On the entropy algorithmics of computable dynamical systems

Silvere Gangloff, Benjamin Hellouin, Cristobal Rojas, Mathieu Sablik

CIRM 2016

21 juin 2016

Given some dynamical system (X, f), the entropy of this system from the viewpoint of a finite open cover is the asymptotic growth rate of the number of possible discrete trajectories, and the topological entropy of the system is the greatest of these entropies.

« Is the entropy computable ? » (J.Milnor)

Sommaire

1 Computability and dynamics

2 Cases of matching

3 Questions

A real number is computable if there is an algorithm that outputs the *nth* bit of the the base two decomposition of this number, with imput n.

A computable metric space is a metric space with a dense set of ideal points, such that there is an algorithm computing the distance of two ideal points, given these two points as imput.

Examples of such spaces : Cantor space, Interval, Continuous functions of the interval.

A function between two computable metric spaces is computable if there is an algorithm that given as imput an ideal ball of the second space (ideal center, rational radius) B, enumerate a set of ideal balls which union is $f^{-1}(B)$.

Examples : Cellular automata, min/max operators for computable functions

A computable dynamical system is some (X, f), where X is some computable compact metric space and f some computable function from X into itself

Arithmetical hierarchy of real numbers (Wheirauch, Zheng) : Δ_1 is the set of computable numbers. Definition by induction : Σ_n (resp. Π_n, Δ_{n+1}) is the image of sup (resp. inf, lim) over sequences of Δ_n-numbers.

Known facts : $\Delta_n \not\subset \Sigma_n \not\subset \Delta_{n+1}$, $\Delta_n \not\subset \Pi_n \not\subset \Delta_{n+1}$

« Given some class of dynamical system, what can we say about the height in the hierarchy of the generic entropy of a system in this class ? »

« Given such a result, can we see every number of this height as the entropy of a system in this class ? »

Theorem

There exists some algorithm such that given as imput a the description of a computable dynamical system, output a computable sequence of rational numbers, such that the entropy of this system is the sup inf of this sequence. **Hence the entropy of a computable system is a Σ_2-number.**

Functional version of this Type 2 theorem ? Generalization of the entropy for compact sets, general obstruction ?

Sommaire

1 Computability and dynamics

2 Cases of matching

3 Questions

An effective dynamical system is the distributed action of the shift on a shift-invariant compact subset of the infinite product of Cantor spaces $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$.

Theorem (Hochman)

Every Σ_2-number is the entropy of an effective dynamical system, and also is the entropy of a computable function of the Cantor space.

Can we realize every Σ_2 number with a minimal EDS ?

A minimal subshift is a subshift which points share the same patterns. The unique ergodicity refers to the possession of a unique invariant probability measure.

Theorem (Grillemberger)

The entropies of **minimal and uniquely ergodic** *effective* **one-dimensional subshifts** *are exactly the* Π_1*-numbers.*

Theorem

The entropies of computable function of $[0,1]$ into itself are exactly the Σ_1 numbers.

This result is « hierarchizable »

More regular functions, as $\mathcal{C}^2$?

A $f(n)$-topologically mixing one-dimensional subshift is a subshift possessing a gluing function that is f.

Theorem

The entropies of $O(n)$- **topologically mixing** *effective one-dimensional subshift are exactly the Π_1 numbers.*

Phenomenon behind distinction of mixing intensities ?

Sommaire

1 Computability and dynamics

2 Cases of matching

3 Questions

A one-dimensional subshift is finitely topologically mixing if there is some integer $k \geq 1$ such that for every two words in the langage of the subshift, and integer $n \geq k$, they can be « glued »together with a word of length n into the langage.

Theorem

The entropy of every **finitely topologically mixing** *effective one-dimensional subshifts is* **computable***.*

No result about realization of computable numbers by topologically mixing subshifts.

Theorem

Every Σ_1-number is the entropy of a surjective computable function of the Cantor space.

No known stronger obstruction than Σ_2 for this class

System class	Restrictions	Obstruction	Realization
General		Σ_2	
SFT		$\log(\text{Perron}) \subset \Delta_1$	$\log(\text{Perron})$
Sshifts 1d	Minimal ue	Π_1	Π_1
	$O(n)$ top mix	Π_1	Π_1
	$O(f(n))$		
	$O(1)$ top mix	Δ_1	
Sshifts $\geq$ 2d		Π_1	Π_1
EDS		Σ_2	Σ_2
	minimalty	Σ_2	
$\mathcal{C}[0,1]$		Σ_1	Σ_1
$\mathcal{C}[0,1]$	regularity		
Cantor space	surjective		Σ_1

« What about surjective/transitive/mixing cellular automata ? »

« Turing machines ? »(E.Jeandel)