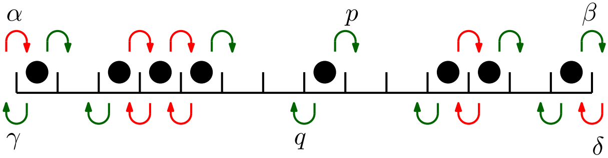


[M. Kardar, G. Parisi, Y.-C. Zhang, **P. R. L.**, 1986]

I - Motivation



Why the open ASEP ?



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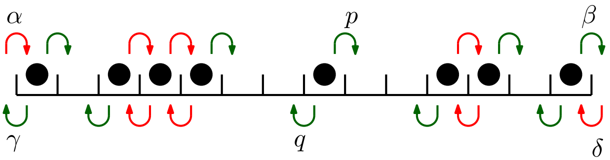

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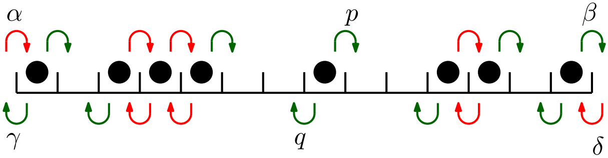

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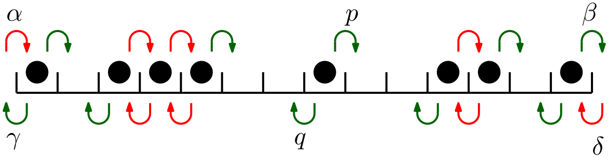

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What in the open ASEP ?

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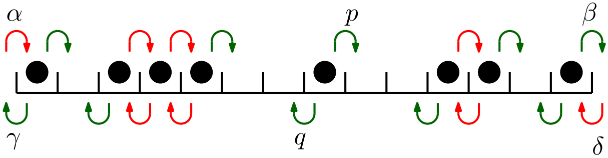

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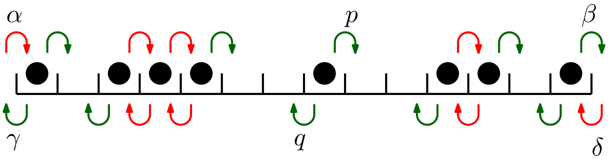
What in the open ASEP ? The macroscopic current of particles, related to the entropy production (irreversibility).

I - Master equation



The probability vector $|P_t\rangle$ which contains the probabilities of observing a configuration $\mathcal{C}$ at time t obeys the master equation

$$\frac{d}{dt}|P_t\rangle = M|P_t\rangle$$

with M being the sum of local matrices M_i (one for each bond $0 \leq i \leq L$) (in bases $\{0,1\}$ and $\{00,01,10,11\}$)

$$M_0 = \begin{bmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{bmatrix}, \quad M_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & p & 0 \\ 0 & q & -p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_L = \begin{bmatrix} -\delta & \beta \\ \delta & -\beta \end{bmatrix}$$

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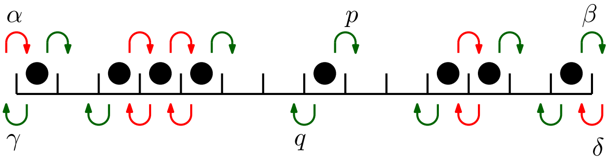

The probability vector $|P_t\rangle$ which contains the probabilities of observing a configuration $\mathcal{C}$ at time t obeys the master equation

$$|P_t\rangle \to |P^\star\rangle$$

with M being the sum of local matrices M_i (one for each bond $0 \leq i \leq L$) (in bases $\{0, 1\}$ and $\{00, 01, 10, 11\}$)

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Exact solution: [B. Derrida, M. R. Evans, V. Hakim, V. Pasquier, 1993]

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Mean field equation: $\boxed{J = (p-q)\rho(1-\rho) - \frac{p+q}{2L}\nabla\rho}$ with ρ_a, ρ_b.

For a given $J = (p-q)\rho_c(1-\rho_c)$, we plot all the possible profiles $\rho(x)$:

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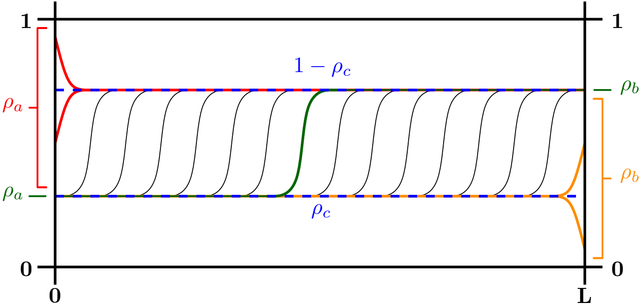
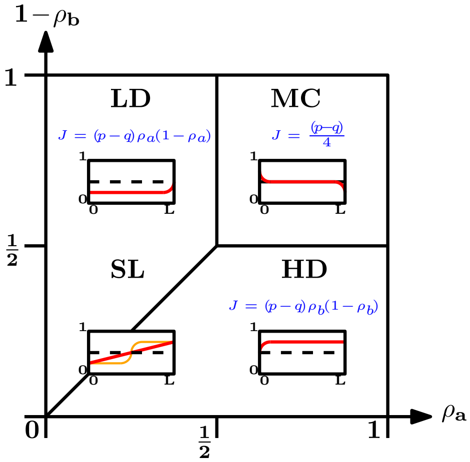


which gives us constraints on ρ_a and ρ_b.

I - Phase diagram of the steady state

Phase diagram of the system (with respect to $\rho_a(\alpha, \gamma, q)$ and $\rho_b(\beta, \delta, q)$):



In each case: $J = (p-q)\rho_c(1-\rho_c)$.

Macroscopic fluctuation theory
[Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; J. Stat. Phys, 2002]

For a diffusive particle gas, the large deviations $g(j,\rho)$ are locally Gaussian around the deterministic equation, i.e. :

$$j = F\sigma(\rho) - D\nabla\rho + \sqrt{\sigma(\rho)}\xi$$

But ASEP is not diffusive : $D \sim \frac{1}{L}$.

II - Macroscopic fluctuations

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- Weakly ASEP: $(p-q) = \frac{\nu}{L}$

We get, for the total current j (through all bonds):

$$g(j,\rho) = \int_0^1 \frac{\left[j - \nu\rho(1-\rho) + \frac{p+q}{2}\nabla\rho\right]^2}{2\rho(1-\rho)} dx$$

Minimising over $\rho \to g(j)$ and optimal profile.

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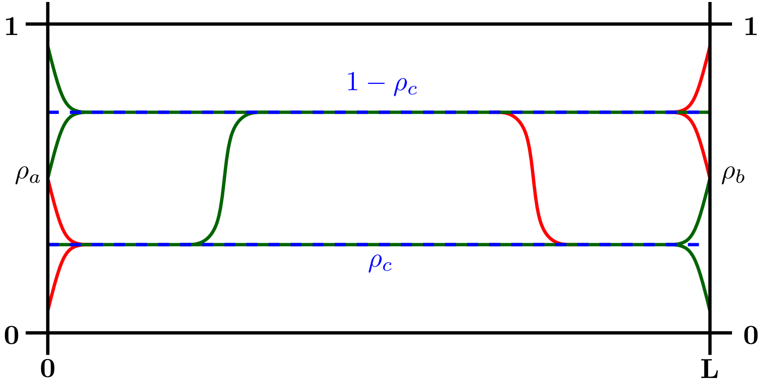
Minimising over $\rho \to g(j)$ and optimal profile.

- [T. Bodineau, B. Derrida, **J. Stat. Phys**, 2006] :

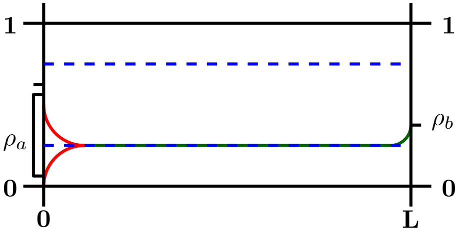
taking $\nu \sim L$ gives correct results for the TASEP.

II - Macroscopic fluctuations

variations of $\rho = \pm$ steady state
for $j = \rho_c(1-\rho_c)$ (TASEP)



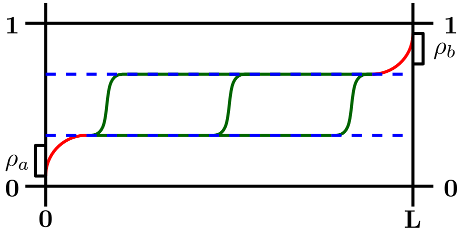
II - LD/HD phases



$$g(j) = j\log\left(\frac{1-\rho_a}{\rho_a}\frac{\rho_c}{1-\rho_c}\right) + \rho_a - \rho_c$$

with $j = \rho_c(1-\rho_c)$

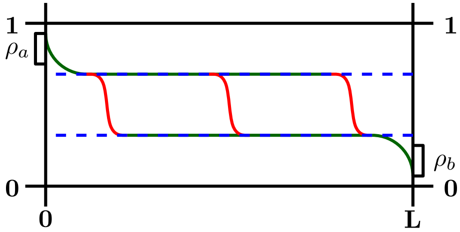
II - Shock phase



$$g(j) = j \log\left(\frac{1-\rho_a}{\rho_a} \frac{\rho_b}{1-\rho_b} \frac{\rho_c^2}{(1-\rho_c)^2} \right) + \rho_a - \rho_b + 1 - 2\rho_c$$

$$\text{with } j = \rho_c(1-\rho_c)$$

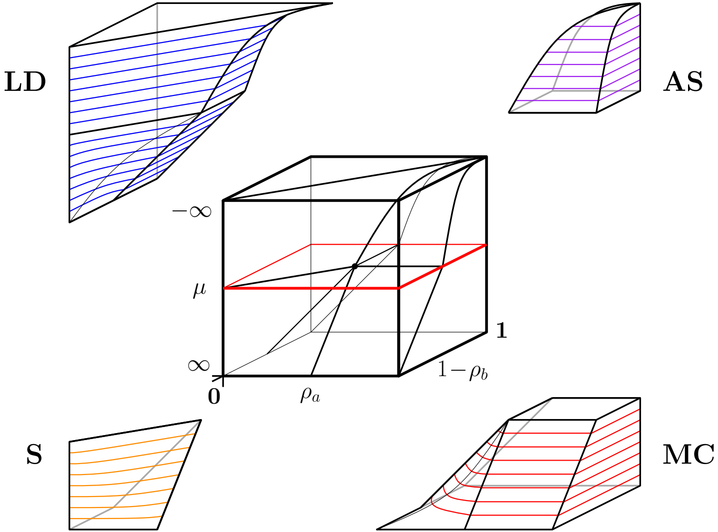
II - Anti-shock phase



$$g(j) = 2j \log\left(\frac{\rho_c}{1-\rho_c}\right) + 1 - 2\rho_c$$

with $j = \rho_c(1-\rho_c)$

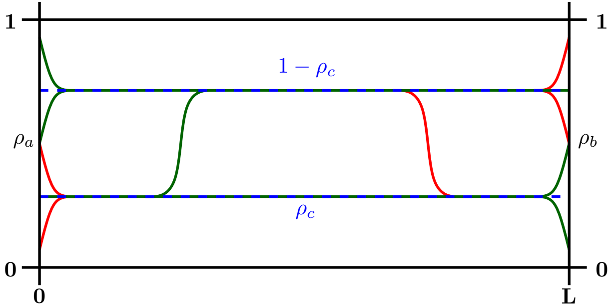
II - Phase diagram



II - Excited states from hydrodynamics

(Work in progress)

$$g(j,\rho) = \int_0^1 \frac{\left[j - \nu\rho(1-\rho) + \frac{1+q}{2}\nabla\rho\right]^2}{2\rho(1-\rho)}\,dx$$



We only looked at the most probable states. What about the others ?

II - Excited states from hydrodynamics

For instance, in the LD phase:

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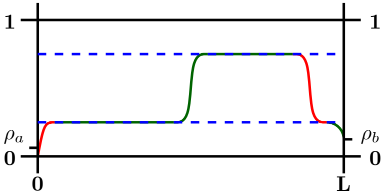

$$g(j) = j \log\left(\frac{1-\rho_a}{\rho_a}\left(\frac{\rho_c}{1-\rho_c}\right)^3\right) + 1 + \rho_a - 3\rho_c$$

for $\rho_c = \left[1 + \left(\frac{1-\rho_a}{\rho_a}\right)^{1/3}\right]^{-1}$

Exists if $\rho_b \leq 1 - \rho_c$.

II - Excited states from hydrodynamics

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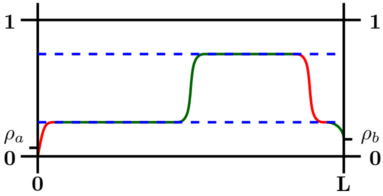

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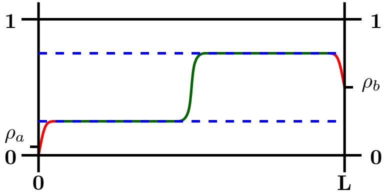

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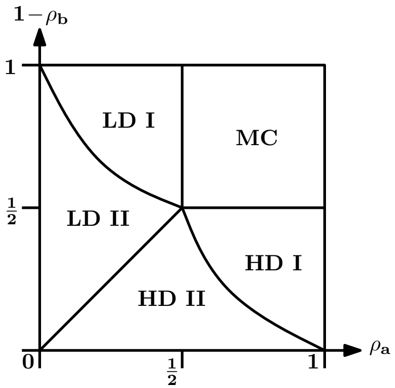
for $\rho_c = \left[1 + \left(\frac{1-\rho_a}{\rho_a} \frac{\rho_b}{1-\rho_b} \right)^{1/2} \right]^{-1}$

Exists only if $\rho_b \geq \rho_c$.

II - Excited states from hydrodynamics

$\rightarrow$ phase diagram for the gap:



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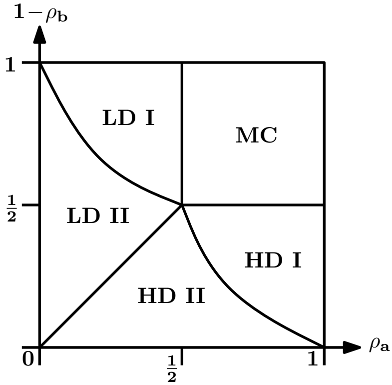
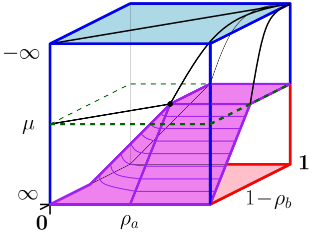


Values for the gap consistent with [F. Essler, J. de Gier, **J. Stat. Mech**, 2006] and [A. Proeme, R. Blythe, M. Evans, **J. Phys. A**, 2011] (numerics).

III - Dynamical phase transition



- Naively applying MFT for $j = \frac{1}{4} + \varepsilon$, $\varepsilon > 0$, gives:

$$g(j) \sim L\ \varepsilon^2.$$

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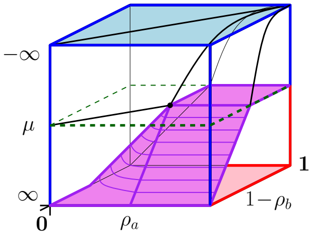

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$$g(j) \sim L\ \varepsilon^{5/2}\ \frac{32\sqrt{3}}{5\pi}.$$

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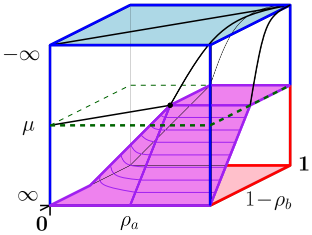

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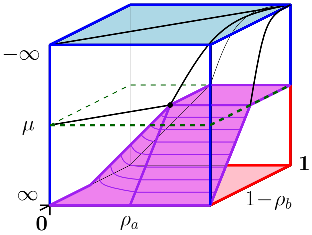

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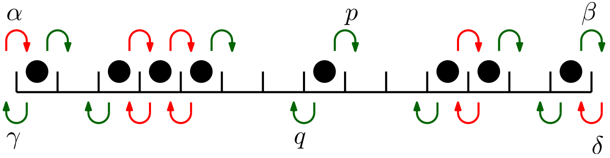
$\rightarrow$ the structure of fluctuating states for $j > \frac{1}{4}$ is not hydrodynamic.

Reason: the hydrodynamic current $j = F\sigma(\rho)$ is bounded from above.

(Soon to be submitted)



If we generalise $p \rightarrow p_i\, \mathrm{e}^{V(C')-V(C)}$, we can show:

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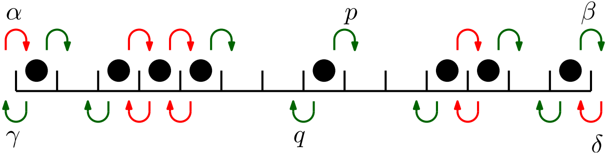

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- For $j \rightarrow +\infty$: Coulomb gas states, and $g(j) \sim L$.

III - Generic dynamical phase transition

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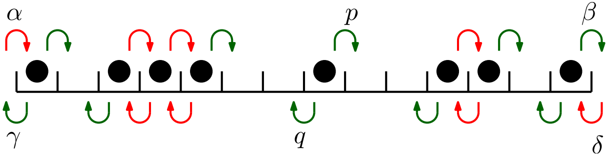

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- For $j \rightarrow -\infty$: anti-shock states, and $g(j)$ independent of L (at least if V has a finite range).

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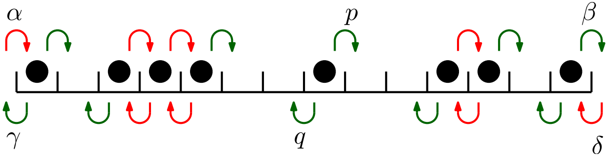

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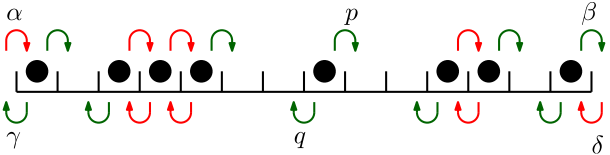
- For $j \to -\infty$: anti-shock states, and $g(j)$ independent of L (at least if V has a finite range).

$\to$ same transition between a hydrodynamic phase and a correlated phase.

III - Generic dynamical phase transition

(Soon to be submitted)



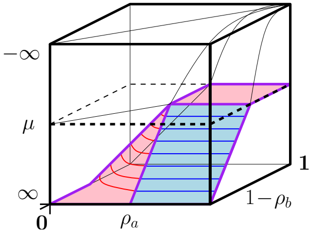
If we generalise $\quad p \quad \to \quad p_i \, \mathrm{e}^{V(C')-V(C)}, \qquad$ we can show:

- For $j \to +\infty$: Coulomb gas states, and $g(j) \sim L$.
- For $j \to -\infty$: anti-shock states, and $g(j)$ independent of L (at least if V has a finite range).

$\to$ same transition between a hydrodynamic phase and a correlated phase.

Consequence of the physical geometry of the model (sites vs. configurations).

III - Connection with KPZ



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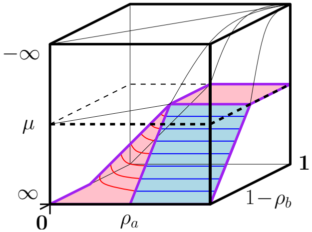

Red area: Tracy-Widom distributions for current fluctuations in the infinite volume setting.
Blue area: probably similar.

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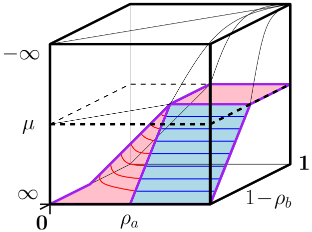

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$$\mathrm{d}_t\rho = -\nabla[(p-q)\rho(1-\rho) - D\nabla\rho + \xi] \to \mathrm{d}_t h = (p-q)(\nabla h)^2 - D\Delta h + \xi.$$

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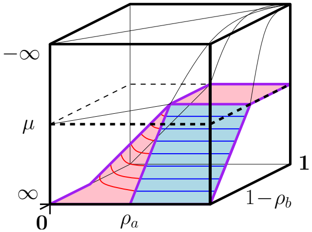

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Hairer and Quastel (Dec. 2015):

$$\mathrm{d}_t h = f(\nabla h) - D\Delta h + \xi \text{ is the same if } f \text{ is even.}$$

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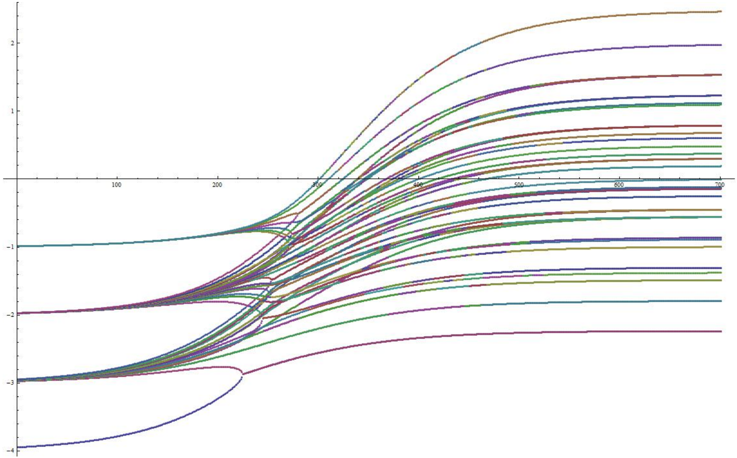
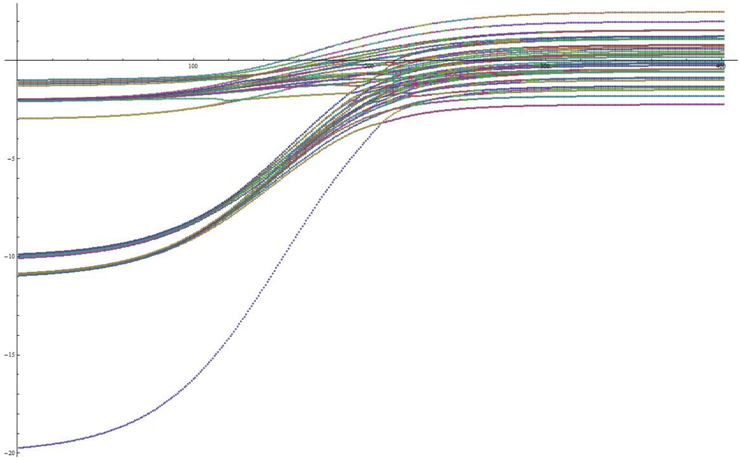
Thank you !

TASEP.



TASEP with Ising interaction.



TASEP with disordered potential.