k-abelian singletons and Gray codes for necklaces

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Motivation

- k-abelian equivalence classes.
- structure of classes.
- singletons.

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Another way to see singletons:

Singletons

k-abelian singletons: the words uniquely characterized by

- prefix of length $k-1$,
- number of occurrences of each x, $x \in \Sigma^k$, as a factor.

Examples

- a^n: singleton for any k.
- $a^n b^m$: singleton for $k \geq 2$.
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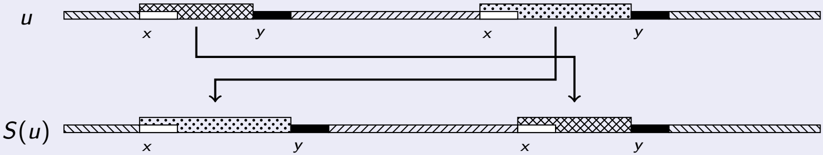
Questions

- What do singletons look like? (structurally)
- How many are there of length n? (for k, Σ fixed)

Another characterization of k-abelian equivalence

Definition 1 (k-switching)



- x and y are words of length $k-1$.

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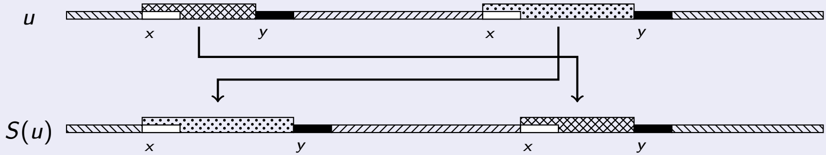

- x and y are words of length $k-1$.

Theorem 2

$$u \equiv_k v \iff u \; R_k^* \; v.$$

Characterization of singletons

- let $u \in A^*$.
- For any $x, y \in A^*$, a *return from* x to y in u:
  - a factor of u,
  - has prefix x,
  - has suffix y,
  - does not contain x or y as a proper factor.

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Theorem 3

u is a k-abelian singleton

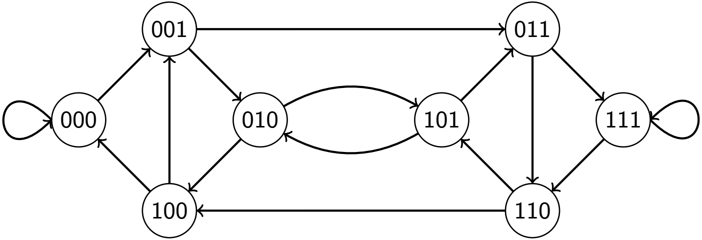
if and only if

at most one return from x to y in u for all $x, y \in A^{k-1}$.

- Implication: Factorization of singletons into runs
- Natural interpretation in de Bruijn graphs

Cycle decompositions and quotient graphs

- Natural interpretation in de Bruijn graphs:
  Cycle decompositions and quotient graphs



Cycle decompositions and quotient graphs

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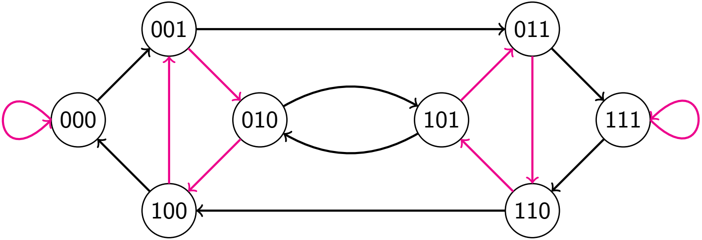

The number of singletons of length n

Theorem 4 (Mykkeltveit 1972)

The maximum size of a cycle decomposition of $dB(k)$ is at most $N(k)$.

Corollary 5

The number of singletons of length n is of order

$$\mathcal{O}\left(n^{N(k-1)-1}\right).$$

The number of singletons of length n

Theorem 6

The number of singletons of length n is of order $\Theta\left(n^{N(k-1)-1}\right)$

if and only if

Exists maximal cycle decomposition C of dB(k-1) such that

$$dB(k-1)/C \text{ contains hamiltonian path.}$$

Necklace graph and Gray codes for necklaces

A Gray code for necklaces:

- list of all necklaces of length k
- consecutive elements have representatives differing in exactly one bit

Observation

Gray codes for necklaces $\leftrightarrow$ Hamiltonian paths in *Necklace graph*

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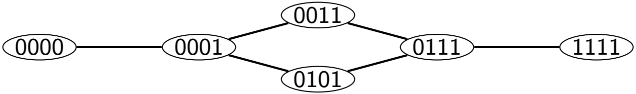
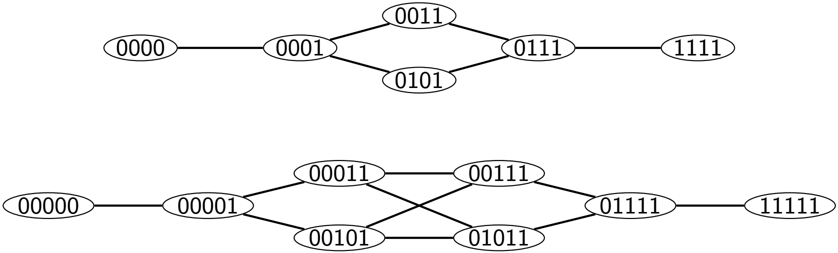
Open Problem 7 (C. Savage, 1997)

For binary alphabet, k odd: Does there exist a Gray code for necklaces of length k?

Necklace graphs



Necklace graphs



Hamiltonian paths in necklace graphs

- Previously known: Gray codes for necklaces of <u>odd</u> length up to 11.
- New bound: up to length 15.

Hamiltonian paths in necklace graphs

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- For even lengths Gray codes not possible.
- Instead: Other maximal cycle decompositions giving hamiltonian path.

Main conjectures

Conjecture 8

Always exists a maximal cycle decomposition C such that dB/C has hamiltonian path.

This is equivalent to

Conjecture 9

The number of singletons of length n is of order $\Theta\left(n^{N(k-1)-1}\right)$.

Thank you!

Merci!