Spatially structured population dynamics: the point of view of PDEs

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Results obtained in collaboration with

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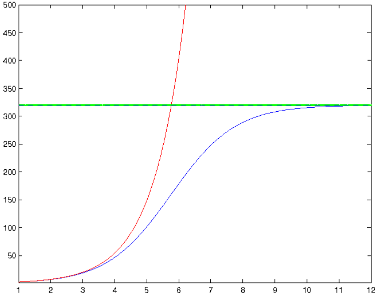


Logistic model (Verhulst, 1838)

$$u'(t) = r_0 \left(1 - \frac{u(t)}{K}\right) u(t), \qquad u(t) = \frac{u(0)\, K}{u(0) + (K - u(0)) \exp(-r_0\, t)}.$$

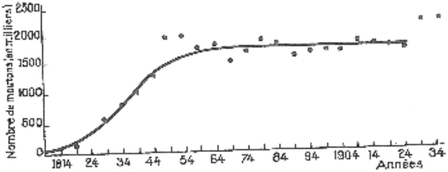
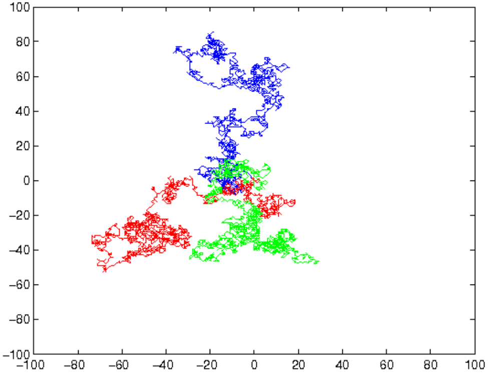


FIG. 62.— *Le développement du mouton après son introduction en Tasmanie.*

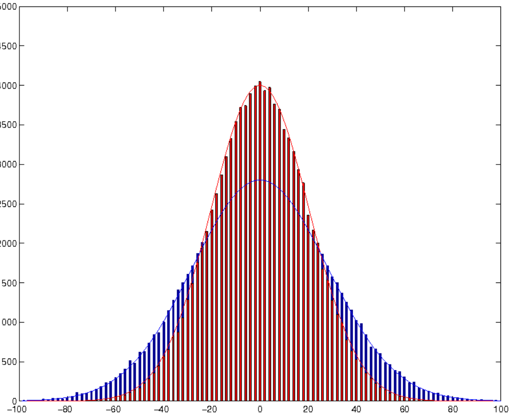
An example of observed data and of logistic "fit"

Dispersion models for populations



Random walk: We assume isotropy, and consider a scale of time and space which is large (w.r.t. the motion of one individual).

Dispersion models for populations



Corresponding law: convergence towards a Gaussian law with variance proportional to time.

Dispersion models for populations

Random walk $S_p = \sum_{i=1}^{p} X_i$, with $X_i = \Delta x$ and $X_i = -\Delta x$ each of probability $1/2$, and independant.

Law of S_p: $P(S_p = q\,\Delta x) = 2^{-p}\, C_p^{\frac{q+p}{2}}$ (when $|q| \leq p$ et $q \equiv p[2]$).

We consider $N(p\Delta t, x) := P(S_p \in [x - \Delta x, x + \Delta x[)$. Then for $t = p\,\Delta t$:

$$N(t, q\,\Delta x) = 2^{-p}\, C_p^{\frac{q+p}{2}}.$$

One uses the following asymptotic expansion:

Lemma:

$$2^{-p}\, C_p^{\frac{q+p}{2}} \sim \frac{2}{\sqrt{2\pi\, p}}\, e^{-\frac{q^2}{2p}}$$

when $p \to +\infty$, $q^3 = o(p^2)$.

Dispersion models for populations

When $\Delta t \to 0$ and $\Delta t^2 << \Delta x^3$,

$$N(t,x) \sim 2\,\Delta x \sqrt{\frac{\Delta t}{\Delta x^2}}\, \frac{e^{-\{\frac{\Delta t}{\Delta x^2}\}\,\frac{x^2}{2t}}}{\sqrt{2\pi t}},$$

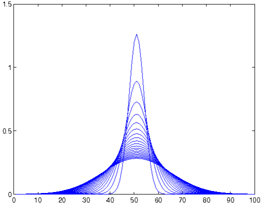
in such a way that $\frac{\Delta t}{\Delta x^2} \to \frac{1}{2D}$ and $\Delta t \to 0$,

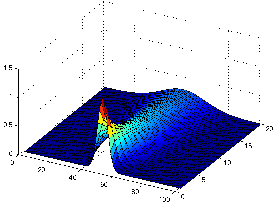
$$\frac{N(t,x)}{2\,\Delta x} \to u(t,x) = \frac{e^{-\frac{x^2}{4D\,t}}}{\sqrt{4\pi\, D\, t}}.$$

This last quantity is the elementary solution of the heat equation in dimension 1 with a diffusion coefficient D:

$$\frac{\partial u}{\partial t}(t,x) = D\,\frac{\partial^2 u}{\partial x^2}(t,x), \qquad u(0,x) = \delta_0(x).$$

Dispersion models for populations





Diffusion (Fourier, 1822): Heat (diffusion) equation and its fundamental solution:

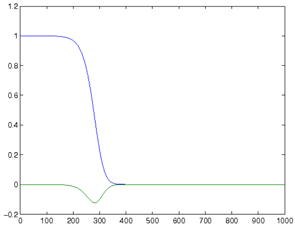
$$\frac{\partial u}{\partial t}(t,x) = D\,\frac{\partial^2 u}{\partial x^2}(t,x), \qquad u(0,x) = \delta_0(x).$$

$$u(t,x) = \frac{e^{-\frac{x^2}{4D\,t}}}{\sqrt{4\pi\,D\,t}}.$$

Invasion model (Fisher; Kolmogoroff-Petrovsky-Piscounoff, 1937)

$$\frac{\partial u}{\partial t}(t,x) = D\,\frac{\partial^2 u}{\partial x^2}(t,x) + r_0 \left(1 - \frac{u(t,x)}{K}\right) u(t,x).$$

Obtained when both diffusion and logistic effects are considered.

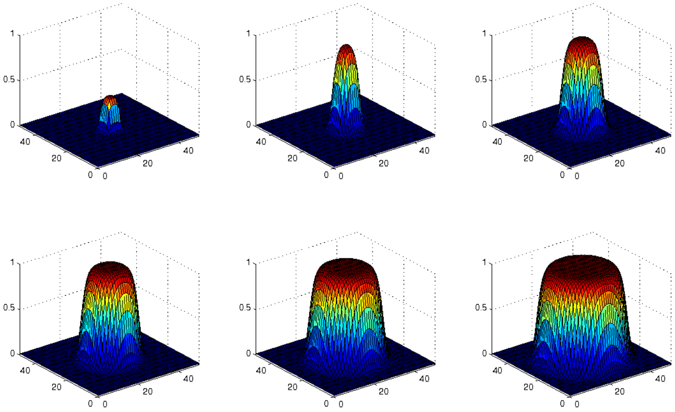


In dimension 1: One looks for $u(t,x) = N(x - ct)$ solution of the PDE:

$$-c\,N'(z) - D\,N''(z) = r_0 \left(1 - \frac{N(z)}{K}\right) N(z); \qquad N(-\infty) = K;\ N(\infty) = 0.$$

Theorem (Kolmogoroff-Petrovsky-Piscounoff, 1937): Solutions to this heteroclinic junction problem in ODEs exist when $c \geq c_0 = \sqrt{2\,r_0\,D}$, critical speed of invasion associated to a population.
Those solutions are stable (in a setting to be made precise...) for the PDE if and only if $c = c_0$.

2D Traveling waves of invasion



Meaning of maps related to a biological invasion

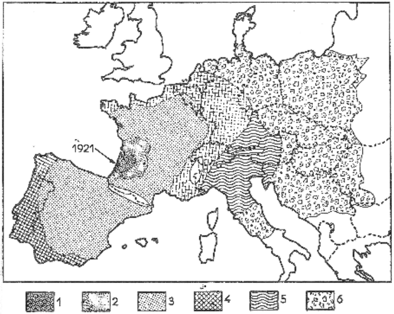


Fig. 63. — *Progression du doryphore en Europe après son arrivée en 1921.*

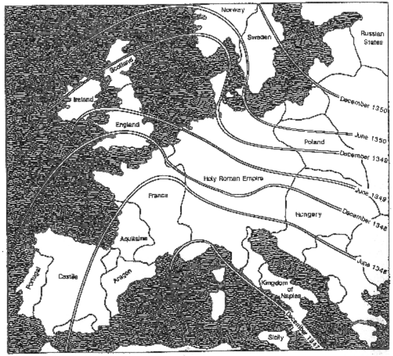


Fig. 20.2. Approximate chronological spread of the Black Death in Europe from 1347–50. (Redrawn from Langer 1964)

Two examples of propagation of a front: invasion of animals and epidemiology

Competition models

Lotka, Volterra, 1925

Unknowns: $u := u(t) \geq 0$, $v := v(t) \geq 0$, for $t \geq 0$.

Equations:

$$u'(t) = (r_1 - S_{11}\, u(t) - S_{12}\, v(t))\, u(t),$$
$$v'(t) = (r_2 - S_{21}\, u(t) - S_{22}\, v(t))\, v(t).$$

$S_{ii} > 0$: intraspecific competition

$S_{ij} \geq 0$, $i \neq j$: interspecific competition

Competition models

Depending on the parameters r_i, S_{ij}, and considering only nonnegative solutions, one has either (up to exchanging n_1 and n_2):

- **Strong competition**: The only stable equilibrium for the system of ODEs is $(u, v) = (n_{10}, 0)$ with $n_{10} > 0$; competitive exclusion.
- **Weak competition**: The only stable equilibrium for the system of ODEs is $(u, v) = (n_{10}, n_{20})$ with $n_{10} > 0$, $n_{20} > 0$; coexistence.

Competition/Diffusion model

Unknowns: $u := u(t,x) \geq 0$, $v := v(t,x) \geq 0$, for $t \geq 0$, $x \in \Omega$.

Equations:

$$\partial_t u - D_1 \, \Delta_x u = (r_1 - S_{11}\, u - S_{12}\, v)\, u,$$

$$\partial_t v - D_2 \, \Delta_x v = (r_2 - S_{21}\, u - S_{22}\, v)\, v.$$

No Turing instability for such models: all steady homogeneous solutions which are stable for the ODEs are also stable for the PDEs; **No segregation of species appears**

Shigesada-Kawasaki-Teramoto (SKT) model, 1979

Equations:

$$\partial_t u - \Delta_x \left(u \left[D_1 + A_{11}\, u + A_{12}\, v \right] \right) = (r_1 - S_{11}\, u - S_{12}\, v)\, u,$$

$$\partial_t v - \Delta_x \left(v \left[D_2 + A_{21}\, u + A_{22}\, v \right] \right) = (r_2 - S_{21}\, u - S_{22}\, v)\, v.$$

$A_{12} \geq 0, A_{21} \geq 0$: cross diffusions (also used in fluid mech. [Maxwell Stefan])

$A_{11} \geq 0, A_{22} \geq 0$: self diffusions (also used in fluid mech. [porous media/fast diffusion])

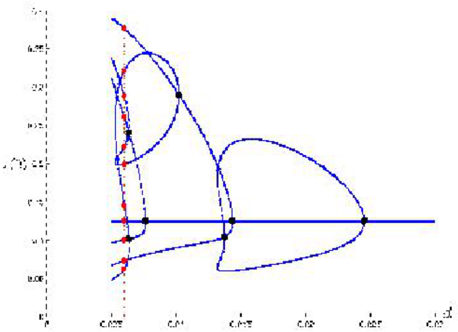
If $A_{21} = 0$, the system is called triangular.

A typical (2D) Turing pattern

Appearance of stable spatially inhomogeneous equilibria for the PDE:

Bifurcation diagram (Turing instability)

At the numerical level, cf. Iida, Izuhara, Mimura, Ninomiya, and at the rigorous level, cf. Breden, Lessard, Vanicat, for a model close to that of SKT.



Why $\Delta_x(u\,v) = \nabla_x \cdot (v\,\nabla_x u) + \nabla_x \cdot (u\,\nabla_x v)$ rather than $\nabla_x \cdot (v\,\nabla_x u)$?

Answer (proposed by Iida; Izuhara; Mimura; Ninomiya) in the case of the triangular system);

$$\partial_t u - \Delta_x(D_1\,u + A_{12}\,u\,v\,) = (r_1 - S_{11}\,u - S_{12}\,v)\,u,$$

$$\partial_t v - D_2\,\Delta_x v = (r_2 - S_{21}\,u - S_{22}\,v)\,v.$$

Possible interpretation in terms of "microscopic" behavior: The first species u exists in two states: quiet (u_A) and stressed (u_B). Individuals switch from one state to the other with a scale of time ε and rates which depend on the density v of the second species (fast reaction).
For a direct passage from an individual model to cross diffusion PDEs, cf. Fontbona; Méléard

Equations for the microscopic model

$$\partial_t u_A - D_1 \Delta_x u_A = (r_1 - S_{11}(u_A + u_B) - S_{12} v) u_A + \frac{1}{\varepsilon}(k(v) u_B - h(v) u_A),$$

$$\partial_t u_B - (D_1 + \alpha) \Delta_x u_B = (r_1 - S_{11}(u_A + u_B) - S_{12} v) u_B - \frac{1}{\varepsilon}(k(v) u_B - h(v) u_A),$$

$$\partial_t v - D_2 \Delta_x v = (r_2 - S_{21}(u_A + u_B) - S_{22} v) v.$$

Possible interpretation in terms of "microscopic" behavior: The first species u exists in two states: quiet (u_A) and stressed (u_B). Individuals switch from one state to the other with a scale of time ε and rates which depend on the density v of the second species (fast reaction).

Formal asymptotics when the microscopic time scale tends to 0

$$\partial_t u_A^\varepsilon - D_1\,\Delta_x u_A^\varepsilon = (r_1 - S_{11}\,(u_A^\varepsilon + u_B^\varepsilon) - S_{12}\,v^\varepsilon)\,u_A^\varepsilon + \frac{1}{\varepsilon}(k(v^\varepsilon)\,u_B^\varepsilon - h(v^\varepsilon)\,u_A^\varepsilon),$$

so that

$$k(v^\varepsilon)\,u_B^\varepsilon - h(v^\varepsilon)\,u_A^\varepsilon = O(\varepsilon).$$

Moreover, by adding the two first equations,

$$\partial_t(u_A^\varepsilon + u_B^\varepsilon) - \Delta_x\bigg(D_1\,(u_A^\varepsilon + u_B^\varepsilon) + \alpha\,u_B^\varepsilon\bigg)$$

$$= (r_1 - S_{11}\,(u_A^\varepsilon + u_B^\varepsilon) - S_{12}\,v^\varepsilon)\,(u_A^\varepsilon + u_B^\varepsilon),$$

$$\partial_t v^\varepsilon - D_2\,\Delta_x v^\varepsilon = (r_2 - S_{21}\,(u_A^\varepsilon + u_B^\varepsilon) - S_{22}\,v^\varepsilon)\,v^\varepsilon.$$

Formal asymptotics when the microscopic time scale tends to 0

Assuming that $u_A^\varepsilon \to u_A$, $u_B^\varepsilon \to u_B$, $v^\varepsilon \to v$,

$$k(v)\, u_B = h(v)\, u_A, \qquad u_B = \frac{h(v)}{k(v)+h(v)}\,(u_A + u_B),$$

and $(u_A + u_B, v)$ satisfy the system of reaction-cross diffusion (of two equations)

$$\partial_t(u_A + u_B) - \Delta_x\left(D_1\,(u_A+u_B) + \alpha\,\frac{h(v)}{k(v)+h(v)}\,(u_A+u_B)\right)$$

$$= (r_1 - S_{11}\,(u_A+u_B) - S_{12}\,v)\,(u_A+u_B),$$

$$\partial_t v - D_2\,\Delta_x v = (r_2 - S_{21}\,(u_A+u_B) - S_{22}\,v)\,v.$$

Formal asymptotics when the microscopic time scale tends to 0

The Shigesada-Kawasaki-Teramoto model can be recovered by observing that for $u = u_A + u_B$,

$$\partial_t u - \Delta_x \left(D_1\, u + \alpha\, \frac{h(v)}{k(v) + h(v)}\, u \right) = (r_1 - S_{11}\, u - S_{12}\, v)\, u,$$

and by choosing h and k such that

$$A_{12}\, v = \alpha\, \frac{h(v)}{k(v) + h(v)}.$$

We observe that the equation for v is kept:

$$\partial_t v - D_2\, \Delta_x v = (r_2 - S_{21}\, u - S_{22}\, v)\, v.$$

Rigorous result for this asymptotics

Iida; Mimura; Ninomiya; *Diffusion, cross-diffusion and competitive interaction*. J. Math. Biol. **53** (2006), no. 4, 617–641.

Izuhara; Mimura; *Reaction-diffusion system approximation to the cross-diffusion competition system*. Hiroshima Math. J. **38** (2008), no. 2, 315–347.

1. $$u_A^\varepsilon - u_A = O(\varepsilon), \quad u_B^\varepsilon - u_B = O(\varepsilon), \quad v^\varepsilon - v = O(\varepsilon)$$
   under the (not known) assumption
   $$||u_A^\varepsilon||_{L^\infty} \le Cst, \quad ||u_B^\varepsilon||_{L^\infty} \le Cst, \quad ||v^\varepsilon||_{L^\infty} \le Cst$$

2. Thorough study of stationary solutions

Fast reaction limit of the reaction-diffusion system with three equations

Theorem (LD, Trescases): Let Ω be a smooth bounded domain of $\mathbb{R}^N$. We assume that $d_A, d_B, d_u, d_v > 0$, $r_u, r_v, r_a, r_b, r_c, r_d > 0$, $a, b, c, d > 0$. We consider functions ϕ, h and k lie in $C^1(\mathbb{R}_+)$ and satisfy, for some $h_0 > 0$,

$$d_A + d_B \frac{h(v)}{h(v) + k(v)} = d_u + \phi(v), \qquad h(v) \geq h_0, \qquad k(v) \geq h_0.$$

Finally, we consider initial data $u_{in} \geq 0$, $v_{in} \geq 0$ such that $u_{in} \in L^2(\Omega)$, $v_{in} \in L^\infty(\Omega) \cap W^{2,1+2/d}(\Omega)$ (and compatibility conditions).

Fast reaction limit of the reaction-diffusion system with three equations

Then, for any $\varepsilon \in]0,1[$, there exists a strong (global, with nonnegative components) solution $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ to system

$$\partial_t u_A^\epsilon - d_A\,\Delta_x u_A^\epsilon = [r_u - r_a\,(u_A^\epsilon + u_B^\epsilon)^a - r_b\,(v^\epsilon)^b]\,u_A^\epsilon + \frac{1}{\epsilon}\,[k(v^\epsilon)\,u_B^\epsilon - h(v^\epsilon)\,u_A^\epsilon],$$

$$\partial_t u_B^\epsilon - (d_A + d_B)\,\Delta_x u_B^\epsilon = [r_u - r_a\,(u_A^\epsilon + u_B^\epsilon)^a - r_b\,(v^\epsilon)^b]\,u_B^\epsilon - \frac{1}{\epsilon}\,[k(v^\epsilon)\,u_B^\epsilon - h(v^\epsilon)\,u_A^\epsilon],$$

$$\partial_t v^\epsilon - d_v\,\Delta_x v^\epsilon = [r_v - r_c\,(v^\epsilon)^c - r_d\,(u_A^\epsilon + u_B^\epsilon)^d]\,v^\epsilon.$$

with adapted initial data

$$u_A^\epsilon(0) = \frac{k(v_{in})}{k(v_{in}) + h(v_{in})}\,u_{in}, \qquad u_B^\epsilon(0) = \frac{h(v_{in})}{k(v_{in}) + h(v_{in})}\,u_{in}, \qquad v^\epsilon(0) = v_{in},$$

and Neumann boundary conditions

$$\text{On } \partial\Omega, \qquad \nabla_x u_A^\epsilon \cdot n = 0, \quad \nabla_x u_B^\epsilon \cdot n = 0, \quad \nabla_x v^\epsilon \cdot n = 0.$$

Fast reaction limit of the reaction-diffusion system with three equations

We assume moreover that $a \leq d$, $a \leq 1$, $d \leq 2$.
Then, when $\varepsilon \to 0$, $(u_A^\epsilon, u_B^\epsilon, v^\epsilon)$ converges, up to extraction of a subsequence, for almost every $(t,x) \in \mathbb{R}_+ \times \Omega$ to a limit (u_A, u_B, v) lying in $L^2([0,T] \times \Omega) \times L^2([0,T] \times \Omega) \times L^\infty([0,T] \times \Omega)$ (for all $T > 0$), and such that $u_A \geq 0$, $u_B \geq 0$, $v \geq 0$. Furthermore,

$$h(v(t,x))\, u_A(t,x) = k(v(t,x))\, u_B(t,x)$$

and $(u := u_A + u_B, v)$ is a (global, with nonnegative components) weak solution of system

$$\partial_t u - \Delta_x(d_u\, u + u\, \phi(v)) = u\,(r_u - r_a\, u^a - r_b\, v^b),$$

$$\partial_t v - d_v\, \Delta_x v = v\,(r_v - r_c\, v^c - r_d\, u^d),$$

with Neumann boundary conditions and initial data u_{in}, v_{in}.

Theorem (LD, Trescases): Let Ω be a smooth bounded domain of $\mathbb{R}^N$ ($N \in \mathbb{N}^*$), and $D_i > 0$, $r_i > 0$, $a, b, c, d > 0$ such that $d < a$ (case 1) or $a \le d$, $a \le 1$, $d \le 2$ (case 2). Consider $\phi \ge 0$ in $W^{1,\infty}_{loc}$, and (nonnegative for each component) initial data such that $u_{in} \in L^{p_0}(\Omega)$, $v_{in} \in L^\infty(\Omega) \cap W^{2,1+p_0/d}(\Omega)$ for some $p_0 > 1$ in case 1 and $p_0 = 2$ in case 2 (+ compatibility conditions).

Then, there exists a (nonnegative for each component) weak solution (u, v) in $L^{p_0+a}([0,T] \times \Omega) \times L^\infty([0,T] \times \Omega)$ for all $T > 0$ (In case 2, $p_0 + a$ is replaced by 2) to the system

$$\partial_t u - \Delta_x \left(u \left[D_1 + \phi(v) \right] \right) = u \,(r_u - r_a\, u^a - r_b\, v^b),$$

$$\partial_t v - D_2\, \Delta_x v = v\,(r_v - r_c\, v^c - r_d\, u^d),$$

with Neumann boundary conditions and initial data u_{in}, v_{in}.

Results of existence for the triangular system

Amann: Local (in time) existence in all cases

Matano, Mimura; Shim Global existence in dimension 1 for the original model

Yagi Global existence in dimension ≤ 2 in the presence of self diffusion, for the original model

Lou, Ni, Wu Global existence in dimension ≤ 2 without restriction, for the original model

Choi, Lui, Yamada Global existence in any dimension for small cross diffusion coefficients, or for any cross diffusion coefficients in the presence of self diffusion in the first equation, or in the second equation in dimension ≤ 6; all for the original model

Phan Global existence in dimension ≤ 10 in the presence of self diffusion in the second equation, for the original model

Yamada Global existence in any dimension under the assumption $a > d$

Back to the non triangular SKT model

Equations:

$$\partial_t u_1 - \Delta_x \left(u_1 \left[D_1 + A_{11}\, u_1 + A_{12}\, u_2 \right] \right) = (r_1 - S_{11}\, u_1 - S_{12}\, u_2)\, u_1,$$

$$\partial_t u_2 - \Delta_x \left(u_2 \left[D_2 + A_{21}\, u_1 + A_{22}\, u_2 \right] \right) = (r_2 - S_{21}\, u_1 - S_{22}\, u_2)\, u_2.$$

Neumann boundary condition (for $t \geq 0$, $x \in \partial\Omega$)

$$\nabla_x u_1(t,x) \cdot n(x) = 0, \quad \nabla_x u_2(t,x) \cdot n(x) = 0.$$

Initial data (for $x \in \Omega$)

$$u_1(0,x) = u_{10}(x), \qquad u_2(0,x) = u_{20}(x).$$

Assumption: $D_1, D_2 > 0,\ A_{12}, A_{21} > 0,\ A_{11}, A_{22} \geq 0.$

Results of existence for the non-triangular SKT system

Amann: Existence of local (in time) solutions

Deuring: Existence of solutions when cross diffusions are small and without self diffusion

Kim; Masuda, Mimura; Shim: Existence of solutions for various types of coefficients in dimension 1

Yagi: Existence of solutions when the self diffusion dominates the cross diffusion

Li, Zhao: Existence of solutions when $D_1 = D_2$.

Chen, Jüngel: Existence of (weak) solutions thanks to the use of the functional $J(u_1, u_2) = c_1 \int_\Omega (u_1 \ln u_1 - u_1 + 1) + c_2 \int_\Omega (u_2 \ln u_2 - u_2 + 1)$.

Computation of the evolution of J

$$\frac{d}{dt} J(u_1,u_2) = \frac{d}{dt} \int \Big[c_1\,(u_1\,\ln u_1 - u_1 + 1) + c_2\,(u_2\,\ln u_2 - u_2 + 1) \Big]$$

$$= \sum_{i=1}^{2} c_i \int (\ln u_i)\,\Delta_x (D_i\,u_i + A_{i1}\,u_1\,u_i + A_{i2}\,u_2\,u_i)$$

$$+ c_1 \int (r_1 - S_{11}\,u_1 - S_{12}\,u_2)\,u_1\,\ln u_1 + c_2 \int (r_2 - S_{21}\,u_1 - S_{22}\,u_2)\,u_2\,\ln u_2$$

$$= -c_1\,D_1 \int \frac{|\nabla_x u_1|^2}{u_1} - c_2\,D_2 \int \frac{|\nabla_x u_2|^2}{u_2} - 2c_1\,A_{11} \int |\nabla_x u_1|^2 - 2c_2\,A_{22} \int |\nabla_x u_2|^2$$

$$- \int \sum_{i=1}^{2} c_i\,(S_{i1}\,u_1 + S_{i2}\,u_2)\,u_i\,\ln u_i + \int (c_1\,r_1\,u_1\,\ln u_1 + c_2\,r_2\,u_2\,\ln u_2)$$

$$- \int u_1\,u_2 \left[c_1\,A_{12} \left| \frac{\nabla_x u_1}{u_1} \right|^2 + (c_1\,A_{12} + c_2\,A_{21})\,\frac{\nabla_x u_1}{u_1} \cdot \frac{\nabla_x u_2}{u_2} + c_2\,A_{21} \left| \frac{\nabla_x u_2}{u_2} \right|^2 \right].$$

$$\Delta = (c_1\,A_{12} - c_2\,A_{21})^2.$$

Computation of the evolution of J

For $c_1 = A_{21}$, $c_2 = A_{12}$,

$$\frac{d}{dt} J(u_1, u_2) = -A_{21}\, D_1 \int \frac{|\nabla_x u_1|^2}{u_1} - A_{12}\, D_2 \int \frac{|\nabla_x u_2|^2}{u_2} - 2\, A_{21}\, A_{11} \int |\nabla_x u_1|^2$$

$$-2\, A_{12}\, A_{22} \int |\nabla_x u_2|^2 - \int A_{21} \left[\sum_{i=1}^{2} S_{1i}\, u_i \right] u_1 \ln u_1 + \int A_{12} \left[\sum_{i=1}^{2} S_{2i}\, u_i \right] u_2 \ln u_2$$

$$-A_{12}\, A_{21} \int u_1\, u_2 \left| \frac{\nabla_x u_1}{u_1} + \frac{\nabla_x u_2}{u_2} \right|^2 + A_{21}\, r_1 \int u_1 \ln u_1 + A_{12}\, r_2 \int u_2 \ln u_2 .$$

After integration in time, for any $T > 0$,

$$\sup_{t \in [0,T]} \int_\Omega \Big(u_1\, |\ln u_1| + u_2\, |\ln u_2| \Big) < \infty, \quad \int_0^T \int_\Omega \Big(|\nabla_x \sqrt{u_1}|^2 + |\nabla_x \sqrt{u_2}|^2 \Big) < \infty$$

$$\int_0^T \int_\Omega \Big(u_1^2\, |\ln u_1| + u_2^2\, |\ln u_2| \Big) < \infty, \quad \int_0^T \int_\Omega \Big(A_{11}\, |\nabla_x u_1|^2 + A_{22}\, |\nabla_x u_2|^2 \Big) < \infty$$

A more recent result

Generalization of SKT model:

$$\partial_t u_1 - \Delta_x \left[(D_1 + A_{11}\, u_1^{\alpha_{11}} + A_{12}\, u_2^{\alpha_{12}})\, u_1 \right] = u_1 \left(r_1 - S_{11}\, u_1^{\beta_{11}} - S_{12}\, u_2^{\beta_{12}} \right),$$

$$\partial_t u_2 - \Delta_x \left[(D_2 + A_{21}\, u_1^{\alpha_{21}} + A_{22}\, u_2^{\alpha_{22}})\, u_2 \right] = u_2 \left(r_2 - S_{21}\, u_1^{\beta_{21}} - S_{22}\, u_2^{\beta_{22}} \right),$$

with Neumann boundary conditions

$$\forall x \in \partial\Omega, \qquad \nabla_x u_1 \cdot n(x) = 0, \qquad \nabla_x u_2 \cdot n(x) = 0,$$

and $A_{12} > 0$, $A_{21} > 0$.

A more recent result

Theorem (LD, Lepoutre, Moussa) We assume that $D_i > 0$, $r_i \geq 0$, $A_{ij} \geq 0$, $A_{12} > 0$, $A_{21} > 0$, and $S_{ij} > 0$.
We assume that $\alpha_{11}, \alpha_{22} \geq 0$, $\alpha_{12}, \alpha_{21} \in]0, 1[$, and, for $i \neq j$,

$$0 < \beta_{ii} < 1, \qquad 0 < \beta_{ij} < \alpha_{ij}/2,$$

Let (u_{10}, u_{20}) be initial data in $L^2(\Omega)$, then there exists a weak solution to the system

$$\partial_t u_1 - \Delta_x \left[(D_1 + A_{11}\, u_1^{\alpha_{11}} + A_{12}\, u_2^{\alpha_{12}})\, u_1 \right] = u_1 \left(r_1 - S_{11}\, u_1^{\beta_{11}} - S_{12}\, u_2^{\beta_{12}} \right),$$

$$\partial_t u_2 - \Delta_x \left[(D_2 + A_{21}\, u_1^{\alpha_{21}} + A_{22}\, u_2^{\alpha_{22}})\, u_2 \right] = u_2 \left(r_2 - S_{21}\, u_1^{\beta_{21}} - S_{22}\, u_2^{\beta_{22}} \right),$$

with Neumann boundary conditions, and these initial data.

Main a priori estimates used in the proof

Entropy (Lyapounov) estimate, $\Omega_T := [0, T] \times \Omega$:

$$J^*(u_1, u_2)(T) + 4 \sum_{i \neq j} A_{ij} D_j \int_0^T \int_\Omega \left| \nabla_x \sqrt{u_j^{\alpha_{ij}}} \right|^2 + 4 A_{12} A_{21} \int_0^T \int_\Omega \left| \nabla_x \sqrt{u_2^{\alpha_{12}} u_1^{\alpha_{21}}} \right|^2$$

$$+ 2 \sum_{i \neq j} A_{ij} \alpha_{ij} A_{jj} \frac{\alpha_{jj} + 1}{\alpha_{jj} + \alpha_{ij}} \int_0^T \int_\Omega \left| \nabla_x \sqrt{u_j^{\alpha_{jj} + \alpha_{ij}}} \right|^2 \leq J^*(u_{10}, u_{20}) + C(T, \Omega),$$

where

$$J^*(u_1, u_2) := \sum_{i \neq j} \frac{A_{ij}\, \alpha_{ij}}{1 - \alpha_{ij}} \int_\Omega \left[\left(u_j - \frac{u_j^{\alpha_{ij}}}{\alpha_{ij}} \right) - \left(1 - \frac{1}{\alpha_{ij}} \right) \right].$$

General equation

$$\partial_t U - \Delta_x[A(U)] = R(U),$$

with $A, R : \mathbb{R}^I \to \mathbb{R}^I$, and $U := U(t,x) : \mathbb{R}_+ \times \Omega \, (\Omega \subset \mathbb{R}^N) \to (\mathbb{R}_+)^I$.

For any $\Phi : (\mathbb{R}_+)^I \to \mathbb{R}_+$, if $R = 0$, and $\langle \;,\; \rangle$ is the Euclidian scalar product on $\mathbb{R}^I$,

$$\frac{d}{dt} \int_\Omega \Phi(U) = \int_\Omega \langle \nabla\Phi(U), \Delta_x[A(U)] \rangle$$

$$= -\sum_{j=1}^N \int_\Omega \langle \partial_{x_j} U, D^2\Phi(U) D(A)(U) \partial_{x_j} U \rangle \leq 0.$$

We say that Φ is an entropy when $(D^2\Phi(U)\, D(A)(U))^{sym} \geq 0$.

Proposition (LD, Lepoutre, Moussa, Trescases): Consider $a_1, a_2 : \mathbb{R}_+^* \to \mathbb{R}_+$ two C^1 functions, and

$$A(X) := \begin{pmatrix} x_1\, a_1(x_2) \\ x_2\, a_2(x_1) \end{pmatrix}.$$

We assume that a_1, a_2 are increasing and $\mathrm{Det}\, D(A) \geq 0$, that is

$$\forall x_1, x_2 > 0, \qquad a_1(x_2)\, a_2(x_1) - x_1\, x_2\, a_1'(x_2)\, a_2'(x_1).$$

Then taking

$$\Phi(X) := \phi_1(x_1) + \phi_2(x_2),$$

where ϕ_i is a nonnegative second primitive of $z \mapsto a_j'(z)/z$ ($i \neq j$), we get an entropy.

The entropic structure

Proof: We compute

$$D(A)(X) = \begin{pmatrix} a_1(x_2) & x_1\, a_1'(x_2) \\ x_2\, a_2'(x_1) & a_2(x_1) \end{pmatrix}, \quad D^2(\Phi)(X) = \begin{pmatrix} \frac{a_2'(x_1)}{x_1} & 0 \\ 0 & \frac{a_1'(x_2)}{x_2} \end{pmatrix},$$

so that

$$M(X) := D^2(\Phi)\, D(A)(X) = \begin{pmatrix} \star & a_2'(x_1) a_1'(x_2) \\ a_1'(x_2) a_2'(x_1) & \star \end{pmatrix}$$

is obviously symmetric. Since the functions a_i are increasing, all the coefficients of $M(X)$ are nonnegative, so that $\mathrm{Tr}\, M(X) \geq 0$; we also see that

$$\mathrm{Det}\, M(X) = \mathrm{Det}\, D^2(\Phi)(X)\, \mathrm{Det}\, D(A)(X) \geq 0.$$

Theorem (LD, Lepoutre, Moussa, Trescases) We assume that $D_i > 0$, $r_i \geq 0$, $A_{ij} \geq 0$, $A_{12} > 0$, $A_{21} > 0$, and $S_{ij} > 0$.
We assume that $\alpha_{11}, \alpha_{22} \geq 0$, $\alpha_{12} > 1$, $\alpha_{21} < 1$, and, for $i \neq j$,

$$0 < \beta_{ii} < 1, \qquad 0 < \beta_{ij} < \alpha_{ij}/2,$$

Let (u_{10}, u_{20}) be initial data in $L^1 \cap H^{-1}(\Omega) \times L^{\alpha_{12}} \cap H^{-1}(\Omega)$, then there exists a weak solution to the system

$$\partial_t u_1 - \Delta_x \left[(D_1 + A_{11}\, u_1^{\alpha_{11}} + A_{12}\, u_2^{\alpha_{12}})\, u_1 \right] = u_1 \left(r_1 - S_{11}\, u_1^{\beta_{11}} - S_{12}\, u_2^{\beta_{12}} \right),$$

$$\partial_t u_2 - \Delta_x \left[(D_2 + A_{21}\, u_1^{\alpha_{21}} + A_{22}\, u_2^{\alpha_{22}})\, u_2 \right] = u_2 \left(r_2 - S_{21}\, u_1^{\beta_{21}} - S_{22}\, u_2^{\beta_{22}} \right),$$

with Neumann boundary conditions, and these initial data.

Other systems for which existence holds

Higher exponents but self-diffusion dominating cross-diffusion

$$\partial_t u_1 - \Delta_x [u_1 \, (D_1 + A_{11} \, u_1^s + A_{12} \, u_2^s)] = 0,$$
$$\partial_t u_2 - \Delta_x [u_2 \, (D_2 + A_{21} \, u_1^s + A_{22} \, u_2^s)] = 0,$$

with $s > 1$ and

$$A_{11} \, A_{22} \geq \left(\frac{s-1}{s+1} \right)^2 A_{12} \, A_{21}.$$

Significantly better than the result obtain by Jüngel.

Other systems for which existence holds

More than two equations

$$\partial_t u_1 - \Delta_x [u_1 \, (D_1 + u_2^s + u_3^s)] = 0,$$

$$\partial_t u_2 - \Delta_x [u_2 \, (D_2 + u_1^s + u_3^s)] = 0,$$

$$\partial_t u_3 - \Delta_x [u_3 \, (D_3 + u_1^s + u_2^s)] = 0,$$

for $0 < s < 1/\sqrt{3}$ and $D_1, D_2, D_3 > 0$.

Possible extensions and conjectures

- Almost nothing is known on systems of three or more equations
- Existence maybe does not hold when $\alpha_{12}\,\alpha_{21} > 1$ and when there is no self diffusion
- Nothing is known (or even conjectured!) for reaction terms with higher exponents