Witten's perturbation on strata with general adapted metrics

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General idea

Setting Strata of compact stratified spaces with general adapted metrics.

Main goal Witten's perturbation of the de Rham complex $\rightsquigarrow$ Morse inequalities.

Main analytic tool A perturbation of the Dunkl harmonic oscillator.

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- $\mathfrak{H}$: graded Hilbert space.
- Hilbert complex (on $\mathfrak{H}$): a differential complex given by a closed densely defined operator $\mathbf{d}$ in $\mathfrak{H}$ (Brüning-Lesch).
- $\rightsquigarrow$ Laplacian: $\boldsymbol{\Delta} = \mathbf{d}\mathbf{d}^* + \mathbf{d}^*\mathbf{d}$ is self-adjoint in $\mathfrak{H}$.
- Smooth core of $\boldsymbol{\Delta}$: $\mathrm{D}^\infty(\boldsymbol{\Delta}) = \bigcap_m \mathrm{D}(\boldsymbol{\Delta}^m)$.
- $\rightsquigarrow$ smooth subcomplex: $(\mathrm{D}^\infty(\boldsymbol{\Delta}), \mathbf{d})$ determines $\mathbf{d}$ and has the same homology.

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Ideal boundary condition

- M : Riemannian manifold, possibly non-complete.
- d, δ : de Rham differential and codifferential on $\Omega_0(M)$ (compactly supported forms).
- Ideal boundary condition (i.b.c.) of d: extension of d to a Hilbert complex $\mathbf{d}$ in $L^2\Omega(M)$.
- $\exists$ min/max i.b.c.: $d_{\min} = \overline{d}$, $d_{\max} = \delta^*$.
- $\rightsquigarrow \Delta_{\min/\max}$.
- M oriented $\implies \star : \Delta_{\min} \xrightarrow{\cong} \Delta_{\max}$.

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Ideal boundary condition (contd.)

- $d_{\mathrm{min/max}} \rightsquigarrow \begin{cases} H^r_{\mathrm{min/max}}(M), \\ \beta^r_{\mathrm{min/max}}(M), \\ \chi_{\mathrm{min/max}}(M) \quad (\text{if } \beta^r_{\mathrm{min/max}} < \infty\ \forall r). \end{cases}$
- These are quasi-isometric invariants.
- $H^r_{\mathrm{max}}(M) = H^r_{(2)}(M)$ (the L^2 cohomology).
- M complete $\implies d_{\mathrm{min}} = d_{\mathrm{max}}$,
  but assume that M may not be complete.

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- (Thom-Mather) stratified space: a space A with a partition into (C^∞) manifolds (strata) satisfying certain conditions.
- In particular, $\forall$ stratum X, $\overline{X} = \bigcup$ strata.
- $\rightsquigarrow$ order relation of the strata: $X \leq Y$ if $X \subset \overline{Y}$.
- $\rightsquigarrow$ depth of a stratum X: maximum ℓ such that $\exists$ a chain of strata $X_0 < X_1 < \cdots < X_\ell = X$.
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Charts

- M: stratum of A. Assume $A = \overline{M}$.
- A chart of A centered at $x \in X < M$:

$$A \underset{\text{open}}{\supset} O \equiv O' \underset{\text{open}}{\subset} \mathbb{R}^m \times c(L) \,,$$

  where
  - L: a compact stratified space of lower depth;
  - $c(L) = \frac{L \times [0,\infty)}{L \times \{0\}}$: cone with link L;
  - $x \equiv (0, *)$: $* = L \times \{0\} \in c(L)$: vertex;
  - $M \cap O \equiv M' \cap O'$, $M' = \mathbb{R}^m \times N \times \mathbb{R}^+$, N stratum of L;
  - $m = m_X = \dim X$, $L = L_X$.
- $\rho : c(L) \to [0, \infty)$: radial function, it's induced by $\mathrm{pr}_2 : L \times [0, \infty) \to [0, \infty)$.
- $\rightsquigarrow (\| \|^2_{\mathbb{R}^m} + \rho^2)^{1/2}$: radial function of $\mathbb{R}^m \times c(L)$.
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Adapted metrics on strata

- A Riemannian metric g on M is called adapted if $\exists$ a chart centered at any $x \in X < M$,

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$$O \cap M \equiv O' \cap M' \;, \quad M' = \mathbb{R}^m \times N \times \mathbb{R}^+ \;, \quad \text{such that}$$
$$g \sim g_0 + \rho^{2u}\tilde{g} + d\rho^2 \quad \text{on} \quad O \cap M \equiv O' \cap M' \;,$$

where
  - g_0: Euclidean metric of $\mathbb{R}^m$,
  - $\tilde{g}$ is an adapted metric on N (induction on the depth),
  - $u = u_k > 0, \quad k := \operatorname{codim} X = \dim N + 1$.
- If A is a pseudomanifold ($\nexists$ strata of codim 1) $\rightsquigarrow \hat{u} := (u_2, \ldots, u_n)$: the type of g.

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Intersection homology

- Perversity : a sequence $\bar{p} = (p_2, p_3, \dots)$ in $\mathbb{N}$ such that $p_2 = 0, \quad p_k \le p_{k+1} \le p_k + 1$.
- Examples : $\bar{0} = (0, 0, \dots)$, $\quad \bar{t} = (0, 1, 2, 3, \dots)$ (top), $\bar{m} = (0, 0, 1, 1, 2, 2, 3, \dots)$, $\quad \bar{n} = (0, 1, 1, 2, 2, 3, 3, \dots)$.
- $\bar{p}$ and $\bar{q}$ are complementary if $\bar{p} + \bar{q} = \bar{t}$.
- $\rightsquigarrow I^{\bar{p}}H_*(A) = I^{\bar{p}}H_*(A; \mathbb{R})$: intersection homology with perversity $\bar{p}$ (Goresky & MacPherson, 1980).
- $\beta_r^{\bar{p}} = \dim I^{\bar{p}}H_r(A)$, $\quad \chi^{\bar{p}} = \sum_{r=0}^{n} (-1)^r \beta_r^{\bar{p}}$.
- $I^{\bar{p}}H_r(A) \cong I^{\bar{q}}H_{n-r}(A)$ $\quad$ if $\bar{p}$ and $\bar{q}$ are complementary and A is oriented (M is oriented).

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- Examples : $\bar{0} = (0, 0, \dots), \quad \bar{t} = (0, 1, 2, 3, \dots)$ (top), $\bar{m} = (0, 0, 1, 1, 2, 2, 3, \dots), \quad \bar{n} = (0, 1, 1, 2, 2, 3, 3, \dots)$.
- $\bar{p}$ and $\bar{q}$ are complementary if $\bar{p} + \bar{q} = \bar{t}$.
- $\rightsquigarrow I^{\bar{p}}H_*(A) = I^{\bar{p}}H_*(A; \mathbb{R})$: intersection homology with perversity $\bar{p}$ (Goresky & MacPherson, 1980).
- $\beta_r^{\bar{p}} = \dim I^{\bar{p}}H_r(A), \quad \chi^{\bar{p}} = \sum_{r=0}^{n} (-1)^r \beta_r^{\bar{p}}$.
- $I^{\bar{p}}H_r(A) \cong I^{\bar{q}}H_{n-r}(A)$ if $\bar{p}$ and $\bar{q}$ are complementary and A is oriented (M is oriented).

Adapted metrics & intersection homology

- A Riemannian metric g on M of type $\hat{u} = (u_2, \ldots, u_n)$ is associated to a perversity $\bar{p} \leq \bar{m}$ if :

$$\begin{array}{ll} \frac{1}{k-1-2p_k} \leq u_k < \frac{1}{k-3-2p_k} & \text{if} \quad 2p_k \leq k-3\,, \\ 1 \leq u_k < \infty & \text{if} \quad 2p_k = k-2\,. \end{array}$$

- $\rightsquigarrow I^{\bar{p}}H_r(A)^* \cong H^r_{(2)}(M) = H^r_{\max}(M) \quad \forall \bar{p} \leq \bar{m}$
Nagase 1983, 1986,
Cheeger-Goresky-MacPherson 1982 in the case of $\bar{m}$ and conic metrics ($u_k = 1 \ \forall\, k$).
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General charts

- The product of two stratified spaces has a non-canonical stratified structure. The product of two cones is a cone.
- $\rightsquigarrow$ general chart centered at $x \in X < N$,

$$A \underset{\text{open}}{\supset} O \equiv O' \underset{\text{open}}{\subset} \mathbb{R}^m \times \prod_{i=1}^{a} c(L_i) ,$$

where
  - L_i: a compact stratified space of lower depth;
  - $x \equiv (0, *_1, \dots, *_a)$, $*_i$: vertex of $c(L_i)$;
  - $O \cap M \equiv O' \cap M'$, $M' = \mathbb{R}^m \times \prod_i (N_i \times \mathbb{R}^+)$, N_i stratum of L_i;
  - $m = m_X$, $a = a_X$, $L_i = L_{X,i}$.
- $\rho_i : c(L_i) \to [0, \infty)$: radial function.
- $\rightsquigarrow (\| \|^2_{\mathbb{R}^m} + \sum_i \rho_i^2)^{1/2}$: radial function of $\mathbb{R}^m \times \prod_{i=1}^a c(L_i)$.

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General adapted metrics on strata

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General adapted metrics on strata (contd)

- The class of general adapted metrics is preserved by products.
- $k_i := \dim N_i + 1$.
- g is called good if

$$u_i \leq 1 ,$$

$$\frac{1}{k_i} \leq u_i < 1 \Longrightarrow \frac{1}{u_i} \in 2\mathbb{Z} + k_i + (0, 1] .$$

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$\rightsquigarrow$ to leave out this condition and make some corrections.

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Spectrum $\Delta_{\min/\max}$

Theorem (up to reform)

For any good general adapted metric on a stratum of a compact stratified space:

(i) $\Delta_{\min/\max}$ *has a discrete spectrum:* $\lambda_{\min/\max,k}$.

(ii) $\exists\, \theta > 0$ *such that* $\liminf_k \frac{\lambda_{\min/\max,k}}{k^\theta} > 0$.

- Cheeger: (i) in the case of pseudomanifolds with conic metrics.
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Contents

Relatively admissible functions

- $f \in C^\infty(M)$ is rel-admissible if $|df|$ y $|\mathrm{Hess}\, f|$ are bounded.
- $\rightsquigarrow$ f has a continuous extension to the metric completion $\widehat{M}$ of M (possibly not to $\overline{M}$).
- $x \in \widehat{M}$ is a rel-critical point of f when $\exists\ (y_k)$ in M such that $y_k \to x$ in $\widehat{M}$ and $|df(y_k)| \to 0$.
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Relatively Morse functions

Rel-Morse function: a rel-admissible Morse function f on M so that, $\forall x \in \mathrm{Crit}_{\mathrm{rel}}(f)$, if $x \in X < M$, then $\exists$ a general chart of $\widehat{M}$ centered at x,

$$A \supset O \equiv O' \subset \mathbb{R}^m \times \prod_{i=1}^{a} c(L_i) , \quad \text{with} \quad O \cap M \equiv O \cap M' ,$$

$$M' = \mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times \prod_{i \in I_+} (N_i \times \mathbb{R}^+) \times \prod_{i \in I_+} (N_i \times \mathbb{R}^+) ,$$

$$\text{such that} \quad f|_O \equiv f(x) + \frac{1}{2}(\rho_+^2 - \rho_-^2)|_{O'} ,$$

where
- $m = m_+ + m_-$, $\quad \{1, \dots, a\} = I_+ \sqcup I_-$,
- N_i : a stratum of L_i,
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Relatively Morse functions

If $x \in \mathrm{Crit}_{\mathrm{rel}}(f) \cap M = \mathrm{Crit}(f)$, $\rightsquigarrow$ the usual description of the Morse function f around x:

$$A \supset O \equiv O' \subset \mathbb{R}^m = \mathbb{R}^{m_+} \times \mathbb{R}^{m_-} , \quad x \equiv 0 ;$$

$$f|_O \equiv f(x) + \frac{1}{2}(\rho_+^2 - \rho_-^2)|_{O'} ,$$

where $\rho_\pm = \| \ \|$ on $\mathbb{R}^{m_\pm}$.

The numbers $\nu^r_{\min/\max}$

- For $x \in \mathrm{Crit}_{\mathrm{rel}}(f)$, $x \in X < M$, as above, and $r = 0, \dots, n$: $\nu^r_{x,\max/\min} = \sum_{(r_1,\dots,r_a)} \prod_{i=1}^a \beta^{r_i}_{\max/\min}(N_i)$, where

$$r = m_- + \sum_{i=1}^a r_i + |I_-| \,,$$

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A version of Morse inequalities

Theorem (up to reform)

For any rel-Morse function on a stratum of dim n of a compact stratified space, equipped with a good general adapted metric,

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U. Ludwig: case of pseudomanifolds with conic metrics.

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A version of Morse inequalities (contd.)

Corollary (up to reform)

A: a compact pseudomanifold of dim n, $\bar{p}$: a perversity. If $\bar{p} \leq \bar{m}$, or if A is oriented and $\bar{p} \geq \bar{n}$, then, for any rel-Morse function on the regular stratum (w.r.t. any good adapted metric),

$$\sum_{r=0}^{k} (-1)^{k-r} \beta_r^{\bar{p}} \leq \sum_{r=0}^{k} (-1)^{k-r} \nu_r^{\bar{p}} \quad (0 \leq k < n) ,$$

$$\chi^{\bar{p}} = \sum_{r=0}^{n} (-1)^r \nu_r^{\bar{p}} .$$

In the Morse inequalities of Goresky-MacPherson, the functions and numbers are different.

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Contents

Witten's perturbation

- M: a Riemannian manifold, $f \in C^\infty(M)$.
- Witten's perturbations on $\Omega_0(M)$ $(s > 0)$:

$$d_s = e^{-sf}\, d\, e^{sf} = d + s\, df\wedge\,,$$

$$\delta_s = e^{sf}\, \delta\, e^{-sf} = \delta - s\, df\lrcorner\,,$$

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Dunkl harmonic oscillator

- Dunkl operator: T_σ on $C^\infty(\mathbb{R})$ defined by:

$$T_\sigma = \begin{cases} \frac{d}{dx} & \text{on even functions} \\ \frac{d}{dx} + 2\sigma\frac{1}{x} & \text{on odd functions.} \end{cases}$$

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Dunkl harmonic oscillator (contd.)

- Generalized Hermite polynomials: the sequence of orthonormal polynomials p_k for the measure $e^{-sx^2}|x|^{2\sigma}\,dx$ on $\mathbb{R}$ with positive leading coefficients.
- $\rightsquigarrow$ generalized Hermite functions: $\phi_k = p_k e^{-sx^2/2}$.
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- $L^2_\sigma := L^2(\mathbb{R}, |x|^{2\sigma}\,dx)$, $\langle\ ,\ \rangle_\sigma$, $\|\ \|_\sigma$.
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Dunkl harmonic oscillator (contd.)

- If $\sigma > -1/2$, then:
  - J, with $\mathrm{D}(J) = \mathcal{S}$, is essentially self-adjoint in L^2_σ.
  - Spectrum of $\overline{J}$: $\begin{cases} \text{eigenvalues} & (2k + 1 + 2\sigma)s \\ \text{eigenfunctions} & \phi_k \end{cases}$
  - $\mathrm{D}^\infty(\overline{J}) = \mathcal{S}$.
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Perturbations of the Dunkl harmonic oscillator

Theorem

Let $0 < u < 1, \quad \xi > 0, \quad \sigma > u - 1/2.$
Then $\exists$ *a positive self-adjoint operator* $\mathcal{U}$ *in* L^2_σ *such that:*

(i) $\mathcal{S}$ *is a core of* $\mathcal{U}^{1/2}$, *and,* $\forall\, \phi, \psi \in \mathcal{S}$,

$$\langle \mathcal{U}^{1/2}\phi, \mathcal{U}^{1/2}\psi\rangle_\sigma = \langle J\phi, \psi\rangle_\sigma + \xi\langle |x|^{-u}\phi, |x|^{-u}\psi\rangle_\sigma \ .$$

(ii) $\mathcal{U}$ *has a discrete spectrum:* λ_k. $\exists\, D = D(\sigma, u) > 0$, *and,* $\forall\, \epsilon > 0, \exists\, C = C(\epsilon, \sigma, u) > 0$ *so that,* $\forall\, k$,

$$\begin{aligned}(2k + 1 + 2\sigma)s + \xi D s^u (k+1)^{-u} &\le \lambda_k \\ &\le (2k + 1 + 2\sigma)(s + \xi\epsilon s^u) + \xi C s^u \ .\end{aligned}$$

Perturbations of the Dunkl harmonic oscillator (contd.)

Theorem (up to reform)

Let $0 < u < 1$, $\xi > 0$, $\eta \in \mathbb{R}$,
$\sigma > u - 1/2$, $\tau > u - 3/2$, $\theta > -1/2$.
A list of conditions is assumed depending on several cases.
Then $\exists$ *a positive self-adjoint operator* $\mathcal{V}$ *in* $L^2_{\sigma,\tau}$ *such that:*

(i) $\mathcal{S}$ *is a core of* $\mathcal{V}^{1/2}$*, and, for all* $\phi, \psi \in \mathcal{S}$*,*

$$\langle \mathcal{V}^{1/2}\phi, \mathcal{V}^{1/2}\psi\rangle_{\sigma,\tau} = \langle J_{\sigma,\tau}\phi, \psi\rangle_{\sigma,\tau} + \xi\langle |x|^{-u}\phi, |x|^{-u}\psi\rangle_{\sigma,\tau} + \eta\left(\langle x^{-1}\phi_{\mathrm{odd}}, \psi_{\mathrm{ev}}\rangle_\theta + \langle \phi_{\mathrm{ev}}, x^{-1}\psi_{\mathrm{odd}}\rangle_\theta\right) .$$

(ii) $\mathcal{U}$ *has a discrete spectrum satisfying estimates similar to the above ones.*

Perturbations of the Dunkl harmonic oscillator (contd.)

- $\mathcal{U} = \overline{U}$ for $U = J + \xi |x|^{-2u}$, $\mathrm{D}(U)$?
- $\mathcal{V} = \overline{V}$, where

$$V = \begin{pmatrix} U_{\sigma,\mathrm{ev}} & \eta |x|^{2(\theta-\sigma)} x^{-1} \\ \eta |x|^{2(\theta-\tau)} x^{-1} & U_{\tau,\mathrm{odd}} \end{pmatrix} = \begin{pmatrix} U_{\sigma,\mathrm{ev}} & \eta |x|^{2(\theta'-\sigma)} x \\ \eta |x|^{2(\theta'-\tau)} x & U_{\tau,\mathrm{odd}} \end{pmatrix},$$

$\theta' = \theta - 1/2 > -3/2$, $\mathrm{D}(V)$?

- Proof: perturbation theory of linear operators (Kato's book).

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Induced operators on $\mathbb{R}^+$

$$\left.\begin{array}{l}\text{Restriction to even/odd functions}\\ \text{restriction to } \mathbb{R}^+\\ \text{conjugation by powers of } x\end{array}\right\}$$

$$\rightsquigarrow \left\{\begin{array}{l}\text{induced operators on } \mathbb{R}^+\\ \text{selfadjoint extensions assuming conditions}\\ \text{a core of (selfadjoint extension)}^{1/2}\\ \text{discrete spectrum}\\ \text{eigenvalue estimates of the above type}\end{array}\right.$$

Induced operators on $\mathbb{R}^+$ (contd.)

Induced operators on $\mathbb{R}^+$:

$$P_0 = H - 2c_1 x^{-1}\frac{d}{dx} + c_2 x^{-2} ,$$

$$Q_0 = H - 2c_1 \frac{d}{dx} x^{-1} + c_2 x^{-2} \quad (c_1, c_2 \in \mathbb{R}) ,$$

$$P = P_0 + \xi x^{-2u} , \quad Q = Q_0 + \xi x^{-2u} \quad (0 < u < 1) ,$$

$$\begin{aligned} W &= \begin{pmatrix} P & \eta x^{2(\theta - c_1) - a - b - 1} \\ \eta x^{2(\theta - d_1) - a - b - 1} & Q \end{pmatrix} \\ &= \begin{pmatrix} P & \eta x^{2(\theta' - c_1) - a - b + 1} \\ \eta x^{2(\theta' - d_1) - a - b + 1} & Q \end{pmatrix} . \end{aligned}$$

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Contents

Local model

- $f = \frac{1}{2}(\rho_+^2 - \rho_-^2)$ on $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times \prod_{i\in I_+}(N_i \times \mathbb{R}^+) \times \prod_{i\in I_+}(N_i \times \mathbb{R}^+)$.
- Künneth formula $\rightsquigarrow f = \pm\frac{1}{2}\rho^2$ on $M = N \times \mathbb{R}^+$ in $c(L)$.
- $\tilde{n} = \dim N, \quad n = \dim M = \tilde{n} + 1$.
- $\tilde{g}$: general adapted metric on N.
- $g = \rho^{2u}\tilde{g} \oplus d\rho^2 \quad$ on $M, \quad u > 0$.
- $\tilde{d}, \tilde{\delta}, \widetilde{\Delta}$ on $N, \quad d, \delta, \Delta$ on M.
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- $f = \frac{1}{2}(\rho_+^2 - \rho_-^2)$ on $\mathbb{R}^{m_+} \times \mathbb{R}^{m_-} \times \prod_{i \in I_+}(N_i \times \mathbb{R}^+) \times \prod_{i \in I_+}(N_i \times \mathbb{R}^+)$.
- Künneth formula $\rightsquigarrow f = \pm\frac{1}{2}\rho^2$ on $M = N \times \mathbb{R}^+$ in $c(L)$.
- $\tilde{n} = \dim N$, $n = \dim M = \tilde{n} + 1$.
- $\tilde{g}$: general adapted metric on N.
- $g = \rho^{2u}\tilde{g} \oplus d\rho^2$ on M, $u > 0$.
- $\tilde{d}, \tilde{\delta}, \widetilde{\Delta}$ on N, d, δ, Δ on M.
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Induction hypothesis

- Assume the 1st main thm holds with smaller depth.
- $\rightsquigarrow$ spectral descomposition of the operator $\widetilde{\Delta}_{\text{min/max}}$ in $L^2\Omega^r(N)$ given by forms:

$$\gamma \in \ker \widetilde{\Delta}_{\text{min/max}} \cap \Omega^r(N) ,$$

$$\alpha \in \operatorname{im} \tilde{d}_{\text{min/max}} \cap \Omega^r(N) , \quad \beta \in \operatorname{im} \tilde{\delta}_{\text{min/max}} \cap \Omega^{r-1}(N) ,$$
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Simple complexes defined by $d_s^{\pm}$

$$0 \longrightarrow C_0^\infty(\mathbb{R}^+)\,\gamma \xrightarrow{d_{s,r}^{\pm}} C_0^\infty(\mathbb{R}^+)\,d\rho\wedge\gamma \longrightarrow 0\,,$$

$$0 \longrightarrow C_0^\infty(\mathbb{R}^+)\,\beta \xrightarrow{d_{s,r-1}^{\pm}} C_0^\infty(\mathbb{R}^+)\,\alpha + C_0^\infty(\mathbb{R}^+)\,d\rho\wedge\beta \xrightarrow{d_{s,r}^{\pm}} C_0^\infty(\mathbb{R}^+)\,d\rho\wedge\alpha \longrightarrow 0\,,$$

Simple complexes defined by $d_s^{\pm}$ (contd.)

They are isomorphic to the following simple elliptic complexes:

$$0 \to C_0^\infty(\mathbb{R}^+) \xrightarrow{\ d\ } C_0^\infty(\mathbb{R}^+) \to 0 ,$$
$$d = \frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho , \quad \kappa = (n-2r-1)u/2 ,$$

$$0 \to C_0^\infty(\mathbb{R}^+) \xrightarrow{\ d_0\ } C_0^\infty(\mathbb{R}^+) \oplus C_0^\infty(\mathbb{R}^+) \xrightarrow{\ d_1\ } C_0^\infty(\mathbb{R}^+) \to 0 ,$$
$$d_0 = \begin{pmatrix} d_{0,1} \\ d_{0,2} \end{pmatrix} , \quad d_1 = \begin{pmatrix} d_{1,1} & d_{1,2} \end{pmatrix} ,$$
$$d_{0,1} = \mu\rho^{-u} , \qquad d_{0,2} = \frac{d}{d\rho} - (\kappa + u)\rho^{-1} \pm s\rho ,$$
$$d_{1,1} = \frac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho , \quad d_{1,2} = -\mu\rho^{-u} .$$

Simple complex of length one

- Laplacian in $L^2(\mathbb{R}^+)$:

$$\Delta_0 = \delta d = H + \kappa(\kappa - 1)\rho^{-2} \mp s(1 + 2\kappa) ,$$
$$\Delta_1 = d\delta = H + \kappa(\kappa + 1)\rho^{-2} \pm s(1 - 2\kappa) .$$

- $\rightsquigarrow P_0 +$ constant.
- $\rightsquigarrow$ self-adjoint operators defined of Δ_0 and Δ_1 in $L^2(\mathbb{R}^+)$, and description of their spectra.

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Simple complex of length one (contd.)

	σ	Condition	Smooth core
$\mathcal{A}_1$	κ	$\kappa > -\frac{1}{2}$	$\mathcal{S}_{\mathrm{ev},+}$
$\mathcal{A}_2$	$1-\kappa$	$\kappa < \frac{3}{2}$	$\rho^{-2\kappa}\mathcal{S}_{\mathrm{odd},+}$

Table: Self-adjoint operators defined by $\Delta_{s,0}$

	τ	Condition	Smooth core
$\mathcal{B}_1$	κ	$\kappa > -\frac{3}{2}$	$\mathcal{S}_{\mathrm{odd},+}$
$\mathcal{B}_2$	$-1-\kappa$	$\kappa < \frac{1}{2}$	$\rho^{-2\kappa}\mathcal{S}_{\mathrm{ev},+}$

Table: Self-adjoint operators defined by $\Delta_{s,1}$

Simple complex of length one (contd.)

$\mathcal{A}_1^+$		0 the 1st one $+$, $O(s)$ the other ones
$\mathcal{A}_1^-$		$+$, $O(s)$
$\mathcal{A}_2^+$	$\kappa > \frac{1}{2}$	$-$ some of them
	$\kappa = \frac{1}{2}$	0 the 1st one $+$, $O(s)$ the other ones
	$\kappa < \frac{1}{2}$	$+$, $O(s)$
$\mathcal{A}_2^-$		$+$, $O(s)$

Table: Eigenvalues of $\mathcal{A}_i$

Simple complex of length one (contd.)

$\mathcal{B}_1^+$		$+$, $O(s)$
$\mathcal{B}_1^-$	$\kappa > -\frac{1}{2}$	$+$, $O(s)$
	$\kappa = -\frac{1}{2}$	0 the 1st one $+$, $O(s)$ the other ones
	$\kappa < -\frac{1}{2}$	$-$ some of them
$\mathcal{B}_2^+$		$+$, $O(s)$
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Table: Eigenvalues of $\mathcal{B}_i$

Simple complex of length one (contd.)

	$\Delta_{s,\max,0}$	$\Delta_{s,\min,0}$	$\Delta_{s,\max,1}$	$\Delta_{s,\min,1}$
$\kappa \geq \frac{1}{2}$	$\mathcal{A}_1$		$\mathcal{B}_1$	
$\lvert\kappa\rvert < \frac{1}{2}$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{B}_1$	$\mathcal{B}_2$
$\kappa \leq \frac{1}{2}$	$\mathcal{A}_2$		$\mathcal{B}_2$	

Table: Description of $\Delta_{s,\max/\min}$

Simple complex of length two

Laplacians in $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R}^+;\mathbb{C}^2)$:

$$\Delta_0 = H + (\kappa+u)(\kappa+u-1)\rho^{-2} + \mu^2\rho^{-2u} \mp s(1+2(\kappa+u))\;,$$

$$\Delta_2 = H + \kappa(\kappa+1)\rho^{-2} + \mu^2\rho^{-2u} \pm s(1-2\kappa)\;,$$

$$\Delta_1 = \begin{pmatrix} \Delta_{1,1} & -2\mu u\rho^{-u-1} \\ -2\mu u\rho^{-u-1} & \Delta_{1,2} \end{pmatrix}\;,$$

$$\Delta_{1,1} = H + \kappa(\kappa-1)\rho^{-2} + \mu^2\rho^{-2u} \mp s(1+2\kappa)\;,$$

$$\Delta_{1,2} = H + (\kappa+u)(\kappa+u+1)\rho^{-2} + \mu^2\rho^{-2u} \pm s(1-2(\kappa+u))\;.$$

Simple complex of length two (contd.)

- Case $u = 1$:
  - $\Delta_{s,0} = P_0 + \text{const.}, \quad \Delta_{s,2} = Q_0 + \text{const.}$, and $\Delta_{s,1}$ can be diagonalized obtaining terms of the same type in the diagonal.
  - $\rightsquigarrow$ self-adjoint extensions of these operators in $L^2(\mathbb{R}^+)$ with discrete spectrum, and description of their eigenvalues.
- Case $u < 1$:
  - $\rightsquigarrow \Delta_{s,0} = P + \text{const.}, \quad \Delta_{s,2} = Q + \text{const.}$, and $\Delta_{s,1} = W + \text{const. diag. matrix.}$
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Simple complex of length two (contd.)

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Simple complex of length two (contd.)

	σ	Condition	Core of $\mathcal{P}_i^{1/2}$
$\mathcal{P}_1$	$\kappa + u$	$\kappa > -\frac{1}{2}$	$\mathcal{S}_{\mathrm{ev},+}$
$\mathcal{P}_2$	$1 - \kappa - u$	$\kappa < \frac{3}{2} - 2u$	$\rho^{-2\kappa-2u}\mathcal{S}_{\mathrm{odd},+}$

Table: Self-adjoint operators defined by $\Delta_{s,0}$

	τ	Condition	Core of $\mathcal{Q}_j^{1/2}$
$\mathcal{Q}_1$	κ	$\kappa > u - \frac{3}{2}$	$\mathcal{S}_{\mathrm{odd},+}$
$\mathcal{Q}_2$	$-1 - \kappa$	$\kappa < \frac{1}{2} - u$	$\rho^{-2\kappa}\mathcal{S}_{\mathrm{ev},+}$

Table: Self-adjoint operators defined by $\Delta_{s,2}$

Simple complex of length two (contd.)

	σ	τ	θ	Condition
$\mathcal{W}_{1,1}$	κ	$\kappa + u$	κ	$\kappa > u - \frac{1}{2}$
$\mathcal{W}_{2,2}$	$1 - \kappa$	$-1 - \kappa - u$	$-\kappa - u$	$\kappa < \frac{1}{2} - 2u$
$\nexists\ \mathcal{W}_{1,2}$	κ	$-1 - \kappa - u$	$-\frac{1}{2} - u$	Impossible
$\mathcal{W}_{2,1}$	$1 - \kappa$	$\kappa + u$	$\frac{1}{2}$	$-\frac{1+u}{2} < \kappa < \frac{1-u}{2}$, $\kappa = -\frac{1}{2} - u, \frac{1}{2}$

Table: Self-adjoint operators defined by $\Delta_{s,1}$

Simple complex of length two (contd.)

	Core of $\mathcal{W}_{i,j}^{1/2}$
$\mathcal{W}_{1,1}$	$\mathcal{S}_{\mathrm{ev},+} \oplus \mathcal{S}_{\mathrm{odd},+}$
$\mathcal{W}_{2,2}$	$\rho^{-2\kappa}\mathcal{S}_{\mathrm{odd},+} \oplus \rho^{-2\kappa-2u}\mathcal{S}_{\mathrm{ev},+}$
$\mathcal{W}_{2,1}$	$\rho^{-2\kappa}\mathcal{S}_{\mathrm{odd},+} \oplus \mathcal{S}_{\mathrm{odd},+}$

Table: Self-adjoint operators defined by Δ_1 (contd.)

Simple complex of length two (contd.)

$\mathcal{P}_1$		$+, O(s)$
$\mathcal{P}_2^+$	$\kappa > \frac{1}{2} - u$	? some of them
	$\kappa \leq \frac{1}{2} - u$	$+, O(s)$
$\mathcal{P}_2^-$		$+, O(s)$
$\mathcal{Q}_1^+$		$+, O(s)$
$\mathcal{Q}_1^-$	$\kappa \geq -\frac{1}{2}$	$+, O(s)$
	$\kappa < -\frac{1}{2}$	? some of them
$\mathcal{Q}_2$		$+, O(s)$
$\mathcal{W}_{i,j}$		$+, O(s)$

Table: Eigenvalues of $\mathcal{P}_i$, $\mathcal{Q}_i$ and $\mathcal{W}_{i,j}$

Simple complex of length two (contd.)

	$\Delta_{s,\max,0}$
$\kappa > -\frac{1}{2}$	$\mathcal{P}_1$
$-\frac{1}{2} - u < \kappa \leq -\frac{1}{2}$	?
$\kappa \leq -\frac{1}{2} - u$	$\mathcal{P}_2$

	$\Delta_{s,\min,0}$
$\kappa \geq \frac{1}{2} - u$	$\mathcal{P}_1$
$\kappa < \frac{1}{2} - u$	$\mathcal{P}_2$

Table: Description of $\Delta_{s,\max/\min,0}$

Simple complex of length two (contd.)

	$\Delta_{s,\max,2}$
$\kappa > -\frac{1}{2}$	$\mathcal{Q}_1$
$\kappa \leq -\frac{1}{2}$	$\mathcal{Q}_2$

	$\Delta_{s,\min,2}$
$\kappa \geq \frac{1}{2}$	$\mathcal{Q}_1$
$\frac{1}{2} - u \leq \kappa < \frac{1}{2}$	?
$\kappa < \frac{1}{2} - u$	$\mathcal{Q}_2$

Table: Description of $\Delta_{s,\max/\min,2}$

Simple complex of length two (contd.)

	$\Delta_{s,\max,1}$
$\kappa > u - \frac{1}{2}$	$\mathcal{W}_{1,1}$
$-\frac{1}{2} < \kappa \leq u - \frac{1}{2}$	?
$-\frac{1+u}{2} < \kappa \leq -\frac{1}{2}$	$\mathcal{W}_{2,1}$
$-\frac{1}{2} - u < \kappa \leq -\frac{1+u}{2}$	?
$\kappa \leq -\frac{1}{2} - u$	$\mathcal{W}_{2,2}$

	$\Delta_{s,\min,1}$
$\kappa \geq \frac{1}{2}$	$\mathcal{W}_{1,1}$
$\frac{1-u}{2} \leq \kappa < \frac{1}{2}$	?
$\frac{1}{2} - u \leq \kappa < \frac{1-u}{2}$	$\mathcal{W}_{2,1}$
$\frac{1}{2} - 2u \leq \kappa < \frac{1}{2} - u$	?
$\kappa < \frac{1}{2} - 2u$	$\mathcal{W}_{2,2}$

Table: Description of $\Delta_{s,\max/\min,1}$

Simple complex of length two (contd.)

- One more trick: in this simple complex of length two,

$$\ker(\Delta_{s,\max/\min,0} \oplus \Delta_{s,\max/\min,2}) = 0 \Longrightarrow \ker \Delta_{s,\max/\min,1} = 0\ .$$

- $\rightsquigarrow$ In this case, $\Delta_{s,\max/\min,0} \oplus \Delta_{s,\max/\min,2}$ and $\Delta_{\max/\min,1}$ have the same eigenvalues, with the same multiplicity.
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Spectrum of $\Delta_{s,\min/\max}$ on our local model

- Assume the adapted metric is good.
- On our local model:

$$d_{s,\min/\max} = \bigoplus \text{min/max i.b.c. of the simple complexes}\,,$$

$$\rightsquigarrow \begin{cases} \Delta_{s,\min/\max} \text{ has a discrete spectrum,} \\ \text{description of } \ker \Delta_{s,\min/\max}\,, \\ \text{the positive eigenvalues are of order } O(s). \end{cases}$$

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Globalization and Witten's arguments

- $\rightsquigarrow$ 1st main thm for $\Delta_{\min/\max}$ after globalization because $\Delta_{s,\min/\max} - \Delta_{\min/\max}$ is bounded.
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Thank you very much!