On k-abelian palindromes

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Framework

Palindromic rich (many palindromes)
and poor (few palindromes) words

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palindrome
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k-abelian equivalence
between abelian equivalence and equality

Palindromes

Palindromes with respect to

- $=$: palindrome: $w = w^R$:

$$ababbaba$$

- $\sim_{ab}$: abelian palindrome: any word is abelian palindrome
- $\equiv_k$: k-abelian palindrome: $w \equiv_k w^R$:

$$abaabaa$$

is a 2-abelian palindrome

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How many palindromes a word can contain?

- rich words: maximal number of palindromes as factors
- poor words: minimal number of palindromes as factors (infinite words with finitely many palindromes)

Rich and poor words (equality)

Rich words

- The maximal number of palindromes a word of length n can contain is $n + 1$.
- There exist infinite rich words (all their factors are rich).

Rich and poor words (equality)

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Poor words

- There exists a reversal closed infinite word containing only finitely many palindromes [Berstel, Boasson, Carton, 2009]
- The smallest number of palindromes such a word can contain [Fici, Zamboni, 2013]

Rich and poor words of different types

	$=$	$\equiv_k$	$\sim_{ab}$	
poor	C	C	∞	for infinite words
rich	$n+1$	$\Theta(n^2)$	$\Theta(n^2)$	for finite words

Rich and poor words of different types

	$=$	$\equiv_k$	$\sim_{ab}$	
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	BBC	our result	trivial	
rich	$n+1$	$\Theta(n^2)$	$\Theta(n^2)$	for finite words
	folklore	our result	trivial	

- for poor words k-abelian equality behaves like equality
- for rich words k-abelian equality behaves like abelian equality

Main result for k-abelian poor words

l: the cardinality of the alphabet.

Characterization of the existence of (k, l)-poor words:

$k \backslash l$	2	3	4	5
1	$-$	$-$	$-$	$-$
2	$-$	$\ominus$	$\oplus$	$+$
3	$-$	$\oplus$	$+$	$+$
4	$\ominus$	$+$	$+$	
5	$\oplus$	$+$		

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5	$\oplus$	$+$		

Full classification for three conditions of reversal-closed (reversal-closed, k-abelian reversal closed or no conditions):

- $+$, $\oplus$: existence with a set of factors closed under reversal
- $\ominus$: existence, but not with a set of factors closed under k-abelian reversal
- $-$: nonexistence

The crucial things to prove are marked $\oplus$ and $\ominus$.

- we modify the construction of sesquipowers:

$$U_0 = w_0,$$
$$U_{i+1} = U_i w_i U_i^R,$$

where w_i is a suitable marker

- E.g., for $(2,4)$-poor word:

$$U_0 = abca\ abda\ acda,$$
$$U_n = U_{n-1}(abca)^{2^{2^n}}(abda)^{2^{2^n}}(acda)^{2^{2^n}}U_{n-1}^R.$$

The required word is obtained as the limit $u = \lim_{n\to\infty} U_n$:

$$abca\ abda\ acda(abca)^4(abda)^4(acda)^4 adca\ adba\ acba(abca)^{16}\cdots$$

- crucial thing: a dominating power in each factor

Construction II for $\oplus$ cases: via a fractal self-avoiding curve

Alphabet $\Delta = \{A, B, C, D\}$.

$w = w_1 w_2 \cdots \in \Delta^\omega \to$ a polygonal line in the lattice $\mathbb{Z}^2$.

Drawing instructions for letters

$$\begin{array}{ccc} A & : & \rightarrow \\ B & : & \uparrow \\ C & : & \leftarrow \\ D & : & \downarrow \end{array}$$

(a line segment of length 1 in the direction of the arrow)

Construction II for $\oplus$ cases

The morphism $\rho : \begin{cases} A \to B \\ B \to C \\ C \to D \\ D \to A \end{cases}$ rotates a curve by 90 degrees.

Lemma

There exists a uniformly recurrent word $w \in \Delta^\omega$ with the following properties:
(i) $F(w)$ is closed under the map $u \mapsto \rho^2(u^R)$;
(ii) consecutive letters of w correspond to orthogonal segments;
(iii) the curve associated to w is self-avoiding.

(i) gives closed under reversal
(iii) long palindrome $\Leftrightarrow$ self-intersection

Construction II for $\oplus$ cases

We construct such word

$$w = ABABCBABABCBCDCBCBABABCBABABCBCDCBCB\cdots$$

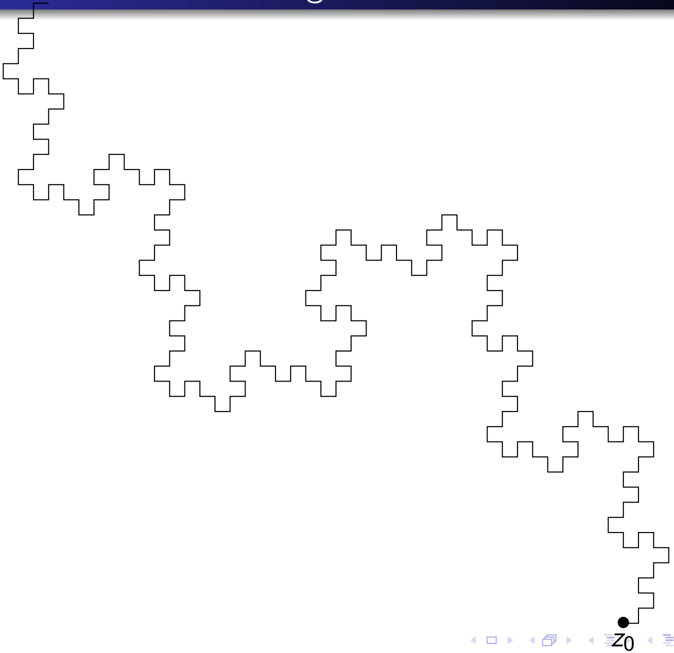
as the fixed point starting with A of φ:

$$\varphi: \begin{array}{l} A \mapsto ABA \\ B \mapsto BCB \\ C \mapsto CDC \\ D \mapsto DAD \end{array}.$$

Alternatively, $w = \lim_{n\to\infty} u_n$:

$$\begin{array}{r} u_0 = A \\ u_{n+1} = u_n\rho(u_n)u_n \end{array}.$$

Construction II: self-avoiding fractal curve



Construction II for $\oplus$ cases

$$\tau_{2,4}: \begin{cases} A \to ab \\ B \to cd \\ C \to ba \\ D \to dc \end{cases} \quad \tau_{3,3}: \begin{cases} A \to abcca \\ B \to abbca \\ C \to accba \\ D \to acbba \end{cases} \quad \tau_{5,2}: \begin{cases} A \to aabbaabaa \\ B \to aabbabaaa \\ C \to aabaabbaa \\ D \to aaababbaa \end{cases}$$

$\tau_{k,l}(w)$

- k-abelian poor over l letter alphabet
- uniformly recurrent

Proving nonexistence

Idea to prove the $\ominus$ cases (example for $k = 2$, $|\Sigma| = 3$):

- Rewriting rules which do not affect the 2-palindromicity:
  - for $x \in \Sigma$, substitute $xx \to x$
  - for $x, y \in \Sigma$, substitute $xyx \to x$
- reduce a word to a reduced form when no rewriting rule can be applied
- the reduced form of any ternary word v is a factor of $(abc)^\infty$ or $(cba)^\infty$
- a ternary word v is a 2-palindrome if and only if its reduced form is a letter
- using the reduced form, we find long palindromes

Rewriting rules for the case $(4,2)$

For $x, y \in \Sigma$, substitute

- $xxx \to xx$ (if this occurrence of xxx is not a prefix or suffix in v)
- $xxyyxx \to xxyxx$
- $xyxyx \to xyx$
- $yxyyxy \to yxy$
- $xxyxxy \to xxy$, $yxxyxx \to yxx$

k-abelian rich words

words of length n containing $\Theta(n^2)$ inequivalent k-abelian palindromes (i.e., proportion of the total number of factors)

Theorem

Let k be a natural number. There exists a positive constant C such that for each $n \geq k$ there exists a word of length n containing at least Cn^2 k-abelian palindromes.

Actually, we can choose $C = 1/4k$.

- Consider words of the form:

$$v = a^l(ba^{k-1})^m$$

- Let $|w| = n$ and $n \geq k$. We can count all k-abelian palindromes and maximaze their number by the choice of l and m:

$$\sharp \text{ of k-abelian palindromes in } w \geq 1/4kn^2$$

- These words are k-abelian rich: $\Theta(n^2)$ k-abelian palindromes and $\Theta(n^2)$ factors in total.

Future work: Problem 1

Optimal numbers of palindromes

What is the exact maximal number of inequivalent k-abelian palindromes a word of length n can contain?

Future work: Problem 2

- there are no infinite words such that each factor contains maximum number of k-palindromes
  - abelian case: finite rich $a^n b^n$

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Infinite k-abelian rich word

Does there exist an infinite word w such that for some constant C each of its factors of length n contains at least Cn^2 inequivalent k-abelian palindromes (infinite k-abelian rich word)?

- there are no binary infinite abelian rich words [P., Avgustinovich]