# The Gamma question

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Connaissance - Action

Given a set  $A \subseteq \mathbb{N}$ . How close is A to being computable?

A recent paradigm : A is coarsely computable. This means there is a computable set R such that the asymptotic density of

$$\{n: A(n) = R(n)\}$$

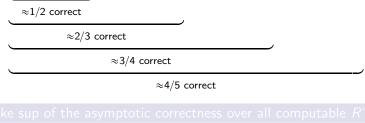
equals 1.

Reference : Downey, Jockusch, and Schupp, Asymptotic density and computably enumerable sets, Journal of Mathematical Logic, 13, No. 2 (2013)

# The $\gamma$ -value of a set $A \subseteq \mathbb{N}$

A computable set R tries to approximate a complicated set A :

- A : 100100100100 000101001001 010101111010 101010101111
- $R: \underbrace{000010110111}_{010101000101} 0100001010101010101010101111$

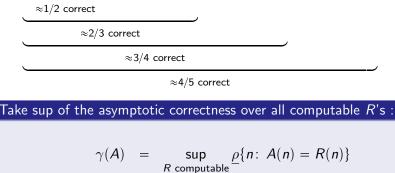


$$\gamma(A) = \sup_{\substack{R \text{ computable} \\ n \text{ of } n}} \underline{\rho}\{n \colon A(n) = R(n)\}$$
  
where  $\underline{\rho}(Z) = \liminf_{n} \frac{|Z \cap [0, n)|}{n}.$ 

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where 
$$\underline{\rho}(Z) = \liminf_{n} \frac{|Z \cap [0, n)|}{n}$$
.

## Recall

$$\begin{split} \gamma(A) &= \sup_{\substack{R \text{ computable}}} \underline{\rho}\{n: A(n) = R(n)\}\\ \text{where } \underline{\rho}(Z) &= \liminf_{n} \frac{|Z \cap [0, n)|}{n}. \end{split}$$

## Some possible values

$$\begin{array}{rcl} A \, {\rm computable} & \Rightarrow & \gamma(A) = 1 \\ A \, {\rm random} & \Rightarrow & \gamma(A) = 1/2. \end{array}$$

# Γ-value of a Turing degree

Andrews, Cai, Diamondstone, Jockusch and Lempp (2013) looked at Turing degrees, rather than sets. They defined

 $\Gamma(A) = \inf\{\gamma(B): B \text{ has the same Turing degree as } A\}$ 

A smaller  $\Gamma$  value means that A is further away from computable.

#### Example

An oracle A is called computably dominated if every function that A computes is below a computable function. *They show :* 

- If A is random and computably dominated, then  $\Gamma(A) = 1/2$ .
- If A is not computably dominated then  $\Gamma(A) = 0$ .

# $\Gamma(A) > 1/2$ implies $\Gamma(A) = 1$

## Fact (Hirschfeldt, Jockusch, McNicholl and Schupp)

If  $\Gamma(A) > 1/2$  then A is computable (so that  $\Gamma(A) = 1$ ).

The idea is to obtain B of the same Turing degree as A by "padding":

- "Stretch" the value A(n) over the whole interval  $I_n = [(n-1)!, n!)$ .
- Since γ(B) > 1/2 there is a computable R agreeing with B on more than half of the bits in almost every interval I<sub>n</sub>.
- So for almost all *n*, the bit A(n) equals the majority of values R(k) where  $k \in I_n$ .

# The **F**-question

## Question ( $\Gamma$ -question, Andrews et al., 2013)

Is there a set  $A \subseteq \mathbb{N}$  such that  $0 < \Gamma(A) < 1/2$ ?

• ????????? • 
$$\times \times \times \times \times \times \times$$
 •  
 $\Gamma = 0$   $\Gamma = 1/2$   $\Gamma = 1$ 

#### Theorem

Let 
$$A \in 2^{\mathbb{N}}$$
. If  $\Gamma(A) < 1/2$  then  $\Gamma(A) = 0$ .

The proof uses the field of error-correcting codes.

# Examples of $\Gamma(A) = 0$ : infinitely often equal

We know that  $A \subseteq \mathbb{N}$  not computably dominated implies  $\Gamma(A) = 0$ .

- We say  $g : \mathbb{N} \to \mathbb{N}$  is infinitely often equal (i.o.e.) if  $\exists^{\infty} n f(n) = g(n)$  for each computable function  $f : \mathbb{N} \to \mathbb{N}$ .
- We say that  $A \subseteq \mathbb{N}$  is i.o.e. if A computes function g that is i.o.e.

Surprising fact : A is i.o.e  $\Leftrightarrow$  A not computably dominated.

 $\Rightarrow$  Suppose A computes a function g that equals infinitely often to every computable function. Then no computable function bounds g.

 $\leftarrow$  *Idea*. Suppose A computes a function g that is dominated by no computable function. Then g is infinitely often above the halting time of any computable total function.

# New Examples of $\Gamma(A) = 0$ : weaken infinitely often equal

We know A not computably dominated implies  $\Gamma(A) = 0$ .

#### Recall

We say that A is infinitely often equal (i.o.e.) if A computes a function g such that  $\exists^{\infty} n \ f(n) = g(n)$  for each computable function  $f : \mathbb{N} \to \mathbb{N}$ .

We can weaken this :

Let  $H: \mathbb{N} \to \mathbb{N}$  be computable. We say that A is *H*-infinitely often equal if A computes a function g such that  $\exists^{\infty} n f(n) = g(n)$  for each computable function f bounded by H.

This appears to get harder for A the faster H grows.

# A i.o.e. implies $\Gamma(A) = 0$

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Theorem (Monin, Nies)

Let A be  $2^{(\alpha^n)}$ -i.o.e. for some  $\alpha > 1$ . Then  $\Gamma(A) = 0$ .

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Recall : A is H-infinitely often equal if A computes a function g such that  $\exists^{\infty} n \ f(n) = g(n)$  for each computable function f bounded by H.

#### Theorem

Let A be  $2^{(\alpha^n)}$ -i.o.e. for some computable  $\alpha > 1$ . Then  $\Gamma(A) = 0$ .

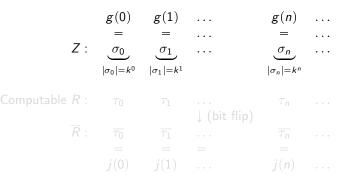
Proof sketch. First step : Let f be  $2^{(\alpha^n)}$ -i.o.e. Then for any  $k \in \mathbb{N}$ , f computes a function g that is  $2^{(k^n)}$ -i.o.e.

f(0) f(1) f(2) f(3) f(4) f(5) ... i.o.e. every comp. funct.  $\leq 2^{(\alpha^n)}$ 

 $\rightarrow \qquad f(0)f(2)f(4)\dots \text{ i.o.e. every comp. funct. } \leqslant n \mapsto 2^{(\alpha^{2n})} \\ \text{or } f(1)f(3)f(5)\dots \text{ i.o.e. every comp. funct. } \leqslant n \mapsto 2^{(\alpha^{2n+1})}$ 

Iterating this  $\rightarrow f \ge_T g$  which i.o.e. every comp. funct.  $\le 2^{(k^n)}$ 

Proof sketch. Second step : g is  $2^{(k^n)}$ -i.o.e. implies  $g \ge_T Z$  with  $\Gamma(Z) \le 1/k$ .

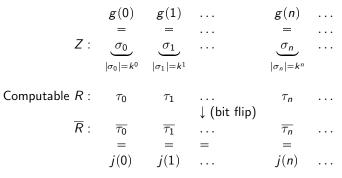


*j* equals *g* infinitely often. Then for infinitely many *n*,  $\tau_n(i) \neq \sigma_n(i)$  everywhere. We have

$$|\tau_n| \ge (k-1) \sum_{i < n} |\tau_i|$$

Then the lim inf of fraction of places where R agrees with Z is bounded by 1/k.

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## Theorem

Let  $X \in \mathbb{N}$ . Suppose that for every  $k \in \mathbb{N}$  and every X-computable sequence  $\{\tau_n\}_{n \in \mathbb{N}}$  with  $|\tau_n| = 2^{n/k}$ , there is a computable sequence  $\{\sigma_n\}_{n \in \mathbb{N}}$  with  $|\sigma_n| = |\tau_n|$  such that for almost every n,  $\sigma_n$  agrees with  $\tau_n$  on a fraction of at least  $\alpha$  bits.

Then  $\Gamma(X) \ge \alpha$ .

Idea : The length of the *n*-th string equals  $2^{1/k} - 1$  times the sum of the length of the previous strings. For *c* as large as we want, let *k* be large enough so that  $2^{1/k} - 1 < 1/c$ .

For  $Y \leq_{\mathcal{T}} X$ , we split Y in strings  $\{\tau_n\}_{n \in \mathbb{N}}$  of length  $2^{n/k}$ . The computable sequence  $\{\sigma_n\}_{n \in \mathbb{N}}$  given above implies  $\gamma(Y) \ge \frac{\alpha}{1+1/c}$ .

If this is true for every c we have  $\gamma(Y) \ge \alpha$ . If this is true for every  $Y \le_T X$  we have  $\Gamma(X) \ge \alpha$ .

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Suppose  $\Gamma(X) < 1/2 - \varepsilon$ . Then there is  $k \in \mathbb{N}$  and an X-computable sequence  $\{\tau_n\}_{n \in \mathbb{N}}$  with  $|\tau_n| = 2^{n/k}$ , such that :

For every computable sequence  $\{\sigma_n\}_{n\in\mathbb{N}}$  with  $|\sigma_n| = |\tau_n|$ , there are infinitely many n such that  $\sigma_n$  agrees with  $\tau_n$  on a fraction of at most  $1/2 - \varepsilon$  bits.

By taking the bitwise complement of every such computable sequence  $\{\sigma_n\}_{n\in\mathbb{N}}$  we get :

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## The error-correcting codes

We want to transmit a message of length m on a noisy chanel. We use an injection  $\Phi : 2^m \to 2^n$  for n > m in such a way that the strings in the range of  $\Phi$  are pairwise as far as possible.

If  $\delta$  is the smallest relative Hamming distance between two strings in the range of  $\Phi$ , we can correct up to a fraction of  $\delta/2$  errors.

We cannot in general correct more than a ratio of 1/4 of errors. To go beyond we need to use List decoding :

#### Theorem (List decoding theorem)

Let  $\varepsilon > 0$  and  $n \in \mathbb{N}$ . For  $L \in \mathbb{N}$  sufficiently large and  $\beta > 0$ sufficiently small, there exists a set C of  $2^{\beta n}$  many strings of length n such that :

For any string  $\sigma$  of length n, there are at most L elements  $\tau$  of C such that  $\sigma$  agrees with  $\tau$  on a fraction of bits of at least  $1/2 + \varepsilon$ .

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For any *n* we compute a sequence  $C_n$  of  $2^{(\beta 2^{n/k})}$  many strings of length  $2^{n/k}$  such that any string  $\sigma$  of length  $2^{n/k}$  agrees with at most *L* elements of  $C_n$  on a fraction of at least  $1/2 + \varepsilon$  bits.

From  $\{\tau_n\}_{n\in\mathbb{N}}$ , we compute the sequence  $\{D_n\}_{n\in\mathbb{N}}$  of all the strings of length  $\beta 2^{n/k}$  whose code in  $C_n$  agrees with  $\tau_n$  on more than  $1/2 + \varepsilon$  bits. We have  $|D_n| \leq L$  for every n.

Claim : For every computable function g bounded by  $2^{(\beta 2^{n/k})}$ , there are infinitely many n such that  $g(n) \in D_n$  (seen as a binary string).

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From this we compute : An X-computable sequence  $\{D_n\}_{n\in\mathbb{N}}$  where  $D_n$  contains at most L strings of length  $L2^n$  and such that : For every computable function g bounded by  $2^{(L2^n)}$ , there are infinitely many n such that  $g(n) \in D_n$  (seen as a binary string).

We see the *i*-th element  $\sigma_i$  of  $D_n$  as an *L*-uplet  $\langle \sigma_i^1, \ldots, \sigma_i^L \rangle$ . Let  $h_i$  be the function which to *n* gives  $\sigma_i^i$  where  $\sigma_i$  is the *i*-th string of  $D_n$ .

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