Planar Hilbert schemes and L^2 cohomology

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Main result

Theorem (Conjecture of Vafa and Witten)

The L^2 cohomology of the planar Hilbert scheme for n points, Hilb^n, *with respect to the Joyce-Nakajima hyper-Kähler metric is concentrated in the middle degree and is the image of the relative in the absolute cohomology.*

- A more general theorem of Hitchin, [4], implies the first part of this result, that the L^2 cohomology is confined to the middle degree.
- The case $n = 1$ is trivial. For $n = 2$ and $n = 3$ this theorem is due to Gilles Carron [3, 2].
- I would like to acknowledge the decisive contributions made by Gilles Carron and Hiraku Nakajima to the proof of this result and note the close relationship with the work of Anda Degeratu and Rafe Mazzeo.

Outline of proof of remaining part

- Using Nakajima's hyper-Kähler construction the centred quotient Hilb_0^n we find a compactification to a manifold with corners $\mathbb{H}_0^n$.
- The boundary hypersurfaces $B_{\bar{n}}$ of $\mathbb{H}^n$ are labelled by partitions $\bar{n}$ of $n = n_1 + \cdots + n_k$.
- These boundary hypersurfaces have an iterated fibration structure, $B_{\bar{n}} = \mathbb{X}_k \times \prod_{i=1}^{k} \mathbb{H}_0^{n_1}$.

Theorem

The Joyce-Nakajima hyper-Kähler metric is an iterated fibred boundary metric with fibre part the same metric for $\mathbb{H}^k$.

- The Laplacian and Hodge operator $d + \delta$ can be fully analyzed using the appropriate iterated calculus and shown to be Fredholm on L^2.
- The crucial Fredholm property for $d + \delta$ can be proved more directly using a simpler adiabatic calculus.

Local form of the metric

- The Euclidean metric is a scattering metric, near the boundary of $\overline{\mathbb{C}^N}$ it takes the form
$$\frac{dx^2}{x^4} + \frac{h}{x^2}$$
with h a tangential metric.
- An iterated fibred boundary metric near a corner takes the form
$$\frac{dX^2}{X^4} + \frac{dx_2^2 + x_2^2\left(h_2 + x_3^2(dx_3^2 + x_3^2 h_3\right)}{X^2}$$
where h_2 and h_3 are metrics in tangential variables $y^{(2)}$ and $y^{(3)}$.
- Here X is a total boundary defining function, either
$$X = x_1 x_2 x_3 \text{ or } X = \zeta x_2 x_3,\ \zeta > 0 \text{ a local coordinate.}$$
- Of course there are additional cross-terms but these vanish at the boundaries.

Stability properties of iterated fibred boundary metrics

- This class of metrics represents, I claim, a smooth realization (and generalization) of the notion of QALE, '*Q*uasi *A*symptotically *L*ocally *E*uclidean', structures due to Joyce.
- They are stable under the blow up of a boundary p-submanifold transversal to the fibrations.
- They are stable under restriction to interior p-submanifolds transversal to all the fibrations.
- They are stable under passage to the quotient if invariant under the free action of a compact Lie group.
- The construction of the full and adiabatic calculi proceeds by quite natural resolutions of the fibrations lifted to the products.

Planar Hilbert schemes

- The planar Hilbert scheme is the algebraic variety formed by the ideals in the ring of polynomials in two complex variables, which have length n :

$$\mathrm{Hilb}^n = \{I \subset \mathbb{C}[z_1, z_2]; \dim\left(\mathbb{C}[z_1, z_2]/I\right) = n\}. \tag{1}$$

- Hilb^n is a smooth algebraic variety (smoothness fails in higher dimension).
- The space $\mathbb{C}^2$ acts freely by translation and the quotient is the centred variety

$$\mathrm{Hilb}^n_0 = \mathrm{Hilb}^n / \mathbb{C}^2. \tag{2}$$

- The Theorem extends unchanged to the centred case.

Hilbert-Chow map

- There is a 'support' or Hilbert-Chow map
$$\mathrm{Hilb}_0^n \longrightarrow \mathrm{Sym}_0^n = \{(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^2)^n; \sum_j \lambda_j = 0\}/\Sigma_n$$
to the configuration space of unordered, centred, n-tuples in $\mathbb{C}^n$.
- This map is a crepant (read small) resolution.
- As a group quotient this is a stratified space with the strata corresponding to cluster data capturing the isotropy types of the action of Σ_n

$$\sigma = \{\sigma_1, \ldots, \sigma_k\},\ \{1, \ldots, n\} = \bigcup_{i=1}^{k} \sigma_i,\ \sigma_i \cap \sigma_j = \emptyset \text{ if } i \neq j,$$
$$D_\sigma = \{(\lambda, \mu) \in D; (\lambda_q, \mu_q) = (\lambda_p, \mu_p) \Longleftrightarrow q, p \in \sigma_i \text{ for some } i\},$$
$$G_\sigma = \prod_i \Sigma_{|\sigma_i|}$$

Resolution of the action of Σ_n

- Take the radial compactification of $(\mathbb{C}^2)^n = \mathbb{C}^{2n}$ to a closed ball, to which the symmetric action extends smoothly with the isotropy types the radial compactifications of the D_σ.
- Pierre Albin and I in [1] have codified a standard real resolution of compact group actions on manifolds with boundary which applies here and in which the $\overline{D_n}$ are iteratively blown up, in the real sense.
- The radial $\mathbb{R}_+$ (and indeed $\mathbb{C}^*$) action extends to give a product decomposition of the resolved space

$$\widehat{\overline{(\mathbb{C}^2)^n}} = \mathbb{X}^n \times [0, \infty) \tag{3}$$

with $\mathbb{X}^n$ a compact manifold with corners of real dimension $4n - 5$ carrying a principal circle action.

Geometry of isotropy resolution of $\overline{(\mathbb{C}^2)^n}$

- The boundary hypersurfaces, $\mathbb{X}^n_\sigma \subset \mathbb{X}^n$, of real dimension $4n-6$, are labelled by the non-singleton cluster data and are trivial bundles with base $\mathbb{X}^{k(\sigma)}$

$$\mathbb{X}^n_\sigma \simeq \left(\prod_{i=1}^{k(\sigma)} \mathbb{X}^{n_i} \right) \times \mathbb{X}^{k(\sigma)}. \tag{4}$$

- Here $\mathbb{X}^{k(\sigma)}$ arises as the base in the blow-up of $\overline{D_\sigma}$ which is blown up corresponding to the cluster data coarser than σ.
- The fibre arises from the blow-ups along the coincidence subspaces corresponding to finer cluster data, i.e. to subclusters within the σ_i, giving the isotropy structure for the action on the normal bundle to $\overline{D_\sigma}$.
- András Vasy has treated similar geometry for the n-body problem.

Resolution of Sym^n

- By construction Σ_n now acts freely and the quotient is a resolution and compactification of Sym_0^n with similar structure.
- Namely it is a manifold with corners which is a product

$$\mathbb{Y}^n \times [0,1] \longrightarrow \overline{\mathrm{Sym}_0^n},\ \mathbb{Y}^n = \mathbb{X}^n/\Sigma_n. \tag{5}$$

- The boundary faces of $\mathbb{Y}^n$ are labelled by the partitions

$$\overline{n} = \{n_1, \dots, n_k\},\ \sum_{i=1}^{k} n_i = n,\ n_i \geq 1 \tag{6}$$

  and arise from the free action of Σ_n on (4) where the isotropy group Σ_k acts on the base
- Again this is a product or trivial bundle

$$\mathbb{Y}^n_{\overline{n}} = \left(\prod_{i=1}^{n} \mathbb{Y}^{n_i}\right) \times \mathbb{Y}^k. \tag{7}$$

Nakajima's construction I

- Nakajima showed how to construct Hilb_0^n as a finite-dimensional hyper-Kähler quotient giving it a hyper-Kähler metric.
- I will only consider the second stage of this, after the passage to the zero level surface of the complex moment map.
- Consider the linear space with $\mathrm{U}(n)$ action

$$\mathbb{M} = \mathbb{M}^n = M_0(n) \times M_0(n) \times \mathbb{C}^n \times [0, \infty),$$
$$M_0(n) = \{\text{Complex } n \times n \text{ matrices, } A, \text{ with } \operatorname{tr}(A) = 0\}$$
$$\mathbb{M} \ni (A, B, x, t) \longrightarrow (UAU^*, UBU^*, Ux, t) \in \mathbb{M}.$$

- For each $t > 0$ consider the smooth submanifold

$$\mathbb{F}_t = \{(A, B, x, t) \in \mathbb{M};\ t > 0,$$
$$[A, B] = 0,\ [A, A^*] + [B, B^*] + xx^* = t^2 \operatorname{Id}\}$$

Nakajima's construction II

- Then
$$\mathrm{Hilb}_0^n \simeq \mathbb{F}_t / \mathrm{U}(n).$$
- The ideal corresponding to a point in $\mathbb{F}_t$ consists of the polynomials $p(z_1, z_2)$ such that
$$p(A, B)x = 0. \tag{8}$$
and this generates the isomorphism.
- The hyper-Kähler metric which is the object of study here is simply the restriction of the $\mathrm{U}(n)$-invariant Euclidean metric on $M_0 \times M_0 \times \mathbb{C}^n$ restricted to $\mathbb{F}_t$.

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