Analysing the effects of nonlocal competition on invasion speed and the formation of patterns

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Centre International de Rencontres Mathmatiques (CIRM)

Outline

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- Discuss Fisher-KPP equation

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This Fisher-KPP equation has local competition.

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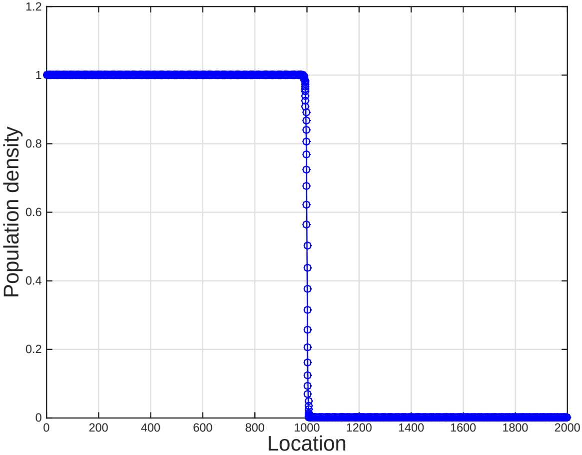

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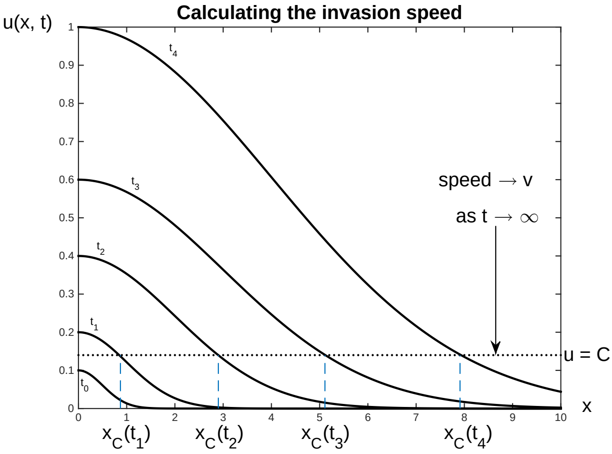

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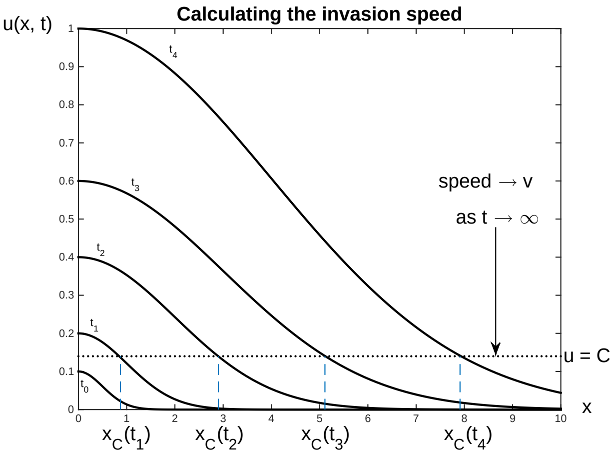

$$v = \lim_{t \to \infty} \frac{dx_C}{dt}$$

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Then, we look for solutions of the form $u(x,t) = \exp(-i\omega(k)t + ikx)$ where $\omega(k)$ is the dispersion relation of the Fourier modes and k is the wavenumber.

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Substituting this solution into the linearised Fisher-KPP equation gives

$$\omega(k) = i(r - Dk^2).$$

[1] W. van Saarloos. Front propagation into unstable states. Physics Reports, 29(222):3042, August 2003

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Equating these two equations and solving for k gives the asymptotic invasion speed $v = 2\sqrt{rD}$.

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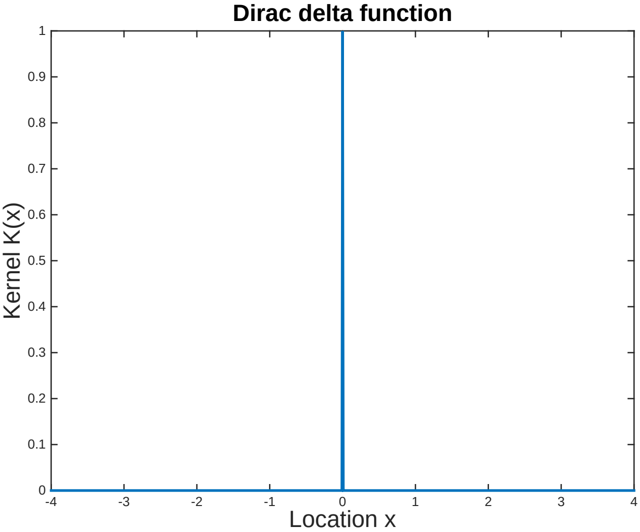

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- $K(x)$ is monotonically decreasing away from the origin
- $K(x)$ integrates to 1

Highlight pattern formation

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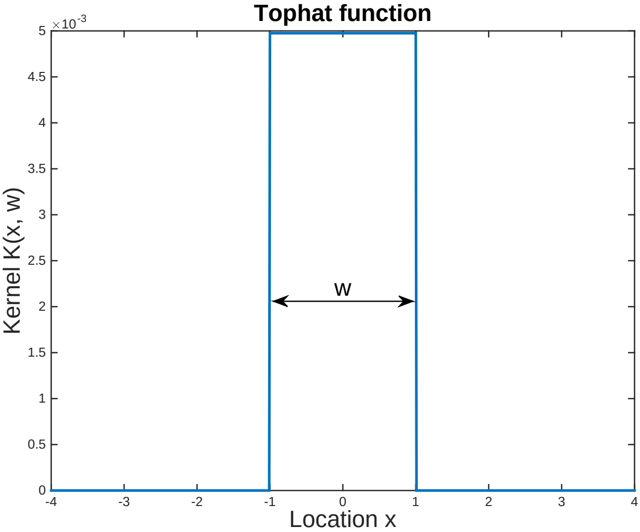
Initially, we set $K(x)$ to be a tophat function with width parameter w.



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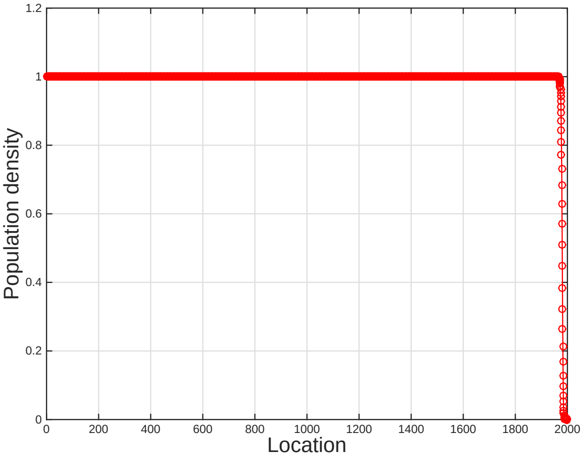

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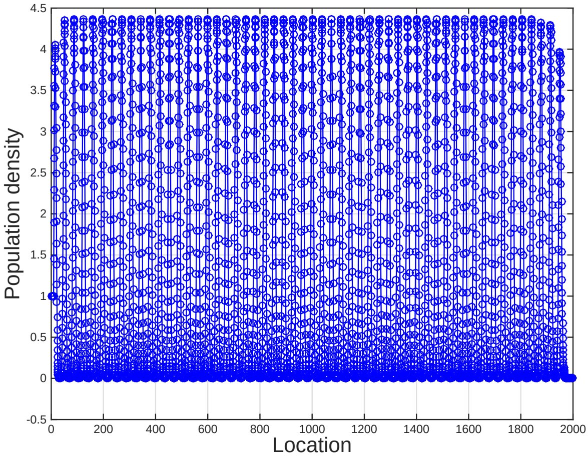

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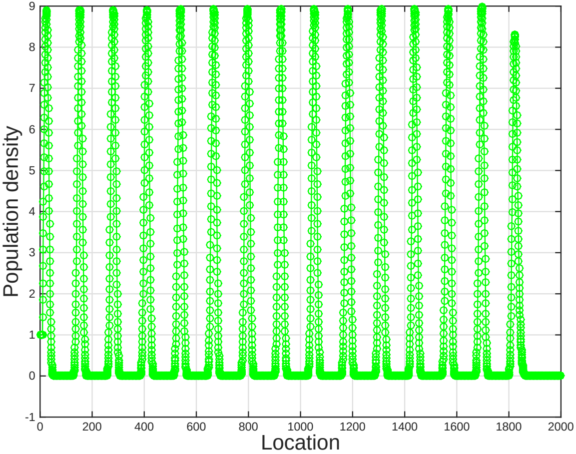

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We are interested in determining how nonlocal competition, and the resulting patterns, affect the population invasion speed.

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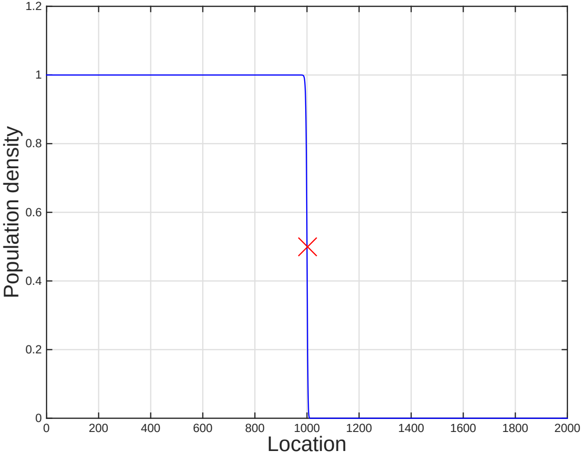

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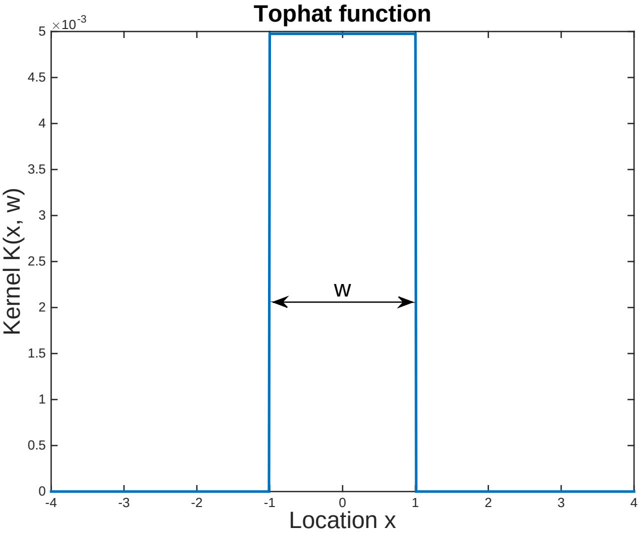

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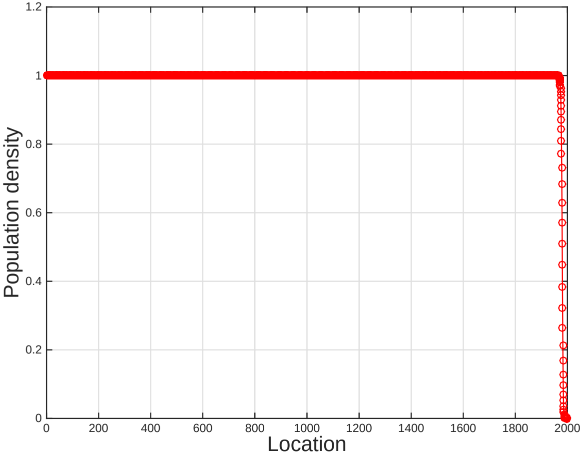

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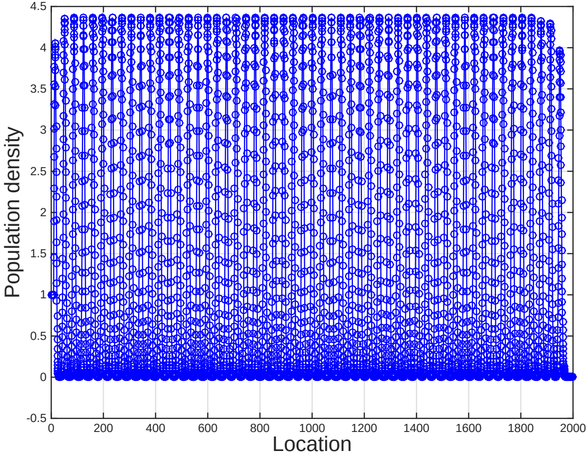

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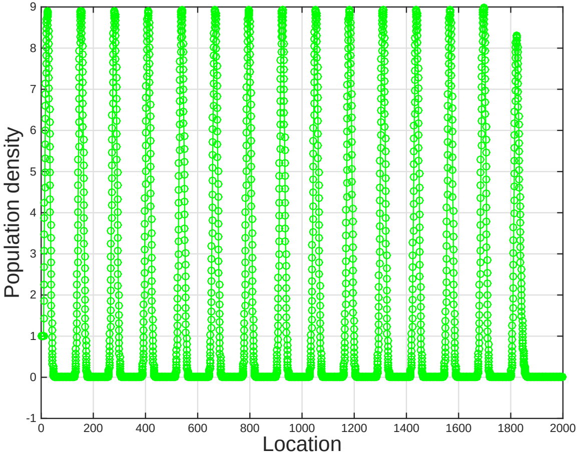

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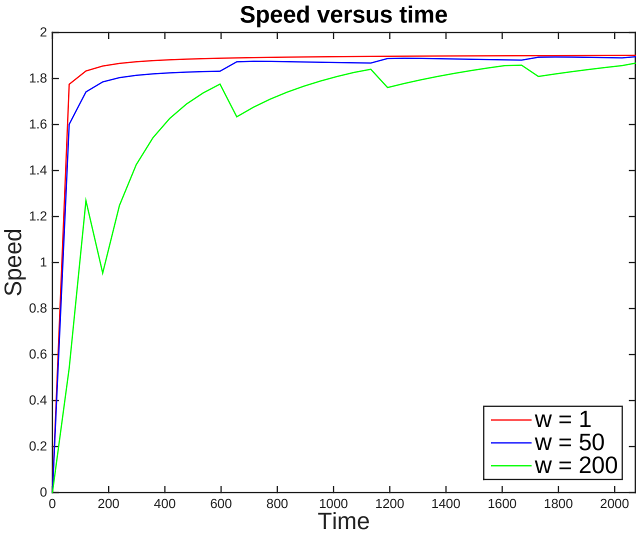

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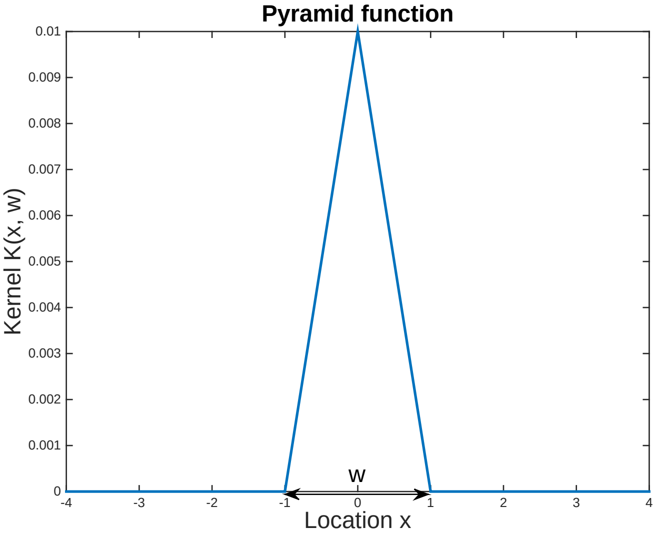

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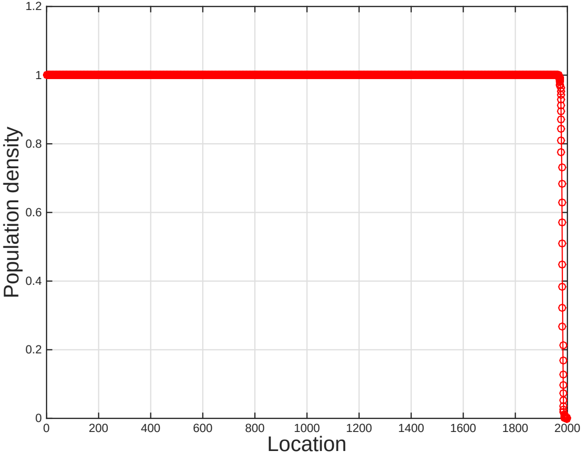

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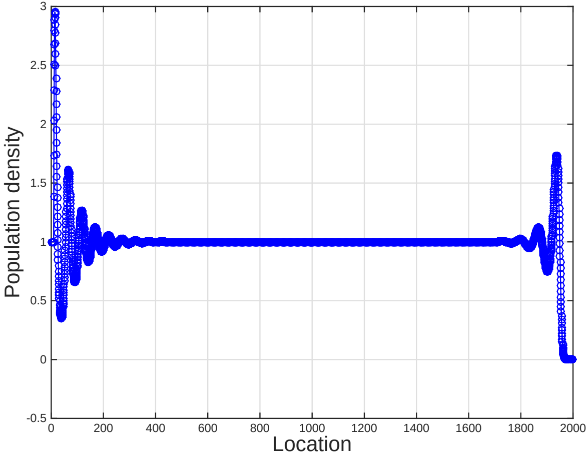

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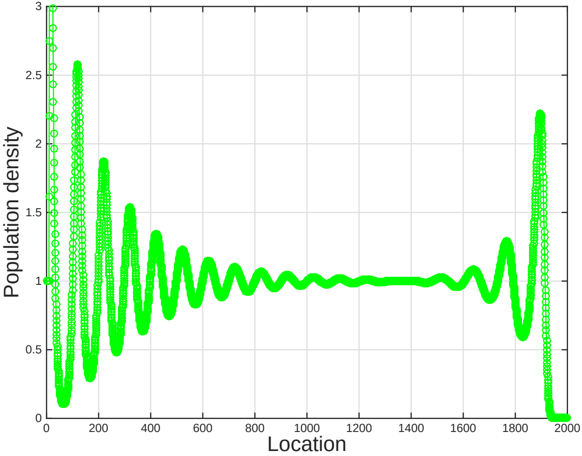

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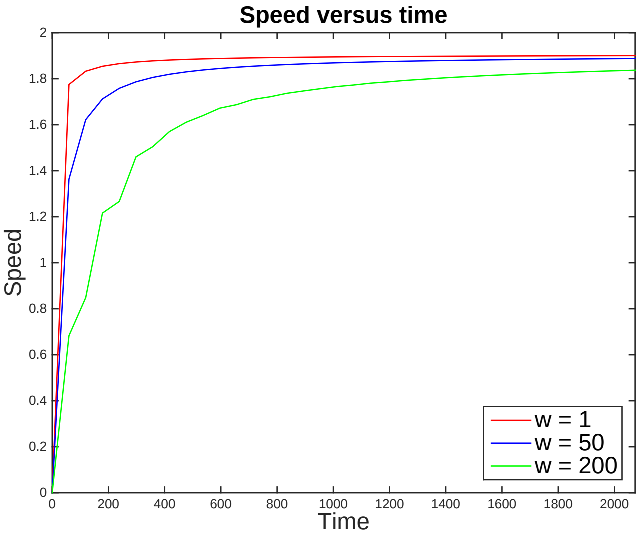

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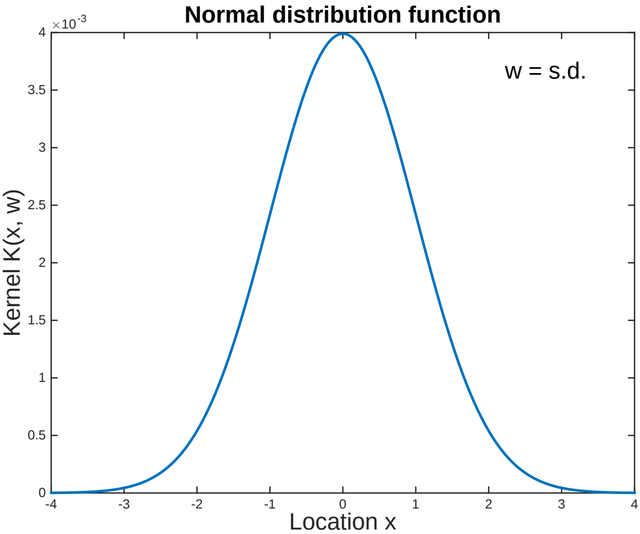

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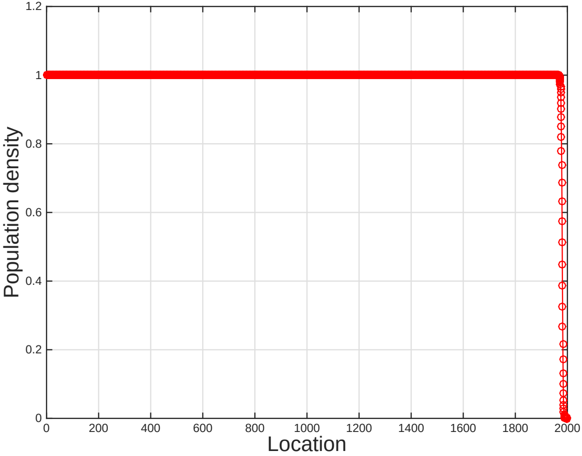

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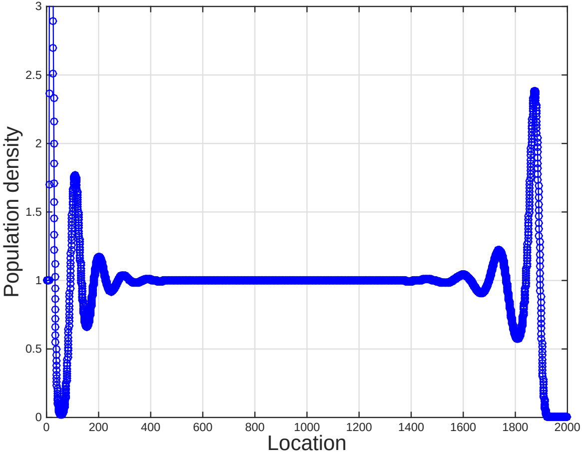

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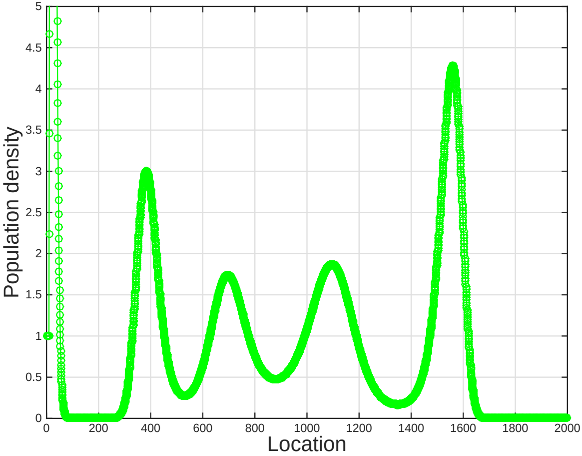

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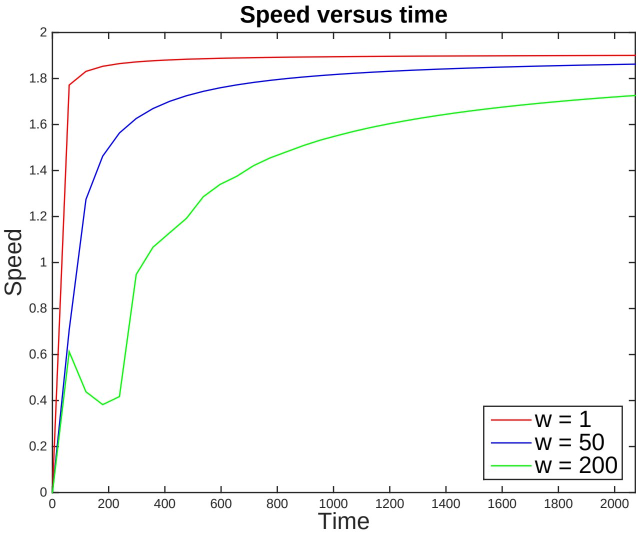

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In particular, a population with nonlocal competition takes longer to reach the asymptotic invasion speed than a population with local competition.

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