Binary periodic trajectories and generalized Collatz functions

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« C'est à travers les mots, entre les mots, qu'on voit et qu'on entend »

Gilles Deleuze (1925-1995)
Critique et clinique (1993)

Let $\mathfrak{M}$ be the (self-generated) set of matrices M with coefficients in $\mathbb{Q}\langle a, b\rangle$ (non-commutative polynomials in the variables a and b) such either

(i) M is a scalar matrix (a matrix of dimensions 1×1),

(ii) $M = \left(\begin{array}{c|c} 0_{n\times n} & A_{n\times n} \\ \hline B_{n\times n} & 0_{n\times n} \end{array}\right)_{2n\times 2n}$ for some $A_{n\times n}, B_{n\times n} \in \mathfrak{M}$,

(iii) $M =$

$$\left(\begin{array}{c|c} 0_{4n\times 4n} & \left(\begin{array}{c|c} 0_{2n\times 2n} & A_{2n\times 2n} \\ \hline B_{2n\times 2n} & 0_{2n\times 2n} \end{array}\right) \\ \hline \left(\begin{array}{c|c} 0_{2n\times 2n} & \left(\begin{array}{c|c} C_{n\times n} & 0_{n\times n} \\ \hline 0_{n\times n} & D_{n\times n} \end{array}\right) \\ \left(\begin{array}{c|c} E_{n\times n} & 0_{n\times n} \\ \hline 0_{n\times n} & F_{n\times n} \end{array}\right) & 0_{2n\times 2n} \end{array}\right)_{4n\times 4n} & 0_{4n\times 4n} \end{array}\right)_{8n\times 8n},$$

for some $A_{2n\times 2n}, B_{2n\times 2n}, C_{n\times n}, D_{n\times n}, E_{n\times n}, F_{n\times n} \in \mathfrak{M}$,

where $0_{m\times m}$ is an $m \times m$ block of zeros.

Our transformation (the Dawkins' mème of this presentation)

We shall use the notation $\mathfrak{M}^+$ for the non-scalar matrices belonging to $\mathfrak{M}$. The *unfold* application $\Phi : \mathfrak{M}^+ \longrightarrow \mathfrak{M}^+$ transforms the matrix

$$\left(\begin{array}{c|c} 0_{n\times n} & A_{n\times n} \\ \hline B_{n\times n} & 0_{n\times n} \end{array}\right)_{2n\times 2n}$$

into the (block) matrix

$$\left(\begin{array}{c|c} 0_{2n\times 2n} & \left(\begin{array}{c|c} 0_{n\times n} & A_{n\times n} \\ \hline B_{n\times n} & 0_{n\times n} \end{array}\right) \\ \hline \left(\begin{array}{c|c} 0_{n\times n} & (A_{n\times n})^2 \\ \hline (B_{n\times n})^2 & 0_{n\times n} \end{array}\right) & 0_{2n\times 2n} \end{array}\right)_{4n\times 4n}$$

Let's see the orbit of

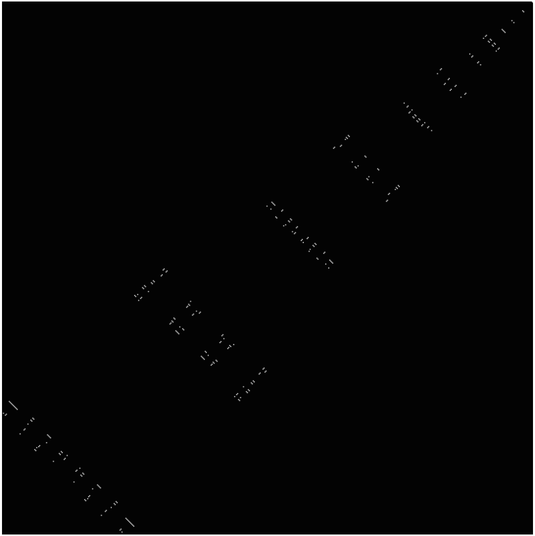
$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & a^2 & 0 & 0 \\ b^2 & 0 & 0 & 0 \end{pmatrix}$$

Orbit : Φ-iteration #2

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & b^2 & 0 & 0 & 0 \\ 0 & 0 & ab & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ba & 0 & 0 & 0 & 0 \\ a^2 b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b^2 a^2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The sequence of iterated “converges” to a configuration like



What kind of words appear in the entries of the matrices from the Φ-orbit of $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$?

Are they well-known ?

The words appearing in the Φ-iteration #2 are

$$a, b, a^2, b^2, ab, ba, a^2b^2, b^2a^2.$$

Let's make the substitution $a \mapsto 1$ and $b \mapsto 2$ and write the exponent 1,

$$1^1, 2^1, 1^2, 2^2, 1^12^1, 2^11^1, 1^22^2, 2^21^2.$$

Let's just write the exponents, using again the exponential notation,

$$1^1, 1^1, 2^1, 2^1, 1^2, 1^2, 2^2, 2^2.$$

After the inverse substitution $1 \mapsto a$ and $2 \mapsto b$ we obtain the words appearing in the Φ-iteration #1 (each one repeated 2 times),

$$a, b, a^2, b^2.$$

A finite word w over the alphabet $\{1,2\}$ is said to be *smooth* if either

(i) $w = 1$;

(ii) the concatenation of the exponents Δw of the letters from w written in exponential notation is a smooth word over the same alphabet. The word Δw is called the *run-length encoding* of w.

For a more general definition see:
Berthé, V., Brlek, S., & Choquette, P. (2005). *Smooth words over arbitrary alphabets*. Theoretical Computer Science, 341(1), 293-310.

A characterization of smooth words over $\{1,2\}$

A word w over the alphabet $\{1,2\}$ is smooth if and only if the word v obtained from w by the substitutions $1 \mapsto a$ and $2 \mapsto b$ appears in one of the entries of the matrices from the Φ-orbit of $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$.

For a graph theoretical characterizations of smooth words, using a very similar idea, see:
Chvátal, V. (1994). *Notes on the Kolakoski sequence.* Rapport, DIMACS Techn. Rep.

Motivated by the work of Marston Morse and Gustav A. Hedlund in symbolic dynamics, Rufus Oldenburger studied the relationship between a periodic trajectory over the alphabet $\{1, 2\}$ and its exponent trajectory :

Oldenburger, R. (1939). *Exponent trajectories in symbolic dynamics*. Transactions of the American Mathematical Society, 46(3), 453-466.

We shall apply our matrices to this setting, but first some preliminary definitions...

Any finite nonempty word $w \in \{1,2\}^+$ over the alphabet $\{1,2\}$ is said to be C^0-*periodic* by default.

A word $w \in \{1,2\}^+$ is said to be a C^{n+1}-*periodic* if the concatenation of the exponents

$$\Delta\left(w^{\pm\infty}\right) = \ldots e_{-2} e_{-1} e_0 e_1 e_2 \ldots$$

of the two-sided infinite periodic word

$$\begin{aligned} w^{\pm\infty} &= \ldots wwwwww \ldots \\ &= \ldots 2^{e_{-2}} 1^{e_{-1}} 2^{e_0} 1^{e_1} 2^{e_2} \ldots \end{aligned}$$

can be written as

$$\Delta\left(w^{\pm\infty}\right) = v^{\pm\infty} = \ldots vvvvvv \ldots$$

for some C^n-periodic word $v \in \{1,2\}^+$.

$w = 112112212212$ is C^4-periodic, because

$$T_4 = ...112112212212\color{orange}{112112212212}11211221221211211...$$
$$T_3 = ...2122121121221211\color{orange}{21221211}21221211212212112...$$
$$T_2 = ...112112112112112112\color{orange}{112}11211211211211211211...$$
$$T_1 = ...121212121212121212\color{orange}{12}121212121212121212121...$$
$$T_0 = ...111111111111111111\color{orange}{1}1111111111111111111111...$$

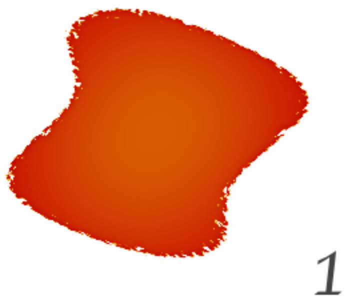
For $n \geq 0$, we define

$$\ell(n) := \min\left\{|w| : \quad w \in \{1,2\}^{+}, \quad w \text{ is } C^{n}-\text{periodic}\right\}.$$

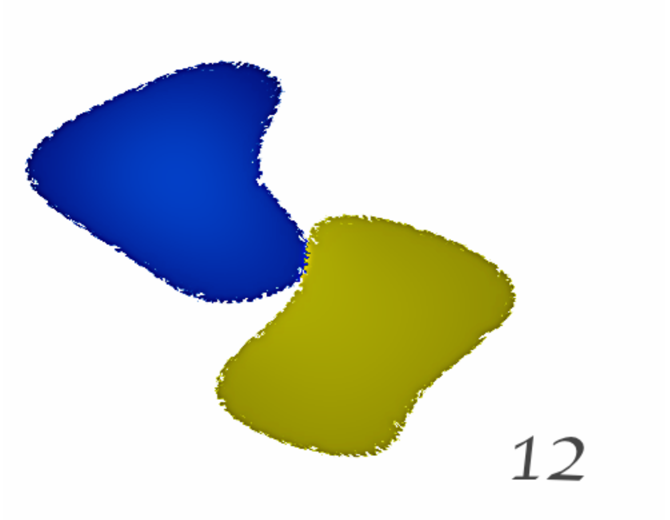
The girth problem.
Find an explicit formula for $\ell(n)$.

$\ell(0) = 1$

...111111111111111111111111111111111111111111111111111...



...1212121212121212121212121212121212121212121212...

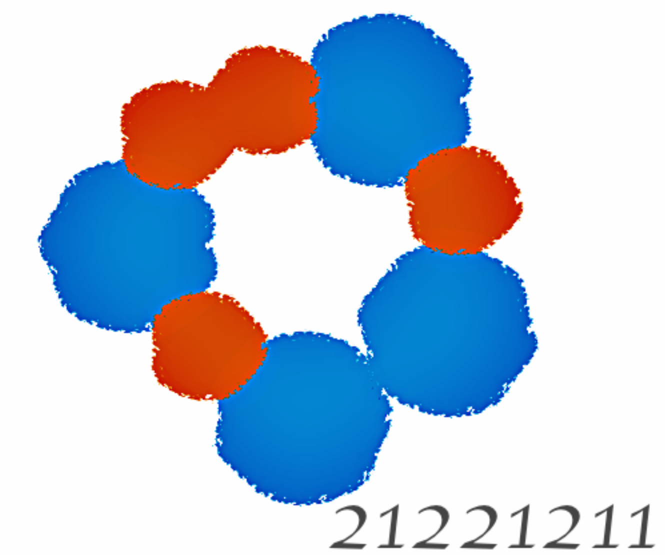


$\ell(2) = 3$

...112112112112112112112112112112112112112112112112...

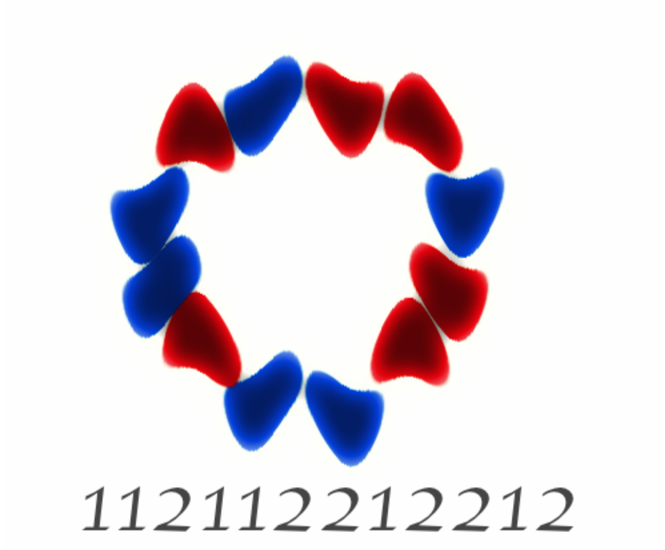
$\ell(3) = 8$

...2122121121221211212212112122121121221211....



$\ell(4) = 12$

...112112212212112112212212112112212212112112212212...



We proved that $\ell(n)$ can be computed as the multiplicity of the root $z = 0$ in the polynomial

$$\det\left(\mathrm{Id}_{2^n\times 2^n} - A_{2^n\times 2^n} - B_{2^n\times 2^n}\right)\Big|_{\substack{a=z\\ b=z}} - 1,$$

where

$$\Phi^n\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}_{2\times 2} = \left(\begin{array}{c|c} 0_{2^n\times 2^n} & A_{2^n\times 2^n} \\ \hline B_{2^n\times 2^n} & 0_{2^n\times 2^n} \end{array}\right)_{2^{n+1}\times 2^{n+1}}$$

The values of $\ell(n)$ for $0 \leq n \leq 9$ are

$$1, 2, 3, 8, 12, 18, 27, 80, 120, 180, \ldots$$

We can obtain the numbers

$$1, 2, 3, 8, 12, 18, 27, 80, 120, 180, \dots$$

as the orbits of 1 by the generalized Collatz function $g : \mathbb{Z} \longrightarrow \mathbb{Z}$ given by

$$g(2n) = 3n, \qquad g(2n-1) = 6n-4.$$

The generalized Collatz functions were defined in : Conway, J. *Unpredictable iterations.* (1972): 49-52.

Too good to be true

$$\ell(10) = 269 \quad \textit{vs} \quad g(10) = 270,$$

$$\ell(11) = 753 \quad \textit{vs} \quad g(11) = 405.$$

The smallest C^{10}-periodic word

211212212112212212112112212112122112112212212112112212112122
122112112212112122112112122122112122121121122122121122122112
122121121122121121221121121221211221221121121221221121221211
221221121121221121122121121122122121121221121122122121122122
112122121121122122121122122111.

The symbols 1 and 0 appear 135 times and 134 respectively.

Let f be a generalized Collatz function, i.e. a function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ of the form

$$f(n) = a_j n + b_j \quad \text{if } n \equiv j \pmod{d},$$

for some $a_j, b_j \in \mathbb{Q}$ for all $0 \leq j < d$.

We conjecture that for all such f, the inequality $\ell(n) \neq f^n(1)$ holds for infinitely many integers $n \geq 0$.

Quelques références

1. Bergeron-Brlek, A., Brlek, S., Lacasse, A., & Provençal, X. (2003, August). Patterns in smooth tilings. In Proc. WORDS (Vol. 3, pp. 10-13).
2. Berstel, J., & Reutenauer, C. (1988). Rational series and their languages.
3. Brlek, S., & Ladouceur, A. (2003). A note on differentiable palindromes. Theoretical Computer Science, 302(1), 167-178.
4. Brlek, S., Dulucq, S., Ladouceur, A., & Vuillon, L. (2006). Combinatorial properties of smooth infinite words. Theoretical Computer Science, 352(1), 306-317.
5. Berthé, V., Brlek, S., & Choquette, P. (2005). Smooth words over arbitrary alphabets. Theoretical Computer Science, 341(1), 293-310.
6. Brlek, S., Jamet, D., & Paquin, G. (2008). Smooth words on 2-letter alphabets having same parity. Theoretical Computer Science, 393(1), 166-181.
7. Brlek, S., & Reutenauer, C. (2011). Complexity and palindromic defect of infinite words. Theoretical Computer Science, 412(4), 493-497.
8. Brlek, S., Hamel, S., Nivat, M., & Reutenauer, C. (2004). On the palindromic complexity of infinite words. International Journal of Foundations of Computer Science, 15(02), 293-306.
9. Paquin, G., Brlek, S., & Jamet, D. (2006). Extremal generalized smooth words. Actes des Journées Montoises d'Informatique Théorique, 13.
10. Fédou, J. M., & Fici, G. (2012). Automata and differentiable words. Theoretical Computer Science, 443, 46-62.
11. Oldenburger, R. (1939). Exponent trajectories in symbolic dynamics. Transactions of the American Mathematical Society, 46(3), 453-466.

According to Martin Heidegger, to think (*Denke*) is to say

danke

thank you

hvala

grazie

...

MERCI

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