The Quintet

Poisson–Mellin–Newton–Rice–Laplace

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Plan of the talk

- General framework.
  - Two probabilistic models,

    the Bernoulli model and the Poisson model.
  - Description of the tools,

    the Poisson transform, the Poisson sequence.

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  - Need : Depoissonization sufficient conditions, well studied.

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- Study of sufficient conditions for tameness,
  - using the inverse Laplace transform

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Begin with (elementary) data

Consider algorithms which use as inputs finite sequences of data

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Not in case (1) where the successive symbols may be strongly dependent.

Two probabilistic models.

The space of inputs is the set $\mathcal{X}^\star$ of the finite sequences of elements of $\mathcal{X}$. There are two main probabilistic models on the set $\mathcal{X}^\star$.

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$\Longrightarrow$ A first study in the Poisson model,

followed with a return to the Bernoulli model

Costs of interest.

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- $R(\boldsymbol{x})$ is the number of vertices of the convex hull built on the sequence $\boldsymbol{x} = (x_1, \ldots, x_n)$ of points x_i
- $R(\boldsymbol{w})$ is a function of the probability $p_{\boldsymbol{w}}$ of the finite prefix $\boldsymbol{w}$, with the word $\boldsymbol{w}$ viewed as a sequence $\boldsymbol{w} := (w_1 \ldots, w_n)$ of symbols w_i.

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Our final aim is the analysis of R in the model $\mathcal{B}_n$,

- We begin with the analysis in the (easier) Poisson model $\mathcal{P}_z$,
- We then wish to return in the (more realistic) Bernoulli model.

Average-case analysis of a cost R defined on $\mathcal{X}^\star$

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$$\mathbb{E}_z[R] = \sum_{n \geq 0} \mathbb{E}_z[R \mid N = n] \, \mathbb{P}_z[N = n]$$

$$= \sum_{n \geq 0} \mathbb{E}_{[n]}[R] \, \mathbb{P}_z[N = n] = e^{-z} \sum_{n \geq 0} r(n) \frac{z^n}{n!}$$

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$\mathbb{E}_z[R]$ is the Poisson transform of the sequence $n \mapsto r(n)$.

- With (properties of) the Poisson transform $P(z)$ of $n \mapsto r(n)$
  return to (the asymptotics of) the sequence $n \mapsto r(n)$

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  - Under this form, it is clear that the map π is involutive.
- Important binomial relation between $f(n)$ and $p(n)$

$$p(n) = \sum_{k=k_0}^{n} (-1)^k \binom{n}{k} f(k), \quad \text{and} \quad f(n) = \sum_{k=k_0}^{n} (-1)^k \binom{n}{k} p(k).$$

An instance of application: Toll functions and tries (I).

A source $\mathcal{S}$ on a finite alphabet Σ

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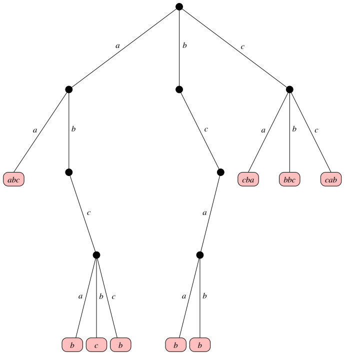
The trie $T(\boldsymbol{x})$ built on $\boldsymbol{x} \in \mathcal{X}^\star$ is a tree :

- If $|\boldsymbol{x}| = 0$, $T(\boldsymbol{x}) = \emptyset$
- If $|\boldsymbol{x}| = 1, \boldsymbol{x} = (x)$, $T(\boldsymbol{x})$ is a leaf labeled by x.
- If $|\boldsymbol{x}| \geq 2$, then $T(\boldsymbol{x})$ is formed with
  – an internal node o
  – and a sequence of tries $T(\underline{\boldsymbol{x}}_{\langle\sigma\rangle})$ for $\sigma \in \Sigma$



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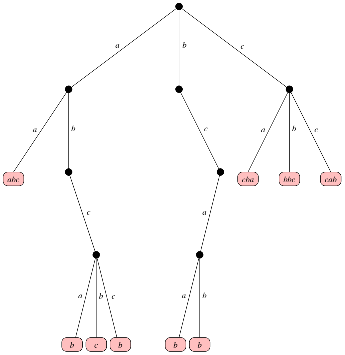

- $\boldsymbol{x}_{\langle\sigma\rangle}$ is the subsequence of $\boldsymbol{x}$ formed with words which begin with σ
- $\underline{\boldsymbol{x}}_{\langle\sigma\rangle}$ is formed with words of $\boldsymbol{x}_{\langle\sigma\rangle}$ stripped of their initial symbol σ.
- If $\boldsymbol{x}_{\langle\sigma\rangle} \neq \emptyset$, the edge $o \to T(\underline{\boldsymbol{x}}_{\langle\sigma\rangle})$ is labelled with σ.

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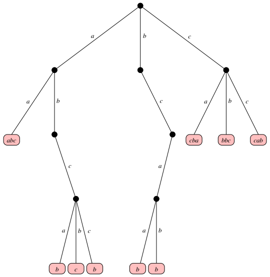
A sequence $n \mapsto f(n)$ with $\mathrm{val}(f) = 2$ plays the role of a toll function.

With the toll f, associate the cost R defined on $\mathcal{X}^\star$

$$R(\boldsymbol{x}) := \sum_{\boldsymbol{w} \in \Sigma^\star} f(|\boldsymbol{x}_{\langle \boldsymbol{w} \rangle}|),$$



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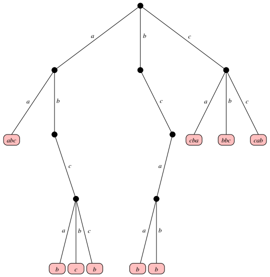

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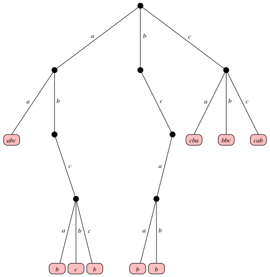

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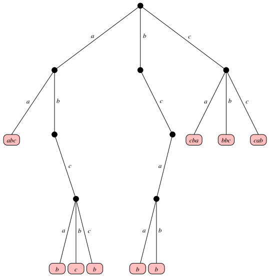

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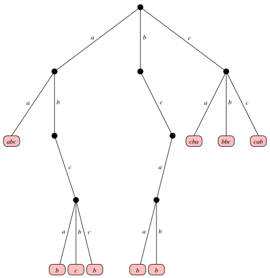

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What is the mean value of the cost $R(\boldsymbol{x})$ when $\boldsymbol{x} \in \mathcal{X}^n$?

An instance of application: Toll functions and tries (III).

Remind: $$R(\boldsymbol{x}) := \sum_{\boldsymbol{w}\in\Sigma^\star} f(|\boldsymbol{x}_{\langle \boldsymbol{w}\rangle}|) = \sum_{\boldsymbol{w}\in\Sigma^\star} f(N_w(\boldsymbol{x}))$$

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$$Q(z) := \mathbb{E}_z[R] = \sum_{\boldsymbol{w}\in\Sigma^\star} \mathbb{E}_z[f(N_{\boldsymbol{w}})] \Longrightarrow Q(z) = \sum_{\boldsymbol{w}\in\Sigma^\star} P(zp_w)$$

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where $N_{\boldsymbol{w}}$ is the number of words which begin with $\boldsymbol{w}$.

What is given? – the source with the probabilities p_w.

– the toll sequence $n \mapsto f(n)$, its transform $P(z)$ and its sequence $\pi[f]$

$$P(z) = \mathbb{E}_z[f(N))] = e^{-z}\sum_{n\geq 2} f(n)\frac{z^n}{n!} = \sum_{n\geq 2}(-1)^n p(n)\frac{z^n}{n!}$$

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What about $r(n) := \mathbb{E}_{[n]}[R]$, its Poisson transform, its Poisson sequence?

$$Q(z) := \mathbb{E}_z[R] = \sum_{\boldsymbol{w}\in\Sigma^\star} \mathbb{E}_z[f(N_{\boldsymbol{w}})] \Longrightarrow Q(z) = \sum_{\boldsymbol{w}\in\Sigma^\star} P(zp_w)$$

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Sequence $f(n)$ and source $\mathcal{S} \Longrightarrow Q(z)$ and $q(n)$

How to return to $r(n)$?

Part II – Description of the two paths. Generic tools.

Two paths from the Poisson model to the Bernoulli model

- Both use the Mellin transform

Description of the two possible paths.

Begin with a sequence $k \mapsto f(k)$,

consider its Poisson transform $P(z)$ and its Poisson sequence $\pi[f] : n \mapsto p(n)$,

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- Rice method (Ri)
  - Deal with the sequence $\pi[f] : n \mapsto p(n)$,
  - and its analytic lifting $\pi[f]$ which is proven to exist
  - Return to the sequence $n \mapsto f(n)$ via the binomial formula which is tranfered into an integral, the Rice integral.

A first technical condition: Valuation-Degree Condition

Definition. For a non zero real sequence $n \mapsto f(n)$, define

$$\mathrm{val}(f) := \min\{k \mid f(k) \neq 0\},$$

$$\deg(f) := \inf\{c \mid f(k) = O(k^c)\} = \limsup \left\{ \frac{\log f(k)}{\log k} \mid k \geq k_0 \right\}.$$

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If $n \mapsto f(n)$ is of polynomial growth, then $\deg(f)$ is finite.

In this case, the `VD`-Condition is not restrictive: Replace f by f_+

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We always assume the VD-Condition to hold,

with a difference $d - k_0$ as smallest as wished.

A second technical tool: the canonical sequence.

When $\mathrm{val}(f) = k_0$, $P(z)$ is written as

$$P(z) = z^{k_0} Q(z), \qquad Q(z) = e^{-z} \sum_{k \geq 0} g(k) \frac{z^k}{k!} = \sum_{n \geq 0} (-1)^n \frac{z^n}{n!} q(n)\,.$$

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The sequence $k \mapsto g(k)$ is the canonical sequence associated with $k \mapsto f(k)$

$$g(k) = \frac{f(k+k_0)}{(k+1)\ldots(k+k_0)} \quad \text{for } k \geq 0\,.$$

It satisfies the `VD`-Condition, with $\mathrm{val}(g) = 0$ and $\deg\, g = d - k_0 < 0$.

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Ex: $f(k) = k \log k$ with $k_0 = 2 \Longrightarrow g(k) = \dfrac{f(k+2)}{(k+1)(k+2)} = \dfrac{1}{k+1} \log(k+2)$

A tool which is used in each of the two paths: The Mellin transform (I).

The Mellin transform $H^* : s \mapsto H^*(s)$ of $x \mapsto H(x)$ is

$$H^*(s) := \int_0^\infty H(x) x^{s-1} dx.$$

If $H(x) = O(x^{-\alpha})$ as $x \to 0$ and $H(x) = O(x^{-\beta})$ as $x \to \infty$,
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When H is defined inside cones $\mathcal{C}(a, \theta) := \{z \mid |\arg(z-a)| < \theta\}$,
an important lemma, often called "Exponential Smallness Lemma".

Lemma. [Flajolet-Gourdon-Dumas (1998)] *If, inside the cone $\mathcal{C}(0, \theta)$ with $\theta > 0$ one has $H(z) = O(|z|^{-\alpha})$ as $z \to 0$ and $H(z) = O(|z|^{-\beta})$ as $|z| \to \infty$, then the following estimate is uniform in $\langle \alpha, \beta \rangle$*

$$H^*(s) = O(e^{-\theta |t|}), \qquad (s = \sigma + it)$$

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An instance of this type of transfer

$$Q(z) = \sum_{\boldsymbol{w} \in \Sigma^\star} P(zp_{\boldsymbol{w}}) \quad \Longrightarrow \quad Q^*(s) = \left[\sum_{\boldsymbol{w} \in \Sigma^\star} p_{\boldsymbol{w}}^{-s}\right] P^*(s)$$

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For the DP path, use also the properties of the inverse Mellin transform:

$$P(z) = \frac{1}{2i\pi} \int_{\uparrow} P^*(s) z^{-s} ds$$

Poles of $P^*(s)$ on the right of the fundamental strip
$\Longrightarrow$ Asymptotic behaviour of $P(z)$ for $z \to \infty$.

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Main contributors

- Jacquet and Szpankowski [1998], two papers...
- Hwang-Fuchs-Zacharovas [2010]
- Jacquet [2014]

Depoissonization path (I). The Charlier-Poisson expansion

introduced in the AofA domain by Hwang-Fuchs-Zacharovas [2010]

$$P(z) = \sum_{j \geq 0} \frac{P^{(j)}(n)}{j!} (z-n)^j \quad \Longrightarrow \quad f(n) := n![z^n]\left(e^z P(z)\right) = \sum_{j \geq 0} \frac{P^{(j)}(n)}{j!} \tau_j(n)$$

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They are called the Charlier-Poisson polynomials. One has $\deg \tau_j = \lfloor j/2 \rfloor$

The first few Poisson-Charlier polynomials are

$$\tau_0(n) = 1, \qquad \tau_1(n) = 0, \qquad \tau_2(n) = -n, \qquad \tau_3(n) = 2n,$$

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$$f(n) = \sum_{j\geq 0} \frac{P^{(j)}(n)}{j!}\tau_j(n)$$

But we wish truncate ... Are the first terms dominant for $n \to \infty$?

We need depoissonnization conditions on the Poisson transform $P(z)$....

Depoissonization path (II). $\mathcal{JS}$ Conditions for depoissonisation

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[$\mathcal{JS}$ admissibility] *An entire function $P(z)$ is $\mathcal{JS}$-admissible with parameters (α, β) if there exist $\theta \in]0, \pi/2[$, $\delta < 1$ for which (for $z \to \infty$)*

(I) *For $\arg z \leq \theta$, one has $|P(z)| = O\left(|z|^\alpha \log^\beta(1 + |z|)\right)$.*

(O) *For $\theta \leq \arg z \leq \pi$, one has $|P(z)e^z| = O\left(e^{\delta|z|}\right)$.*

Depoissonization Path (III) : the main result.

Theorem. (Jacquet-Szpankowski[1998] Hwang-Fuchs-Zacharovas[2010])
If the Poisson transform $P(z)$ *of* $f(n)$ *is* $\mathcal{JS}(\alpha, \beta)$ *admissible, then*

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$(O) \Longrightarrow$ the integral over $\{|z| = n, \epsilon \le |\arg z| \le \pi\}$ is negligible.
$(I) \Longrightarrow$ smooth estimates for all derivatives $P^{(k)}(z)$

$$P^{(k)}(z) = O\left(\int_{|w-z|=\epsilon|z|} \frac{|w|^{\alpha} \log^{\beta}(1+|w|)}{|w-z|^{k+1}} |dw|\right) = O\left(|z|^{\alpha-k} \log^{\beta}(1+|z|)\right)$$

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Let $P(z)$ be the Poisson transform of $f(n)$ assumed to be entire.

- *The two conditions are equivalent*
  - (i) $P(z)$ *is* $\mathcal{JS}$*-admissible*
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  and used for proving $(i) \to (ii)$.
- Jacquet proves that the analytic lifting exists in the cone $\mathcal{C}(0, \theta)$.
  He does not remark that it exists in fact in the cone $\mathcal{C}(-1, \theta)$...
  It will be important for us in the following...

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The analytical lifting $\varphi(z)$ is obtained with an extension of the Cauchy formula

$$f(n) = \frac{n!}{2i\pi} \int_{|z|=n} \frac{P(z)e^z}{z^{n+1}} dz$$

as: $$\varphi(z-1) = \frac{z\Gamma(z)}{2\pi} z^{-z} \int_{-\pi}^{+\pi} e^{i\theta} P(ze^{i\theta}) \exp[z(e^{i\theta} - i\theta)]\, d\theta$$

The integral part is an analytical function of x on the whole complex plane

The function $z\Gamma(z)z^{-z}$ is analytical in $\mathcal{C}(0, \theta)$ with $\theta < \pi$.

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- Flajolet and Sedgewick [1995]
- Many other
- A first attempt here for the last item...

The Mellin-Newton-Rice path (I). Mellin–Newton

If $n \mapsto f(n)$ has $\mathrm{val}(f) = 0$, $\deg(f) = c < 0$,
the sequence $\pi[f]$ has an analytic lifting on $\Re s > c$ of polynomial growth,

$$\pi[f](s) = \sum_{k \geq 0} (-1)^k f(k) \frac{s(s-1)\ldots(s-k+1)}{k!}\,.$$

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In the strip $\langle 0, -c \rangle$, the Mellin transform $P^*(s)$ of $P(z)$ exists and satisfies

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Exchange of integration and summation is justified

- each $\Gamma(s+k)$ is well defined for $k \geq 0$ as soon as $\Re s > 0$.
- $P^*(s)/\Gamma(s)$ is convergent for $\Re s + c < 0$ due to the estimate

$$\frac{1}{k!} \frac{\Gamma(s+k)}{\Gamma(s)} = \frac{s(s+1)\ldots(s+k-1)}{k!} = \frac{k^{s-1}}{\Gamma(s)} \left[1 + O\left(\frac{1}{k}\right) \right] \qquad (k \to \infty),$$

The equality holds on the strip $\langle c, 0 \rangle$

$$\varpi(s) := \frac{P^*(-s)}{\Gamma(-s)} = \sum_{k \geq 0} (-1)^k f(k) \frac{s(s-1)\dots(s-k+1)}{k!} \,.$$

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For any $a \in]c, 0[$ and $n \geq 0$, one has:

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} p(k) \Longrightarrow f(n) = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} L_n(s) \cdot \pi[f](s)\, ds$$

with the Rice kernel $\quad L_n(s) = \dfrac{(-1)^{n+1}\, n!}{s(s-1)(s-2)\ldots(s-n)}.$

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This integral representation is valid for $a \in [c, 0]$.

We now shift to the left ... and we need tameness conditions on $\pi[f]$,

and thus sufficient conditions on the sequence $n \mapsto f(n)$.

The Mellin-Newton-Rice path (III). Rice – Tameness and shifting to the left?

Definition. A function ϖ analytic and of polynomial growth on $\Re s > c$ is tame if there exists a region $\mathcal{R}$ between a curve $\mathcal{C} \subset \{\Re s < c\}$ and the line $\Re s = c$ for which ϖ is meromorphic and of polynomial growth on $\mathcal{R}$.

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$$f(n) = -\left[\sum_{k \mid s_k \in \mathcal{R}} \mathrm{Res}\left[L_n(s) \cdot \pi[f](s); s = s_k\right] + \frac{1}{2i\pi} \int_{\mathcal{C}} L_n(s) \cdot \pi[f](s)\, ds\right],$$

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And polynomial growth? True for $P^*(-s)$ – But with the factor $1/\Gamma(-s)$??

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There are sequences $f(k)$ which resemble geometric sequences...
when the function f is a Laplace transform of some function $\widehat{f}$. Then

$$f(k) = \int_0^\infty e^{-ku}\widehat{f}(u)du, \qquad \pi[f](s) = \int_0^\infty \widehat{f}(u)\,(1-e^{-u})^s du$$

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- proves the tameness condition to hold on $\pi[g]$ in this basic case.

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- proves the tameness condition to hold on $\pi[g]$ in this basic case.
- (in progress) extends the previous study to "generic" sequences f

Expression for the $\pi[f]$ lifting

Proposition. Consider a sequence $f \mapsto f(n)$ which is extended into a function $f : [0, +\infty] \to \mathbb{R}$. Assume the following

f is the Laplace transform of a function $\widehat{f} : [0, +\infty] \to \mathbb{R}^+$ integrable on $[0, +\infty]$ and continuous on $]0, +\infty[$.

Denote by σ the infimum of the reals τ for which $\int_0^\infty u^\tau \, \widehat{f}(u) du < \infty$

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Then, the $\pi[f]$ function admits an integral form on the halfplane $\Re s > \sigma$

$$\pi[f](s) = \int_0^\infty \widehat{f}(u) \, (1 - e^{-u})^s du = \int_0^\infty u^s \, \widehat{f}(u) \left(\frac{1 - e^{-u}}{u} \right)^s du$$

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Proposition. Consider a sequence $f \mapsto f(n)$ which is extended into a function $f : [0, +\infty] \to \mathbb{R}$. Assume the following
f is the Laplace transform of a function $\widehat{f} : [0, +\infty] \to \mathbb{R}^+$ integrable on $[0, +\infty]$ and continuous on $]0, +\infty[$.
Denote by σ the infimum of the reals τ for which $\int_0^\infty u^\tau \widehat{f}(u)du < \infty$
Then, the $\pi[f]$ function admits an integral form on the halfplane $\Re s > \sigma$

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A nice alternative integral expression of $\pi[f]$

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A nice alternative integral expression of $\pi[f]$

- It is important to characterise such functions f
  and compare σ to $c = \deg(f)$
- We do not know yet such a precise charecterisation...
- We limit ourselves to a class of particular functions...

A particular class of interest : Basic functions.

Consider a triple (k_0, d, b) with an integer $b \geq 0$, an integer $k_0 \geq 1$ which satisfies $k_0 \geq 2$ for $b > 0$ and a real $d < k_0$.

A sequence $k \mapsto f(k)$ is called basic with the triple (k_0, d, b) if it satisfies

$$f(k) = k^d \log^b k \; S\left(\frac{1}{k}\right) \quad \text{for } k \geq k_0, \qquad f(k) = 0 \quad \text{for } k < k_0$$

S is analytic at 0 with a convergence radius $r = 1/(k_0 - 1)$, and $S(0) = 1$.

The VD condition $d < k_0$ holds with $\mathrm{val}(f) = k_0$ and $\deg(f) = d$.

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The canonical sequence $g : k \mapsto g(k)$ is extended into $g :]-1, +\infty] \to \mathbb{R}$

$$g(x) = (x + k_0)^{d-k_0} \log^b(x + k_0) \; T\left(\frac{1}{x + k_0}\right)$$

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Ex: $f(k) = k \log k$ with $k_0 = 2 \Longrightarrow g(x) = \dfrac{\log(x+2)}{x+1} = \dfrac{\log(x+2)}{x+2}\left(1 - \dfrac{1}{x+2}\right)^{-1}$

Expression of $\pi[g]$ for canonical sequences associated with basic functions f.

Proposition. Consider the canonical sequence $g(n)$ associated with a basic sequence of the form $f(k) = k^d \log^b k \; S(1/k)$. Let $c = d - k_0 < 0$.

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- Its inverse Laplace transform $\widehat{g}$ is a linear combination of functions

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  where V_ℓ satisfy $V_\ell(0) \neq 0$ and $|V_\ell(u)| \leq e^{(u/2)(2k_0-1)}$.

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- On $\Re s > c$, its $\pi[g]$ function is a linear combination of functions

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Proof. $\quad g_t = (1+x_0)^{-(t+c)} \quad \Longrightarrow \quad \widehat{g}_t(u) = \dfrac{u^{c+t-1}}{\Gamma(c+t)}$

We deal with the $\log$ factors via the derivative wrt to t at $t = 0$

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Proof. Consider the derivatives of the Γ function and the linear form $\mathcal{I}_s$

$$\Gamma^{(\ell)}(s) = \int_0^\infty e^{-u} u^{s-1} \log^\ell u \, du \qquad \mathcal{I}_s[h] := \int_0^\infty h(u) u^s \left(\frac{1 - e^{-u}}{u}\right)^s du$$

On any halfplane $\Re s + c \geq \sigma_0 > -1$, the difference, for any fixed ℓ,

$$\mathcal{I}_s[u^{c-1} \log^\ell u] - \Gamma^{(\ell)}(s + c) = \int_0^\infty e^{-u} u^{c+s-1} \log^\ell u \left[\left(\frac{1 - e^{-u}}{u}\right)^s - 1\right] du$$

defines a normally convergent integral.

Tameness of $\pi[g]$ for canonical sequences associated with a generic functions f.

We have shown tameness of $\pi[f]$ for basic sequences,

We solve our problem for $f(k) = k \log k$

where we prove that the Newton-Rice path may be used.

We expect a general result which validates the Newton–Rice path for generic sequences in the same general framework as Depoissonization.

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Remind : If the sequence f satisfies the $\mathcal{JS}$ condition, then

- the sequence f admits an analytical lifting $f(z)$ in any cone $\mathcal{C}(-1, \theta)$ which is of polynomial growth in a cone $\mathcal{C}(0, \theta_0)$
- the canonical sequence g admits an analytical lifting $g(z)$ in any cone $\mathcal{C}(-1, \theta)$ which is of polynomial growth in a cone $\mathcal{C}(0, \theta_0)$. We can choose the polynomial growth $c < -1$.

We thus need a result for general canonical sequences g which

- describes the properties of their inverse Laplace transform $\widehat{g}$
- makes possible the extension of our previous study

Tameness of $\pi[g]$ for canonical sequences associated with a generic functions f.

We thus need a result as follows (but yet not completely proven)

Proposition (?). Consider a function g as follows

- it is analytic in the cone $\mathcal{C}(-1, \theta)$ with any $\theta < \pi$.
- For $c < -1$, $g(z)$ is $O(z+1)^c \log^{\ell}(z+1)$ in a cone $\mathcal{C}(0, \theta)$, $\theta > \pi/2$

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Then its inverse Laplace transform $\widehat{g}(u)$ exists, is analytic on the real line $[0, +\infty]$ and satisfies for some $a \in]0, 1[$ the estimate $O\left(e^{-au} u^{c-1} \log^\ell u\right)$

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It would validate our approach for the Newton-Rice path.

Proposition/Conjecture. Consider a sequence $n \mapsto f(n)$ which satisfies the $\mathcal{JS}$ condition, namely, it admits an analytical lifting $f(z)$ in any cone $\mathcal{C}(-1, \theta)$ which is of polynomial growth in a cone $\mathcal{C}(0, \theta_0)$.
If moreover, the angle θ_0 satisfies $\theta_0 > \pi/2$, then

- the method of the inverse Laplace transform may be used,
- the Newton-Rice path may be used

Conclusion : Comparison between the two paths.

Tools used in Depoissonization.

first derive asymptotics of $P(z)$ for large $|z|$ by the inverse Mellin integral

$$P(z) = \frac{1}{2i\pi} \int_{\uparrow} P^*(s) z^{-s} ds = \frac{1}{2i\pi} \int_{\uparrow} P^*(-s) z^{s} ds \,, \tag{1}$$

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and use the Cauchy integral formula

$$f(n) = \frac{n!}{2i\pi} \int_{|z|=r} P(z)\, e^z \frac{1}{z^{n+1}} dz \,.$$

Compare with the Newton-Rice approach.

As $P(z)e^z$ is entire, replace the contour $\{|z| = r\}$ by a Hankel contour

$$f(n) = \frac{n!}{2i\pi} \int_{\mathcal{H}} P(z)\, e^z \frac{1}{z^{n+1}} dz \tag{2}$$

High-level and formal view

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and use the equality $\quad \dfrac{1}{\Gamma(n+1-s)} = \dfrac{1}{2i\pi} \displaystyle\int_{\mathcal{H}} e^z \frac{z^s}{z^{n+1}} dz \,,$

we obtain the representation

$$f(n) = \frac{n!}{2i\pi} \int_{\uparrow} P^*(-s) \frac{1}{\Gamma(n+1-s)} ds = \frac{1}{2i\pi} \int_{\uparrow} \pi[f](s) \frac{(-1)^{n+1}\, n!}{s(s-1)\dots(s-n)} ds \,.$$

Comparison of analytic tools. Conclusion

– A priori not the same tools in the two paths

– It is interesting to compare these two paths (not generally done...)

– The method of the inverse Laplace transform seems powerful
(when it may be used) for the two paths

– It remains to completely validate this approach....