k-ABELIAN EQUIVALENCE – AN EQUIVALENCE RELATION IN BETWEEN THE EQUALITY AND THE ABELIAN EQUIVALENCE

Motto: "When does the k-abelian equivalence resemble the equality and when the abelian equivalence"

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Presentations:

- M. Rosenfeld, M. Rao: k-abelian avoidability
- S. Puzynina: k-abelian palindromicity
- A. Saarela: k-abelian complexity and fluctuation
- M. Whiteland: k-abelian singletons and Gray codes

Basic properties

Definition 1

u and $v \in A^*$ are *k-abelian* equivalent if

- $|u|_x = |v|_x$ for all $x \in A^{\leq k}$.

Denoted by $u \equiv_k v$.

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- k-abelian equivalence is a congruence on A^*.
- $k = 1$ is just abelian equivalence
- $k = \infty$ is the equality of words.
- A hierarchy of approximations of the equality

$$= \quad \subseteq \quad \dots \quad \subseteq \quad \equiv_k \quad \subseteq \quad \equiv_{k-1} \quad \subseteq \quad \dots \quad \subseteq \quad \equiv_0$$

  where $\equiv_0$ denotes the equal length relation.
- $u = v \iff \forall k : u \equiv_k v$

Background

Parikh Theorem

- not true for the equality.
- true for the abelian equivalence.
- not true for k-abelian equivalence.

Background

PCP and D0L

PCP

$$\exists w, w' \colon h(w) \equiv_k g(w) \text{ or } h(w) = g(w') \text{ and } w \equiv_k w'?$$

D0L

$$h^n(w) \overset{?}{\equiv}_k g^n(w) \quad \forall n$$

also for morphic images!

Equivalent definitions

Lemma 2 (K., Saarela, & Zamboni 13)

$u \equiv_k v$ *if and only if*

- $|u|_x = |v|_x$ *for all* $x \in \Sigma^k$,
- $\mathrm{pref}_{k-1}(u) = \mathrm{pref}_{k-1}(v)$, *and*
- $\mathrm{suff}_{k-1}(u) = \mathrm{suff}_{k-1}(v)$.

Equivalent definitions

Let $a \in \Sigma$. Let Z_k be the set of non-empty words:

- of length at most k;
- do not begin or end with a:

$$Z_k = \{u \in \Sigma^* \mid 1 \leq |u| \leq k, \mathrm{pref}_1(u) \neq a \neq \mathrm{suff}_1(u)\}.$$

A refinement of the previous

The k-abelian equivalence class of u is determined by

- $|u|$,
- $\mathrm{pref}_{k-1}(u)$ and $\mathrm{suff}_{k-1}(v)$, and
- $|u|_z$ for $z \in Z_k$.

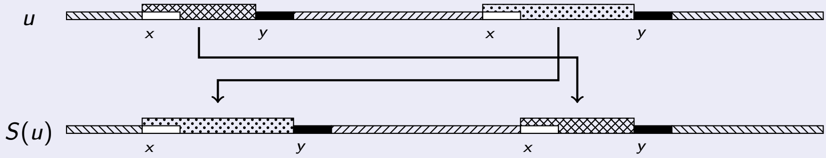
Equivalent definitions

k-switchings

A characterization by rewriting.

k-switchings

Definition 3 (k-switching)



- x and y are words of length $k-1$.

Theorem 4

$u \equiv_k v$ iff $uR_k^* v$.

Applications

Singletons $\Rightarrow$ regular expression (for complement)
Minimal $\Rightarrow$ regular expression (for complement)

Example 5

Expression for minimals:

$$\left(\bigcup_{\substack{x,y \in A^{k-1} \\ a,b \in A \\ b<a}} A^*(((xbA^* \cap A^*y)A^* \cap A^*x)aA^* \cap A^*y)A^* \right)^c$$

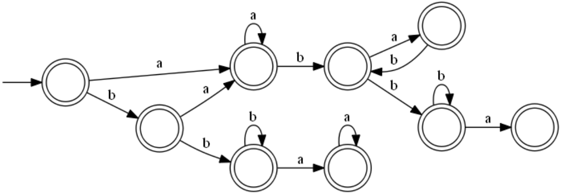
Corollary 6

$(a_n)_{n\geq 0}$ is rational (for any A), where

$$a_n = \#\{w \mid w \textit{ is minimal } \& \ |w| = n\} = \mathcal{P}_k(n).$$

Examples of automata

DFA accepting minimal representatives; $k = 2$, $\Sigma = \{a, b\}$



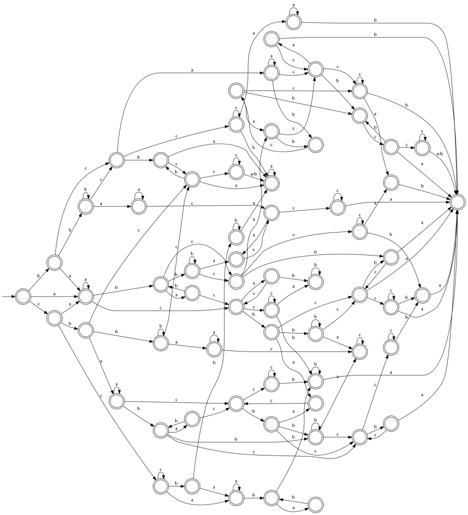
- The sink state is not illustrated.
- All other states are final, since we are avoiding patterns.

Example 7

- *abbba* is accepted; it is a minimal representative.
- *abbbab* is not accepted; minimal representative is *ababbb*.

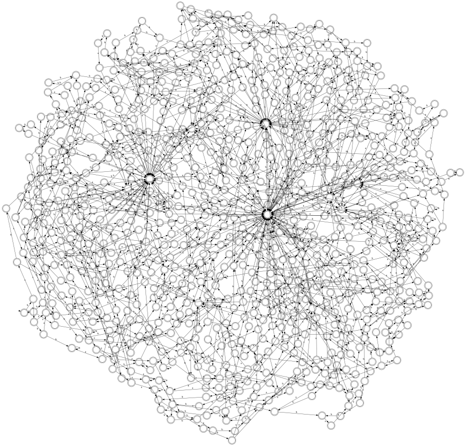
Examples of automata

DFA accepting minimal representatives; $k = 2$, $\Sigma = \{a, b, c\}$



Examples of automata

DFA accepting minimal representatives; $k = 4$, $\Sigma = \{a, b\}$

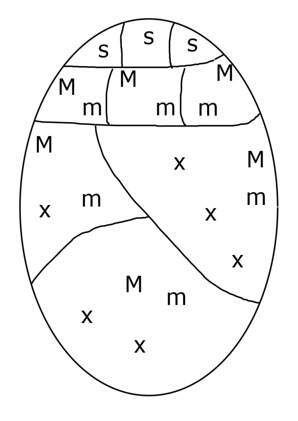


Regular languages closed under R_k

Theorem 8

L is a regular language $\Rightarrow R_k(L)$ is regular.

Application: Words admitting classes of size 2



Open problems

- When is $\mathcal{P}_k(n) \sim C \cdot n^{m^{k-1}(m-1)}$?
- Exact complexity for small values of k?
- Same for singleton classes?
- Regularity of classes of size l (classes of bounded size)?