Concentration of Measure

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CIRM workshop, 29 Jan 2016

What is concentration?

"A random variable that depends in a smooth way on many independent random variables (but not too much on any of them) is essentially constant."
- M. Talagrand, 1996.

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- We saw how the entropy method and log-Sobolev inequalities showed sub-Gaussian tails
- Today, we'll study the transportation method which uses a beautiful idea of coupling

Transportation lemma

Let P be a probability measure on Ω and $Z : \Omega \mapsto \mathbb{R}$ be any random variable.

The following are equivalent:

$$\log \mathbb{E} e^{\lambda(Z-\mathbb{E}Z)} \leq \frac{\lambda^2\sigma^2}{2}, \ \ \forall\, \lambda > 0$$

$$\mathbb{E}_Q Z - \mathbb{E}Z \leq \sqrt{2\sigma^2 D(Q||P)}, \ \ \forall\, Q << P$$

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Coupling

Given two probability distributions P and Q on Ω, a coupling M of P and Q is a probability distribution on $\Omega \times \Omega$ whose marginals are respectively P and Q.

Let $\mathscr{C}(P, Q)$ be the set of all couplings of P and Q

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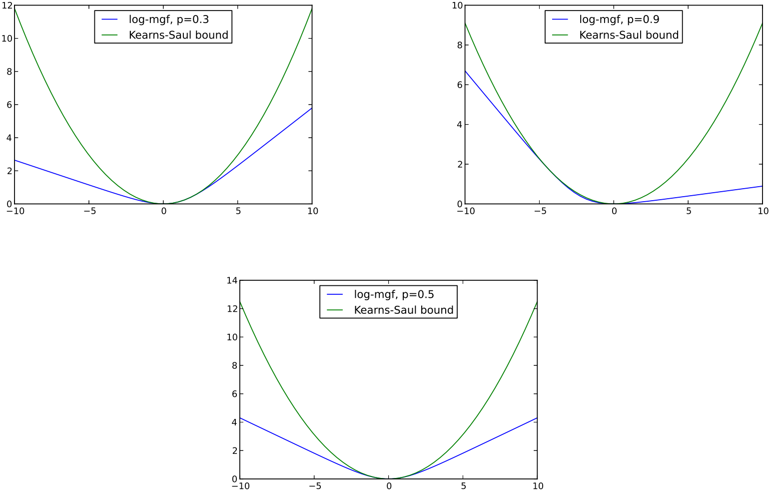
The two statements above are equivalent.

Kearns-Saul bound plots



Transportation method for bounded differences inequality

Suppose f satisfies the bounded difference property: a change in the i^{th} co-ordinate can change f by at most c_i.

$$\begin{aligned}
\mathbb{E}_Q Z - \mathbb{E} Z =& \mathbb{E}_M[f(Y) - f(X)] \\
\leq& \mathbb{E}_M[\sum_{i=1}^{n} c_i 1_{X_i \neq Y_i}] \\
=& \sum_{i=1}^{n} c_i \mathbb{P}_M[X_i \neq Y_i] \\
\leq& \sqrt{\sum_{i=1}^{n} c_i^2} \sqrt{\sum_{i=1}^{n} \mathbb{P}_M[X_i \neq Y_i]^2}
\end{aligned}$$

Marton's transportation inequality

If $P = P_1 \times P_2 \times \ldots \times P_n$, and $Q << P$, and

$$X = (X_1, X_2, \ldots, X_n) \sim P,$$

$$Y = (Y_1, Y_2, \ldots, Y_n) \sim Q,$$

then

$$\inf_{M \in \mathscr{C}(P,Q), (X,Y) \sim M} \sum_{i=1}^{n} \mathbb{P}_M (X_i \neq Y_i)^2 \leq \frac{1}{2} D(Q||P),$$

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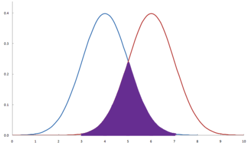

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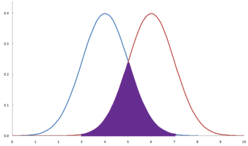

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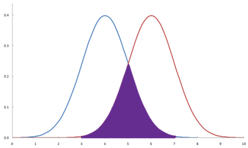

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\begin{aligned}
& M(x,y) \\
=& (1-V) \cdot \frac{\min\{p(x), q(x)\} 1_{x=y}}{1-V} \\
& + V \cdot \frac{(p(x) - q(x))_+}{V} \cdot \frac{(q(y) - p(y))_+}{V}
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Average over y_1, apply Jensen on LHS and chain rule of relative entropy on RHS:

$$\sum_{i=1}^{2} \mathbb{P}_M(X_i \neq Y_i)^2 \leq \frac{1}{2} D(Q||P)$$

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If $Z = f(X_1, \ldots, X_n)$ and $X_1, X_2 \ldots, X_n$ are independent, then Z is σ^2-sub-Gaussian:

$$\log \mathbb{E} e^{\lambda(Z - \mathbb{E}Z)} \leq \frac{\lambda^2 \sigma^2}{2}, \;\; \forall \;\; \lambda \in \mathbb{R}$$

Then, $\mathbb{P}[Z - \mathbb{E}Z \geq t], \mathbb{P}[Z - \mathbb{E}Z \leq -t] \leq e^{-\frac{t^2}{2\sigma^2}}, \; \forall t > 0$

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Prove sub-Gaussianity by the equivalent transportation description:

$$\mathbb{E}_Q Z - \mathbb{E}Z \le \sqrt{2\sigma^2 D(Q||P)}, \ \ \forall \ \ Q << P$$

$\mathbb{E}_Q Z - \mathbb{E} Z$

$$\mathbb{E}_Q Z - \mathbb{E} Z = \mathbb{E}_M[f(Y) - f(X)]$$

$$\begin{aligned}\mathbb{E}_Q Z - \mathbb{E} Z &= \mathbb{E}_M[f(Y) - f(X)] \\ &\leq \mathbb{E}_M[\sum_{i=1}^{n} c_i(X) 1_{X_i \neq Y_i}]\end{aligned}$$

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Need to show the second quantity in $\sqrt{}$ is $\leq 2D(Q||P)$

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This gives BOTH left and right tails!

Spectral norm of a random matrix

Populate an $m \times n$ matrix A by independent entries, each taking values in $[0, 1]$.

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So, $\mathbb{P}[|Z - \mathbb{E}Z| \geq t] \leq 2e^{-t^2/32} \ \forall t > 0$

Talagrand's convex distance inequality

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then $d_T(x, A)$ is the minimum Euclidean distance of the origin from the convex hull of $V(x, A)$ (hence the name 'convex distance').

Application: Longest increasing subsequence problem

Let $X_1, X_2, \ldots, X_n$ be independent, each drawn uniformly from $[0, 1]$.

A sequence $i_1 < i_2 < \ldots < i_r$ constitutes an increasing subsequence if $X_{i_1} < X_{i_2} < \ldots < X_{i_r}$.

What is the behavior of the length $Z = f(X_1, X_2, \ldots, X_n)$ of the longest increasing subsequence?

Erdös-Szekeres theorem (1935)

If m^2+1 people of different heights stand in a line, there exists among them either a monotonically increasing subsequence of length $m+1$ or a monotonically decreasing subsequence of length $m+1$

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Therefore, $\mathbb{E}Z \geq \frac{\lfloor \sqrt{n-1} \rfloor}{2}$

$$\mathbb{P}[Z \geq k]$$

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\begin{aligned}
&\mathbb{P}[Z \geq k] \\
= &\mathbb{P}[\text{at least one increasing subsequence of length} = k]
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\mathbb{E}Z \leq \mathbb{P}[Z \leq k]k + \mathbb{P}[Z \geq k]n
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\mathbb{E}Z &\leq \mathbb{P}[Z \leq k]k + \mathbb{P}[Z \geq k]n \\
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Thus, $\mathbb{E}Z = \Theta(\sqrt{n})$.

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Talagrand's convex distance inequality can show sub-Gaussian tail bounds with typical deviation $O(n^{1/4})$

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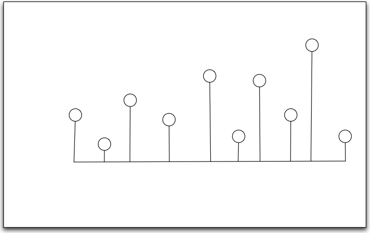

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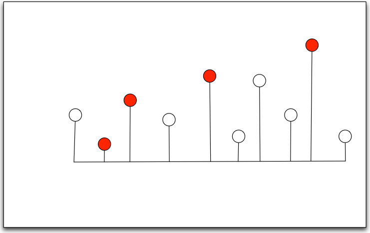

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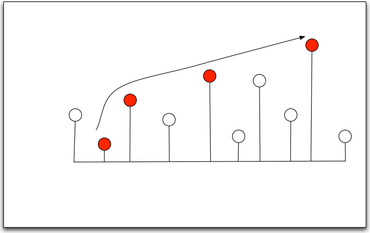

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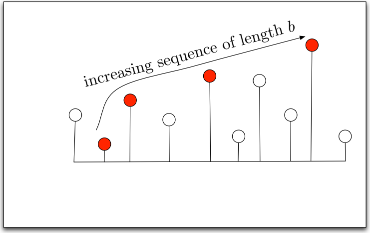

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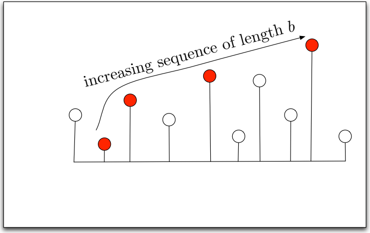

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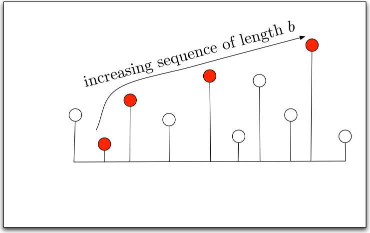

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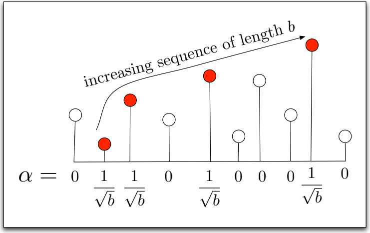

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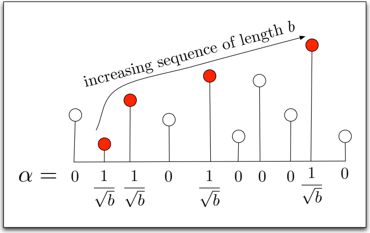

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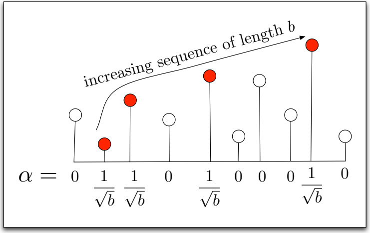

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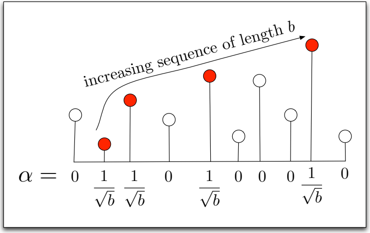

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Thus, $d_T(x, A) > t$ and so, $B \subseteq A_t^c$.

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Actually, $Z = 2\sqrt{n} + O(n^{1/6})$ and the limiting distribution of $\dfrac{Z - 2\sqrt{n}}{n^{1/6}}$ is known (Baik, Deift, Johansson, 1999)

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(All slides available on my webpage)