Zero sets and local time of algorithmically random Brownian motion

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LOCAL TIME in Physics, Signal Processing and its computational aspects..
Consider heuristically (mathematician) or formally (physicist) these two intriguing if somewhat crazy symbolic expressions.

$$L(t,\omega,x) := \int_0^t \delta(X_\omega(s) - x)ds,$$

where

$$\delta(X_\omega(s) - x) = \int_{\mathbb{R}} e^{i(X_\omega(s)-x)}ds.$$

Intuitively, $L(t,\omega,x)$ is the time that the sample path $s \mapsto X_\omega(s)$ spends "infinitesimally close" to x during the time interval $0 \leq s \leq t$.
(Of course, we are not working with Lebesgue integrals here.)

We shall also look at the measure (when it makes sense)

$$\mu_\omega(a, b) = L(b, \omega, 0) - L(a, \omega, 0), \quad 0 \leq a < b \leq 1$$

and its Fourier transform.
This especially when X is an algorithmically random Brownian motion.
We will provide evidence that ideas from Kolmogorov complexity yield new insights on these intuitions and even new (and mathematically solid) results on the local time of Brownian motion and the geometry of its sample paths.
Our arguments rely heavily on Fourier analysis.

If (X, d) is a metric space, a *regular* Borel measure μ on X is a Borel measure with the property that for every subset A of X, there is a Borel set B containing A such that $\mu(B) = \mu^*(A)$. Here μ^* is the outer measure associated with μ, in other words, writing $\mathcal{B}$ for the Borel algebra on X,

$$\mu^*(A) = \inf\{\mu(B) : A \subset B \in \mathcal{B}\},\ A \subset X.$$

A measure on Euclidean space $\mathbb{R}^d$ is a *Radon measure* iff it is regular Borel and assumes finite values on compact subsets of $\mathbb{R}^d$.

HISTORY

The *Fourier dimension* of a compact set E in $\mathbb{R}$ is the supremum of positive real numbers $\alpha < 1$ such that for some non-zero Radon measure μ supported by E , it is the case that

$$|\hat{\mu}(\xi)|^2 \leq \frac{1}{|\xi|^\alpha},$$

for $|\xi|$ sufficiently large. The Fourier dimension of E is denoted by $\dim_f(E)$. It can be shown that

$$\dim_f(E) \leq \dim_h(E),$$

for all compact sets E. Here $\dim_h(E)$ denotes the Hausdorff dimension of E.

The set E is called a *Salem* set if $\dim_f(E) = \dim_h(E)$.

The following question posed by Beurling was addressed and solved in the positive by Salem in 1950. (On singular monotonic functions whose spectrum has a given Hausdorff dimension By R. Salem (1950), Ark Mat 1,353-365.)
Given a number $\alpha \in (0, 1)$, does there exist a closed set on the line whose Hausdorff dimension is α that carries a Radon measure μ whose Fourier transform

$$\hat{\mu}(u) = \int_{\mathbb{R}} e^{iux} d\mu(x)$$

is dominated by $|u|^{-\alpha/2}$ as $|u| \to \infty$?

Salem proved this result by constructing for every α in the unit interval, a random measure μ (over a convenient probability space) whose support has Hausdorff dimension α and which satisfies the Beurling-requirement with probability one.
It was recently shown (published in 2014) in collaboration with George Davie and Safari Mukeru that such sets can also be constructed by looking at Cantor ternary sets E with computable ratios ξ and then to consider the image of E under a complex oscillation or, equivalently, a Martin-Löf Brownian motion.

COMPLEX OSCILLATIONS

For any finite binary word α we denote its (prefix-free) Kolmogorov complexity by $K(\alpha)$. Recall that an infinite binary string α is Kolmogorov-Chaitin complex if

$$\exists_d \forall_n \; K(\overline{\alpha}(n)) \geq n - d.$$

In the sequel, we shall denote this set by KC and refer to its elements as KC-strings.

For $n \geq 1$, we write C_n for the class of continuous functions on the unit interval that vanish at 0 and are linear with slopes $\pm\sqrt{n}$ on the intervals $[(i-1)/n, i/n]$, $i = 1, \ldots, n$. With every $x \in C_n$, one can associate a binary string $a = a_1 \cdots a_n$ by setting $a_i = 1$ or $a_i = 0$ according to whether x increases or decreases on the interval $[(i-1)/n, i/n]$. We call the sequence a the code of x and denote it by $c(x)$. The following notion was introduced by Asarin and Prokovskii. (1987).

Definition

A sequence (x_n) in $C[0,1]$ is *complex* if $x_n \in C_n$ for each n and there is a constant $d > 0$ such that $K(c(x_n)) \geq n - d$ for all n. A function $x \in C[0,1]$ is a *complex oscillation* if there is a complex sequence (x_n) such that $\|x - x_n\|$ converges effectively to 0 as $n \to \infty$.

The class of complex oscillations is denoted by $\mathcal{C}$. It was shown by Asarin and Prokovsky that the class $\mathcal{C}$ has Wiener measure 1. In fact, they implicitly showed that the class corresponds exactly, in the broad context and modern language of Hoyrup and Rojas, to the Martin-Löf random elements of the computable measure space

$$\mathcal{R} = (C_0[0,1], d, B, W),$$

where $C_0[0,1]$ is the set of continuous functions on the unit interval that vanish at the origin, d is the metric induced by the uniform norm, B is the countable set of piecewise linear functions f vanishing at the origin with slopes and points of non-differentiability all rational numbers and where W is the Wiener measure.

Theorem

(F:2000).There is a bijection $\Phi : KC \rightarrow \mathcal{C}$ and a uniform algorithm that, relative to any KC-string α, with input a dyadic rational number t in the unit interval and a natural number n, will output the first n bits of the the value of the complex oscillation $\Phi(\alpha)$ at t.

It was shown in 2013, in collaboration with Davie that the construction of Φ is layerwise computable in α.

Let $\mathcal{U} = (U_n)$ be a universal Martin-Löf test. Define

$$\mathrm{LAY}_{\mathcal{U}} : \mathrm{ML} \rightrightarrows \mathbb{N}$$

by

$$n \in \mathrm{LAY}_{\mathcal{U}}(p) \longleftrightarrow p \notin U_n$$

. Then

$$\Phi \equiv_W \mathrm{LAY}.$$

(Davie, F, Pauly 2015.)

Definition

Let $0 < \alpha < 1$. We call a Radon (probability) measure on the unit interval an α-*Frostman measure* if for some constant $C > 0$, we have $\nu(I) \leq C|I|^\alpha$ for all dyadic intervals I contained in the unit interval.
An α-Frostman measure ν is called an *effective* α-Frostman measure if the function

$$\mathbb{D} \to \mathbb{R}, \; d \mapsto \nu(I_d)$$

is computable. ($I_d = (\frac{\ell}{2^k}, \frac{\ell+1}{2^k})$, when ($d = \frac{\ell}{2^k}$).)

Definition

Let E be a compact subset of Hausdorff dimension $\beta > 0$. We say that the Hausdorff dimension of E is *effectively witnessed*, if for each rational $0 < \alpha < \beta$, there is an effective α-Frostman measure which supports E.

Theorem

(F, Mukeru, Davie) Let $0 < \alpha \leq 1$*. Suppose* ϕ *is a complex oscillation and* μ *is an effective* α*-Frostman measure on* $[0, 1]$ *and* $\epsilon > 0$*. Then for all reals u such that* $|u|$ *is sufficiently large (depending on* ϵ*),*

$$\left| \int_0^1 e^{iu\phi(t)} d\mu(t) \right| \leq \frac{1}{|u|^{\alpha-\epsilon}}. \tag{1}$$

Theorem

(follows from folklore in geometric measure theory) Suppose E is a compact subset of reals such that, for every $\epsilon > 0$, there is some $\mu \in M_+(E)$ and $0 < \alpha < 1$, such that, for some constant $C = C(\epsilon)$, it is the case that

$$|\hat{\mu}(\xi)|^2 \leq C|\xi|^{-\alpha+\epsilon},$$

as $|\xi| \to \infty$. Then, if k is a natural number such that $k\alpha > 1$, it will follow, upon writing

$$E_k = E + \cdots + E \ \ (k \text{ times}),$$

that

$$\mathbb{R} = \bigcup_{n<\omega} n(E_k - E_k).$$

$< E >_{\mathbb{Z}} = \mathbb{R}$.
This implies, algebraically, and quite explicitly, that the set E generates $\mathbb{R}$ as an abelian group!

Theorem

(FM 2013) Let X be one-dimensional Brownian motion and let μ be the "Dirac measure" of X. For a natural number $q \geq 1$, write $\mathcal{U}_{2q}$ for the set of quadratic forms with integral coefficients of the form

$$\sum_{j=1}^{2q} \alpha_j x_j^2$$

with $\alpha_1 = 0$ and $\alpha_{2q}, \alpha_{j+1} - \alpha_j \in \{-1, 1\}$ for all $j \geq 1$. Then, for all $u \in \mathbf{R}$,

$$E\left(|\hat{\mu}(u)|^{2q}\right) = \frac{(q!)^2}{(2\pi)^q} \sum_{Q \in \mathcal{U}_{2q}} \int_{\mathcal{B}_{2q}} e^{iuQ(x)} dx \qquad (2)$$

where $\mathcal{B}_{2q}$ is the unit ball in $\mathbf{R}^{2q}$.

Theorem

(F. 2013) For a continuous version X of Brownian motion over the unit interval, we have, almost surely,

$$\mathbb{R} = \bigcup_{n=1}^{\infty} n(Y_X - Y_X),$$

where

$$Y_X = Z_X + Z_X + Z_X,$$

and Z_X is the zero set of X. Moreover, almost surely, for any finite set A of real numbers, the set Y_X will contain an affine (rescaled and translated) copy of A.

This is because the zero set of X is a Salem set of Fourier dimension $\frac{1}{2}$, almost surely.

Heuristically, we say that a function

$$X : \mathbb{R} \to \mathbb{R}$$

admits a Dirac function $\delta(X(t))$, if "sense can be made of" the formal expression

$$\delta(X(t)) = \int_{\mathbb{R}} e^{i\alpha X(t)} d\alpha.$$

For more about this see the final chapter in Kahane's "Some random series of functions".

BIG OPEN QUESTION: Is the zero set of each complex oscillation a Salem set?
It is well-known that such a zero set has Hausdorff dimension $\frac{1}{2}$.
But is this also its Fourier dimension???
Is it true that almost surely, there are distinct $x, y, z \in Z_X$ such that $y - x = z - y$? (The Roth phenomenon in Ramsey theory.)
What if X is a complex oscillation?

ADDITIVE STRUCTURE OF ZERO SETS

The zero-set of a complex oscillation has the following diophantine property:

Theorem

(F. 2014) If x is a complex oscillation and r is a real number then

$$\forall_{\ell}\exists_{n}\exists_{t_1,\ldots,t_6\in[0,1]\cap\mathbb{Q}}\ \Big[|n((t_1+t_2+t_3)-(t_4+t_5+t_6))-r|<\frac{1}{\ell}\Big]$$

$$\wedge\forall_{1\le i\le 6}|x(t_i)|<\frac{1}{\ell}.$$

Can we interchange the first two quantifiers here????

LOCAL TIME

The *occupation measure* of Brownian motion $X : \mathbb{R} \times \Omega \to \mathbb{R}$ up to time t, is the random Borel measure defined by

$$\mu(t, \omega, A) = \lambda\{s \in [0, t] : X(s, \omega) \in A\}, \; A \text{ Borel in } \mathbb{R}, \; \omega \in \Omega.$$

Here λ is the Lebesgue measure.

REMARK: One can analogously define the occupation measure $\mu(t, f, A)$ of any ("deterministic") Borel function $f : \mathbb{R} \to \mathbb{R}$.

Abuse of notation: $\mu(t, \omega, A) = \mu(t, X(., \omega), A)$.

Lévy (1940 ...) proved that for almost all $\omega \in \Omega$, the occupation measure $\mu(t, \omega, .)$ is absolutely continuous (with respect to Lebesgue measure), that is, there exists a function

$$\mathcal{L}(t, \omega, .) : \mathbb{R} \to \mathbb{R},\ x \mapsto \mathcal{L}(t, \omega, x),\ t \geq 0$$

such that

$$\mu(t, \omega, A) = \int_A \mathcal{L}(t, \omega, x) dx,\ (A \text{ Borel in } \mathbb{R}).$$

The number $\mathcal{L}(t, \omega, x)$ is called the "local time of ω at x up to time t".
Lévy referred to his construct as the "mesure du voisinage" suggesting to me at least that it might represent, for $t > 0$, the "time that the Brownian path ω spends at the point x or infinitesimally around the point x during the time interval $[0, t]$".

It is clear, by the Lebesgue density theorem that, for almost every $x \in \mathbb{R}$ (with respect to the Lebesgue measure), and *each* t,

$$\mathcal{L}(t, \omega, x) = \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \lambda\{0 \leq s \leq t : |X(s, \omega) - x| \leq \epsilon\}, \qquad (3)$$

almost surely.

Trotter (1958) proved later that the occupation measure $\mu(t, \omega, .)$ has continuous density for almost all ω, that is,

$$\mathcal{L}(., \omega, .) : [0, 1] \times \mathbb{R} \to \mathbb{R}$$

$$(t, x) \mapsto \mathcal{L}(t, \omega, x)$$

is continuous for almost all $\omega \in \Omega$.

This has the implication that, for every $x \in \mathbb{R}$, almost surely, for *all* $t \in [0, 1]$,

$$\mathcal{L}(t, \omega, x) = \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \lambda\{0 \leq s \leq t : |X(s, \omega) - x| \leq \epsilon\}. \qquad (4)$$

In 2014 , F (with George Davie and Safari Mukeru) proved that for *each* complex oscillation ω, the occupation measure $\mu(t,\omega,.)$ of ω up to time t is such that its Fourier transform

$$\hat{\mu}(t,\omega,u) = \int_0^t \exp(i\,u\,\omega(s))ds,\;\; u \in \mathbb{R}$$

satisfies

$$|\hat{\mu}(t,\omega,u)|^2 = \ll_\epsilon |u|^{-2+\epsilon},\;\; |u| \to \infty$$

for all $\epsilon > 0$. This has the implication that $\hat{\mu}(t,\omega,.) \in L^2(\mathbb{R})$ and by a standard argument (Parseval), $\mu(t,\omega,.)$ is absolutely continuous and its Radon-Nikodym derivative $L(t,\omega,.)$ is in $L^2(\mathbb{R})$. Hence for *almost every* $x \in \mathbb{R}$ (with respect to the Lebesgue measure), and every complex oscillation ω, for all t in the unit interval:

$$L(t,\omega,x) = \lim_{\epsilon\to 0^+} \frac{1}{2\epsilon}\lambda\{0 \le s \le t : |\omega(s) - x| \le \epsilon\}.$$

Subsequently Safari Mukeru and F proved that for *any* complex oscillation ω and any *computable* real number x, and all t, the limit

$$L(t,\omega,x) = \lim_{n\to+\infty} \frac{1}{2\epsilon_n}\lambda\{s \leq t : |\omega(s) - x| \leq \epsilon_n\},$$

where $\epsilon_n = 2^{-n}$, exists, and the function

$$L(.,\omega,x) : [0,1] \longrightarrow [0,+\infty),\ \ t \mapsto L(t,\omega,x)$$

is continuous.
We call $L(t,\omega,x)$ the *effective* local time of ω at x up to time t.

Eventually we found (paper in preparation)

Theorem

(FM ???) For any complex oscillation ω and for any computable real number a, and all t

$$L(t,\omega,a) = 2 \lim_{m\to\infty} \sum_{k\in S_m} |\omega(k/2^m) - a|$$

where S_m is the subset of $\{1,2,\ldots,\ell\}$, $\ell = \lfloor t2^m \rfloor$, defined by

$$k \in S_m \textit{ iff } sign(\omega(k/2^m) - a) \neq sign(\omega((k-1)/2^m) - a)$$

i.e., $k \in S_m$ iff ω assumes the value a in the interval $(\frac{k-1}{2^n}, \frac{k}{2^n})$.

Note that the value of ω at a dyadic rational can never be computable. Even the almost sure version for just $a = 0$ of this result is new, it would appear.

I conclude with two interesting, I think, challenges.
Understand Borel functions $f : [0,1] \to \mathbb{R}$ which has a local time $L(t, f, x)$ which is real analytic in x for each t.
Such a function f will have the following interesting property: It would be *extremely discontinuous* (turbulent/Pascal infinite) in the following sense:
Given any open interval I in the unit interval and D a Borel set of positive Lebesgue measure. Then $f^{-1}(D)$ will meet any interval I in a significant manner, in the sense that

$$\lambda(f^{-1}(D) \cap I) > 0.$$

Problem: Does every extremely discontinuous function have a local time ... and if so is it real analytic, ... when is it computable??

Consider the Weierstrass function

$$X(t) := \Sigma_{k\geq 0} b^k \cos(a^k \pi t),\ 0 < b < 1,\ ab \geq 1.$$

Does this function have a local time, i.e., is the occupation measure of the Weierstrass function absolutely continuous, and, if so, how can this local time be computed from a, b?

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