An Application of Word Equations to Group Theory

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1.Introduction

1.1 3 problems on the plane

Suppose we are given two plane figures (domains) D and M on the integral grid of the plane, defined by the words $d = uur\bar{u}\bar{u}\bar{r}$ and $m = uuuurrrr\bar{u}\bar{u}\bar{r}\bar{r}\bar{u}\bar{u}\bar{r}\bar{r}$, respectively, while u means "one step up", r means "one step to the right" and $\bar{u}$ and $\bar{r}$ are the inverses of u and r, respectively.

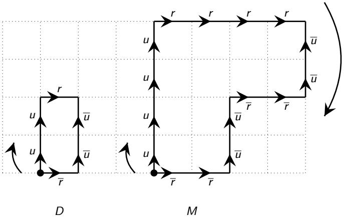


Fig. 1

There are at least three natural questions one can ask about the interrelation between D and M:

I Can we fill in M with copies of D?

II If YES, then in how many ways?

III If YES, then how many copies of D are needed?

1.Introduction

1.2-I generalization of the problems

This scenario occurs in a more general setting in group theory: we remain on the plane $\mathbb{E}^2$, but forget about the grid and about the shapes of D and M, as well forget about the geometrical interpretation of the letters u and r, so that we are left with regions D and M on the plane, the interiors of which are homeomorphic to open discs and their boundaries are labelled by arbitrary words over an alphabet.

1 Introduction

1.2-II generalization of the problems

Example 1

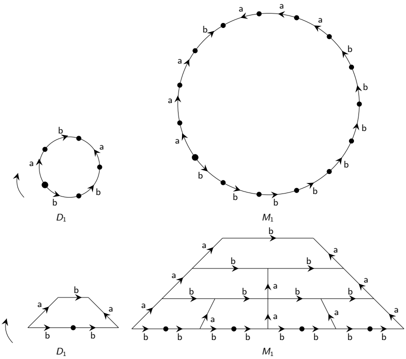


Fig. 2

Now we can repeat questions I,II and III in this new context, for D_1 and M_1. These questions represent three fundamental problems in Group Theory.

1.Introduction

1.3 Group Theory

Thus, let X be a set, $F(X)$ the free group, freely generated by X and let R be a cyclically reduced word on X. This data uniquely defines a group G presented by

$$\langle X|R\rangle \tag{0}$$

as the quotient of $F(X)$ by the normal closure N of R in $F(X)$. It is well known and easy to see that the elements of N are precisely the words of $F(X)$ which represent 1 in G and each such word W can be written by

$$W = C_1 \cdots C_k, \quad C_i = f_i^{-1} R^{\epsilon_i} f_i, \quad f_i \in F(X), \epsilon_i \in \{1,-1\}, 1 \leq i \leq k \tag{1}$$

1.Introduction

1.4-I Cancellation Diagrams (van Kampen Diagrams)

Equality (1) can be interpreted in the plane graphically as follows:

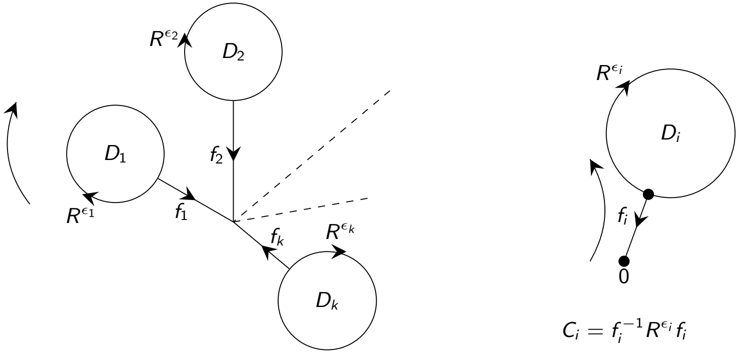


Fig. 3

1 Introduction

1.4-II Cancellation Diagrams (van Kampen Diagrams)

While W is a cyclically reduced word in $F(X)$, the product on the right-hand side of (1) need not be.

Example 2 Let $X = \{a, b, c\}$, let $R = cba\bar{c}\bar{b}\bar{a}$ and let $W = cba\bar{c}a\bar{c}\bar{b}\bar{a}\bar{b}\bar{a}cb$

We have in F

$$W = \underset{C_1}{(R)} \cdot \underset{C_2}{\left(aba\bar{c}\bar{b}\bar{a} \cdot R \cdot abc\bar{a}\bar{b}\bar{a}\right)} \cdot \underset{C_3}{\left(a\bar{c}\bar{b}\bar{a}Rabc\bar{a}\right)} \tag{2}$$

$f_1 = 1$, $f_2 = aba\bar{c}\bar{b}\bar{a}$ and $f_3 = abc\bar{a}$.

Now $C_2 = aba\bar{c}\bar{b}\bar{a} \cdot c\underbrace{ba\bar{c}\bar{b}\bar{a} \cdot abc\bar{a}\bar{b}}_{\text{cancellation}}\bar{a} = aba\bar{c}\bar{b}\bar{a}c\bar{a}$

and $C_3 = a\bar{c}\bar{b}\bar{a} \cdot cb\underbrace{a\bar{c}\bar{b}\bar{a} \cdot abc\bar{a}}_{\text{cancellation}} = a\bar{c}\bar{b}\bar{a}cb$

Hence $W = cba\bar{c}\underbrace{\bar{b}\bar{a} \cdot ab}_{\text{cancellation}}a\bar{c}\bar{b}\bar{a}\underbrace{c\bar{a} \cdot a\bar{c}}_{\text{cancellation}}\bar{b}\bar{a}cb = (cba\bar{c})(a\bar{c}\bar{b}\bar{a})\bar{b}\bar{a}cb$

The graphical interpretation of (2) depicted below.

1 Introduction

1.4-III Cancellation Diagrams (van Kampen Diagrams)

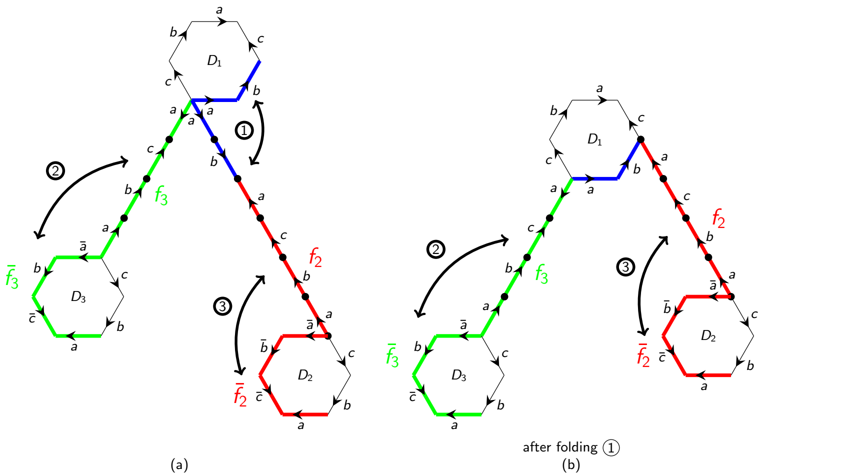
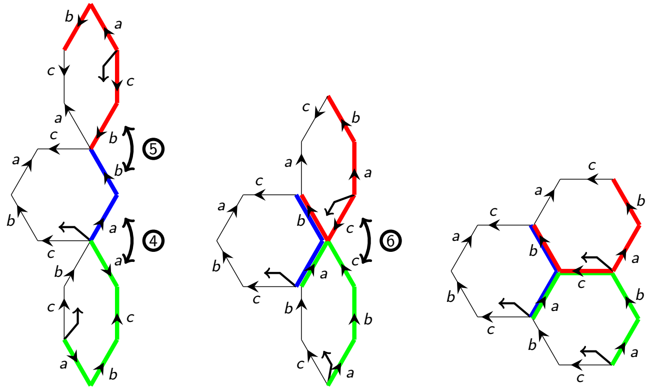


Fig. 4

1 Introduction

1.4-IV Cancellation Diagrams (van Kampen Diagrams)



after folding ② and ③
(c)

after folding ④ and ⑤
(d)

after folding ⑥
(e)

Fig. 4

1 Introduction

1.4-V Cancellation Diagrams (van Kampen Diagrams)

Cancellation of words is interpreted graphically as folding adjacent edges with the same label having a common endpoint.

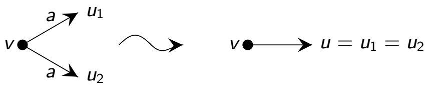


Fig. 5

Carrying out all possible foldings leads to a 2-complex on the plane which spells out the reduced word W on its boundary. See Fig. 4(e). Such 2 complexes are called *van Kampen diagrams* and they constitute one of the main tools in the branch of combinatorial group theory, called *small cancellation theory*.

The three problems above are known in this context in group theory under the following names:

1.Introduction

1.5 Three problems in Group Theory

(I) The word Problem (Decide whether a word W in F represents 1_G)

(II) The Identity Problem (How many essentially different minimal expressions like (1) are there for W?)

(III) The Isoperimetric Problem Find k in terms of the length of the boundary of M for a shortest expression like (1), for W.

The first problem was solved by W. Magnus in 1932 and the second problem was solved by Roger Lyndon in 1961. However, the third problem is still widely open. The present work is devoted to this problem in the particular case when R is a positive word. To formulate the result, we recall a few standard notions concerning isoperimetric functions.

1.Introduction

1.6 Isoperimetric Functions

An *isoperimetric function* of a presentation $\langle X|R\rangle$ is a monotone non decreasing function $f:\mathbb{N}\to[0,\infty)$ such that if $W=1$ in G, W reduced, then k in (1) satisfies $k\le f(|W|)$, where $|W|$ denotes the word length of W. f is a *Dehn-function for* $\langle X|R\rangle$ if $f(n)\le g(n)$, for every other isoperimetric function g for $\langle X|R\rangle$ and for every $n\ge 0$. Since practically it is impossible to calculate Dehn-functions precisely, they are computed up to an equivalence. Thus if given two isoperimetric functions f and g, we say that f *is dominated by* g if there exists a constant $C\ge 1$ such that $f(n)\le Cg(Cn+C)+Cn+C$. Say that f is *equivalent to* g (write $f\approx g$), if f is dominated by g and g is dominated by f. For example, $1\approx n$ and for every polynomial f of degree m, $f(n)\approx n^m$. Isomorphic groups have equivalent Dehn-functions.

1 Introduction

1.7-I Examples

Examples 3

1. Consider the presentation $\mathcal{P} = \langle a, b | a^{-1}ba = b^2 \rangle$. A van Kampen diagram for it is given in Fig. 2 . We see that the length of the upper horizontal segment has length 1, the next has length 2, the next has length 4 and it is not difficult to see that the nth horizontal line has length 2^n. Hence, if we have n horizontal layers in our diagram then the perimeter is $P = n + 1 + n + 2^n = 2^n + 2n + 1$. The number of regions A is $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$. Hence $A = 2P - (4n + 3) < 2P$, i.e. A is bounded by a linear function of P for such diagrams. However, we can attach two such diagrams M_1 and M_2 along the long horizontal segment, shifted by 1. See Fig 6.

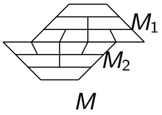


Fig. 6

1 Introduction

1.7-II Examples

Then the perimeter is $n+1+n+1+n+1+n+1=4n+4$, while the number of regions is $2\cdot(2^{n+1}-1)=2^{n+2}-2$. Hence, A is an exponential function of P. Consequently, the isoperimetric function of $\mathcal{P}$ is at least exponential. It turns out that the Dehn function is 2^n. The groups $B(m,n)=\langle a,b|a^{-1}b^m a=b^n\rangle \quad |m|\neq|n|$ all have exponential Dehn functions.

2. This trick can be repeated to construct a one-relator group with Dehn function $f(n)=\left.2^{2^{2^{2^{\cdot^{\cdot}}}}}\right\}$ n times.
3. If G is a one-relator group with torsion then it has linear Dehn function.

1 Introduction

1.8-I Small Cancellation Theory

Consider the group $G = \langle a, b, c | cbac^{-1}b^{-1}a^{-1} \rangle$ of Example 2. Fig. 4(e) is a finite part of the tessellation of the plane by hexagons and it can be shown that each van Kampen diagram related to this group is a finite part of the tessellation of the plane by hexagons. Since on $\mathbb{E}^2$ we have quadratic isoperimetric function, this implies that G has quadratic isoperimetric function. Similarly, if a group has a presentation such that every corresponding van Kampen diagram is a finite part of the tesselation of the plane by triangles or squares then it has quadratic isoperimetric function. Actually, it has quadratic Dehn function. By the same idea, groups having diagrams which are parts of a tessellation of the hyperbolic plane have linear Dehn functions. Hence the fundamental groups of oriented closed surfaces with genus at least 2 have linear Dehn functions.
These results have been extended by Roger-Lyndon (1966) as follows.

Theorem (Lyndon's Theorem)

Let $\mathcal{P} = \langle X | R \rangle$ be a group presentation. If any of the following conditions is satisfied then $\mathcal{P}$ has quadratic (linear) Isoperimetric functions:

1 Introduction

1.8-II Small Cancellation Theory

(i) *in every van Kampen diagram for $\mathcal{P}$, every inner region has at least 6(7) neighbouring regions.*

(ii) *In every van Kampen diagram for $\mathcal{P}$, every inner region has at least 4 neighbouring regions and every boundary vertex of it has valency at least 4.*

(iii) *In every van Kampen diagram for $\mathcal{P}$, every inner region has at least 3 neighbouring regions and every boundary vertex of it has valency at least 6(7) .*

These conditions, which generalize the regular tessellations of the plane, are called the *classical small cancellation conditions*. Conditions (i),(ii)(iii) are called C(6),C(4) & T(4) and C(3) & T(6), respectively.

1 Introduction

1.9 The Main Result

This examples raise the question: what are all the possible Dehn functions for one-relator groups? This is a very difficult (active) open problem, hence it is reasonable to subdivide it into two subproblems:

1. What are the polynomial functions which are Dehn functions of one-relator groups ?
2. What are the non-polynomial Dehn functions for one-relator groups?

In the present work we solve Problem 2 for positive words.

Main Theorem

Let $G_R = \langle X | R \rangle$ be a torsion-free one-relator group. If R is a positive word then G_R has an isoperimetric function which is either polynomial or exponential.

2 Idea of the proof

2.1-I Admissible Sets of Words for W.

We start with some introductory remarks.

1. Let $W \in F(X)$ be a reduced word. Suppose that there are words $A_1, \ldots, A_m$ in $F(X)$ such that

   1.1 $\mathbf{A} = \{A_1, \ldots, A_n\}$ freely generates a subgroup H of $F(X)$.

   1.2 $W \in H$. Hence W can be uniquely written as a word in $\mathbf{A}$.

   Then we can consider the presentation $\widetilde{H} = \langle \mathbf{A} | W(A_1, \ldots, A_m) \rangle$. Every diagram corresponding to $\widetilde{H}$ is also a diagram of $\langle X|W \rangle$.

2. Suppose that $U, V \in H$, reduced words, If $V = C * U * D$ is an occurrence of U in V in $F(X)$ but not in H then we say that that occurrence of U is *non H-standard in V*.
   Example 5: $\mathbf{A} = \{A, B\}$ $\quad |A||B| \geq 5, \quad A = aBb$ in $F(X)$. Then the occurrence of B in A is not H-standard.

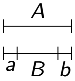


Fig. 7

2 Idea of the proof

2.1-II Admissible Sets of Words for W.

3. Now assume that each of the following holds
   3.1 Every diagram for a word in H representing 1 in $\widetilde{H}$ satisfies the small cancellation condition C(6).
   3.2 If $U = C * V * D, \quad U, V \in H$, is a non H-standard occurrence of V in U then $|V|_H \leq l$, for some integer l.
   3.3 Each diagram for a word in H which represents 1 in $\widetilde{H}$ has boundary of length (in H) at least $6l$.

Then we call $\mathbf{A} = \{A_1, \dots, A_m\}$ an *admissible set for* W.

2 Idea of the proof

2.2-I Isoperimetric Functions for $\langle X|R\rangle$.

Lemma

Suppose that $\mathbf{A} = \{A_1, \ldots, A_m\}$ is an admissible set for R. Then $f(n) = n^4$ is an isoperimetric function for $\langle X|R\rangle$.

proof:

Define an equivalence relation on $Reg(M)$-the regions of M- as follows: adjacent regions D_1 and D_2 are *H-equivalent* if their common edge μ is a word on $\mathbf{A}$ in standard position.

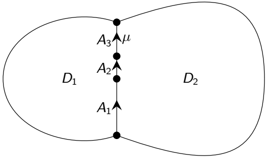


Fig. 8

2 Idea of the proof

2.2-II Isoperimetric Functions for $\langle X|R\rangle$.

Let "$\sim_H$" be the transitive closure of this relation. Then it partitions $Reg(M)$ into equivalence classes $\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_s$. Let Δ_i be the subdiagram of M consisting of the regions in $\widetilde{\Delta}_i$. It follows from standard arguments relying on the assumptions of the Lemma that each Δ_i is homeomorphic to a disc. Hence, we may consider the Δ_i as regions. Denote the diagram obtained from M this way by M_H. Then due to parts (b) and (c) in the definition of admissibility, M_H satisfies the small cancellation condition $C(6)$, hence due to Lyndon's Theorem

$$s \leq |\partial M|^2 \tag{$\star$}$$

Due to part(a) there,

$$|\Delta_i| \leq |\partial \Delta_i|^2 \tag{$\star\star$}$$

and it follows again by standard arguments that

$$|\partial \Delta_i| \leq |\partial M| \tag{$\star\star\star$}$$

Combining $\star$ and $\star\star$ with $\star\star\star$ implies that $|M| \leq s|\partial M|^2 \leq |\partial M|^2 |\partial M|^2$. Thus $f(n) = n^4$ is an isoperimetric function for $\langle X|R\rangle$. $\square$

So we have to find an admissible set of words for R.

2 Idea of the proof

2.3-I Integral Piecewise Rotations and word Equations

We observe that we may assume that $\langle X|R\rangle$ does not satisfy the condition $C(4)$ since the positivity of R implies that each inner vertex v has even valency which is at least 3. i.e. at least 4.

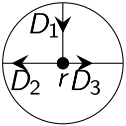
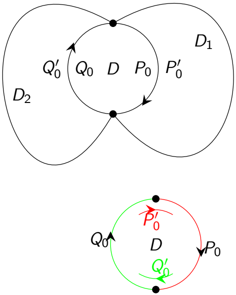


Fig. 9

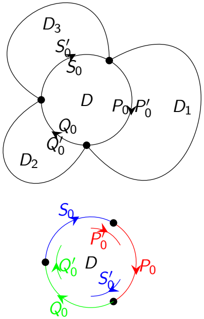
Hence if it satisfies the condition $C(4)$ then it satisfies the condition $C(4)\&T(4)$ and hence Lyndon's Theorem applies. In other word, we may assume that either there is a diagram with an inner region with 2 neighbours or there is a diagram with an inner region with 3 neighbours.

2 Idea of the proof

2.3-II Integral Piecewise Rotations and word Equations



a 2-system



a 3-system

Fig. 10

2 Idea of the proof

2.3-III Integral Piecewise Rotations and word Equations

We call such systems *2-systems* and *3-systems*, respectively and denote $\mathcal{P}_0 = (P_0, Q_0 | P'_0, Q'_0)$ or $\mathcal{P}_0 = (P_0, Q_0, S_0 | P'_0, Q'_0, S'_0)$, respectively.

They can be described by linear word equations:

$$\begin{cases} R = P_0 Q_0 \\ RR = H_1 P'_0 T_1 \\ RR = H_2 Q'_0 T_2 \end{cases} \qquad \begin{cases} R = P_0 Q_0 S_0 \\ RR = H_1 P'_0 T_1 \\ RR = H_2 Q'_0 T_2 \\ RR = H_3 S'_0 T_3 \end{cases} \tag{E}$$

Word equations for 2-systems

Word equations for 3-systems

2 Idea of the proof

2.4-I The Main Proposition

Proposition *Let $\mathcal{P}_0$ be an r-system as above, $r = 2$ or $r = 3$*

1. *The cyclic word $\widehat{R}$ contains a canonically defined subword $P_1 Q_1$ if $r = 2$ and canonically defined subword $P_1 Q_1 S_1$ if $r = 3$, which constitutes an r-system, $\mathcal{P}_1 = (P_1, Q_1 | P_1', Q_1')$ and $\mathcal{P}_1 = (P_1, Q_1, S_1 | P_1', Q_1', S_1')$, respectively. We call $\mathcal{P}_1$ the derived system of $\mathcal{P}_0$ and denote by $\mathbf{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_k)$ the sequence of successively derived systems. Also, we define $\kappa : F(X) \to \mathbb{N} \cup \{\infty\}$ by $\kappa(R) = \infty$ if R does not satisfy equations (E) and $\kappa(R) = k$ if R satisfies one of the systems of equations.*

2 Idea of the proof

2.4-II The Main Proposition

2. *Let* **P** *be the derived sequence of* $\mathcal{P}_0$ *as in part (a). If* $r = 2$ *then* $P_k Q_k$ *is a cyclic conjugate of* $W_a(A, B)$*, by a word in* $\langle A, B\rangle$*, where* $W_a = (AB)^a A$ *and if* $r = 3$ *then the subword* $Q_k S_k$ *constitutes a 2-system* $(Q_k, S_k | Q_k', S_k')$ *such that* $Q_k S_k$ *is a cyclic conjugate of* $W_{\alpha_1} \cdots W_{\alpha_p}, \quad W_{\alpha_i} = W_{\alpha_i}(A, B)$*, such that* A *and* B *decompose into* $A = A_0 A_1$ *and* $B = B_0 B_1$ *respectively and* $P_k, Q_k, S_k \in \langle A_0, A_1, B_0, B_1\rangle . P_k \subset Q_k S_k$*. Moreover, if* $\kappa \geq 4$ *then* $\{A, B\}$ *and* $\{A_0, A_1, B_0, B1\}$ *respectively is an admissible set of words for* R*.*

2 Idea of the proof

2.5 Proof of the Main Theorem.

Proof of the Theorem

If $\kappa(R) \geq 4$ then by the Proposition and Lemma $G = \langle X | R \rangle$ has polynomial isoperimetric function f. If $\kappa(R) \leq 3$ then we may list all the possibilities for R due to the canonicity of $\mathcal{P}_i$. We find by identifying G with known classes of groups, up to isomorphism, that f is either linear or quadratic or exponential. $\square$