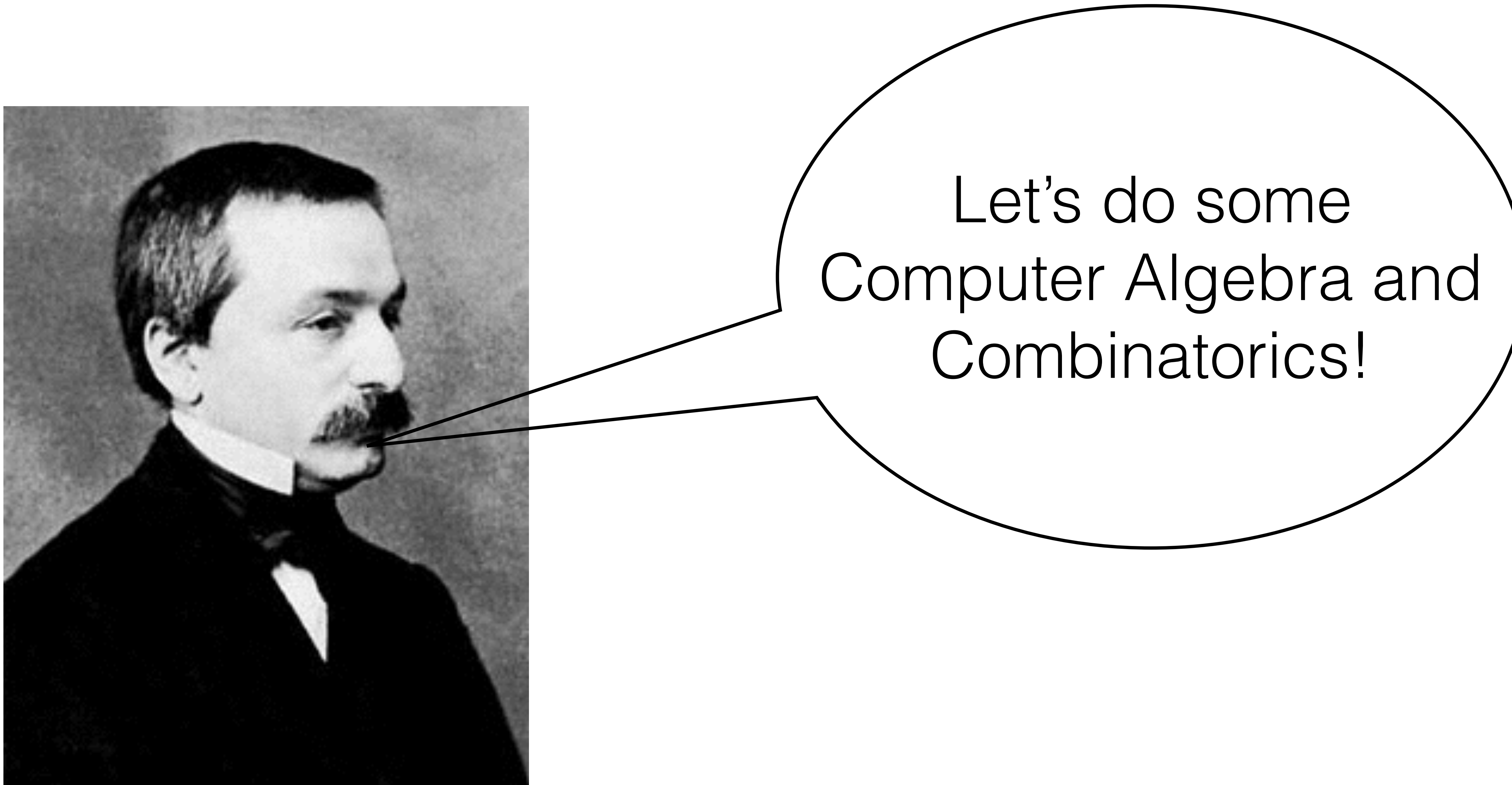
SYMBOLIC-NUMERIC TOOLS FOR ANALYTIC COMBINATORICS IN SEVERAL VARIABLES

Stephen Melczer
ENS Lyon / Inria & University of Waterloo

Joint work with Bruno Salvy



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Lassen Sie uns einige Computer-Algebra und Kombinatorik tun!

Motivation: Asymptotics of Diagonals

Input: Rational function $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ with power series

$$F(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}^n} c_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$$

Goal: Asymptotics of the *diagonal sequence* $c_{k,k,\ldots,k}$ as $k \to \infty$

Example (Simple lattice walks)

$$F(x,y) = \frac{1}{1-x-y}$$

Here $(c_{k,k})_{k\geq 0}$ counts the number of ways to walk from (0,0) to (k,k) using the steps (0,1) and (1,0).

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Goal: Asymptotics of the *diagonal sequence* $c_{k,k,\ldots,k}$ as $k \to \infty$

Example (Restricted factors in words)

$$F(x,y) = \frac{1 - x^3y^6 + x^3y^4 + x^2y^4 + x^2y^3}{1 - x - y + x^2y^3 - x^3y^3 - x^4y^4 - x^3y^6 + x^4y^6}$$

Here $(c_{k,k})_{k \geq 0}$ counts the number of binary words with k zeroes and k ones that do not contain 10101101 or 1110101.

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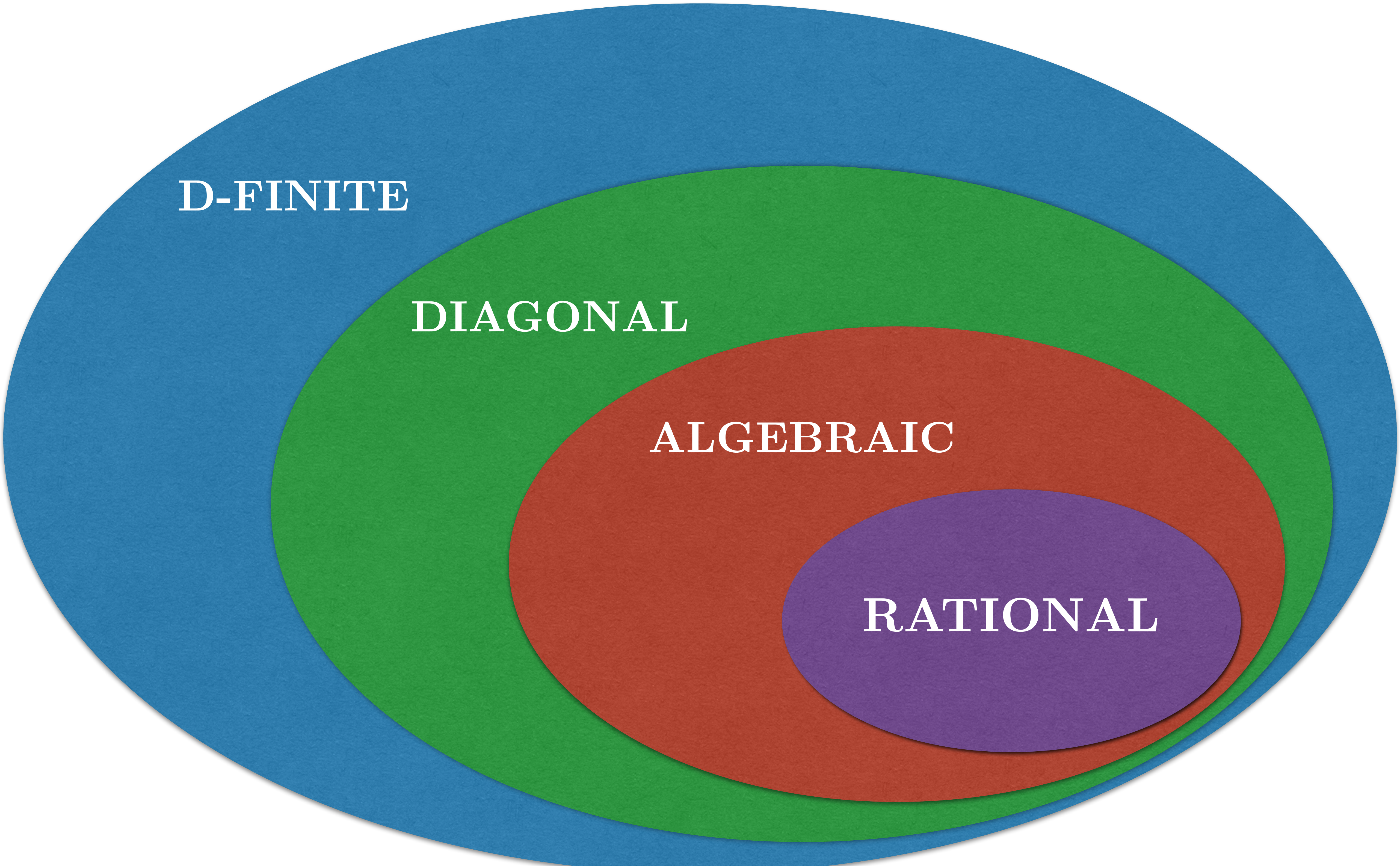
Goal: Asymptotics of the *diagonal sequence* $c_{k,k,\dots,k}$ as $k \to \infty$

Example (Apéry)

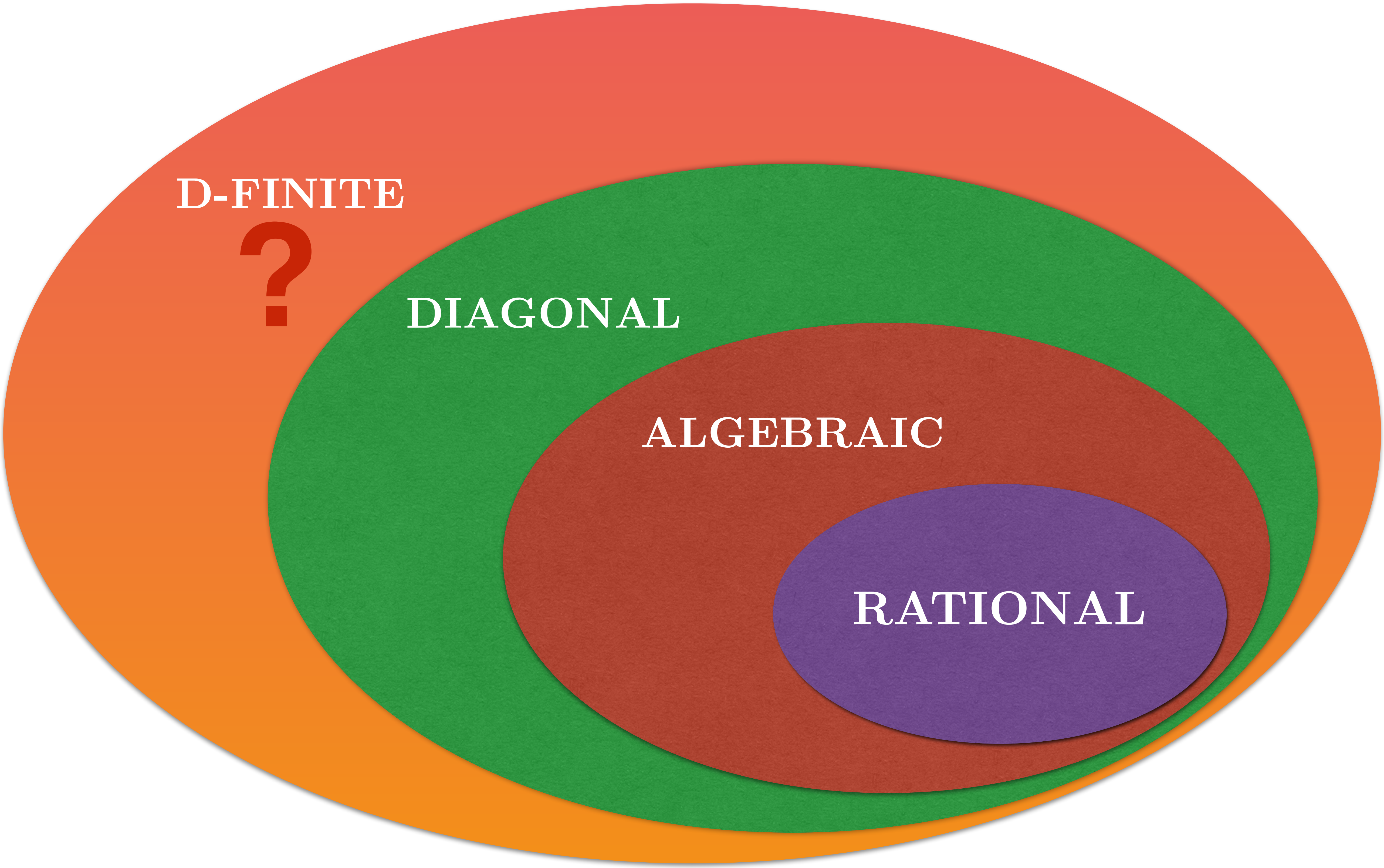
$$F(w,x,y,z) = \frac{1}{1 - z(1+w)(1+x)(1+y)(wxy + xy + x + y + 1)}$$

Here $(c_{k,k,k,k})_{k \geq 0}$ determines Apéry's sequence, related to his celebrated proof of the irrationality of $\zeta(3)$.

Connections to Univariate Theory



Connections to Univariate Theory



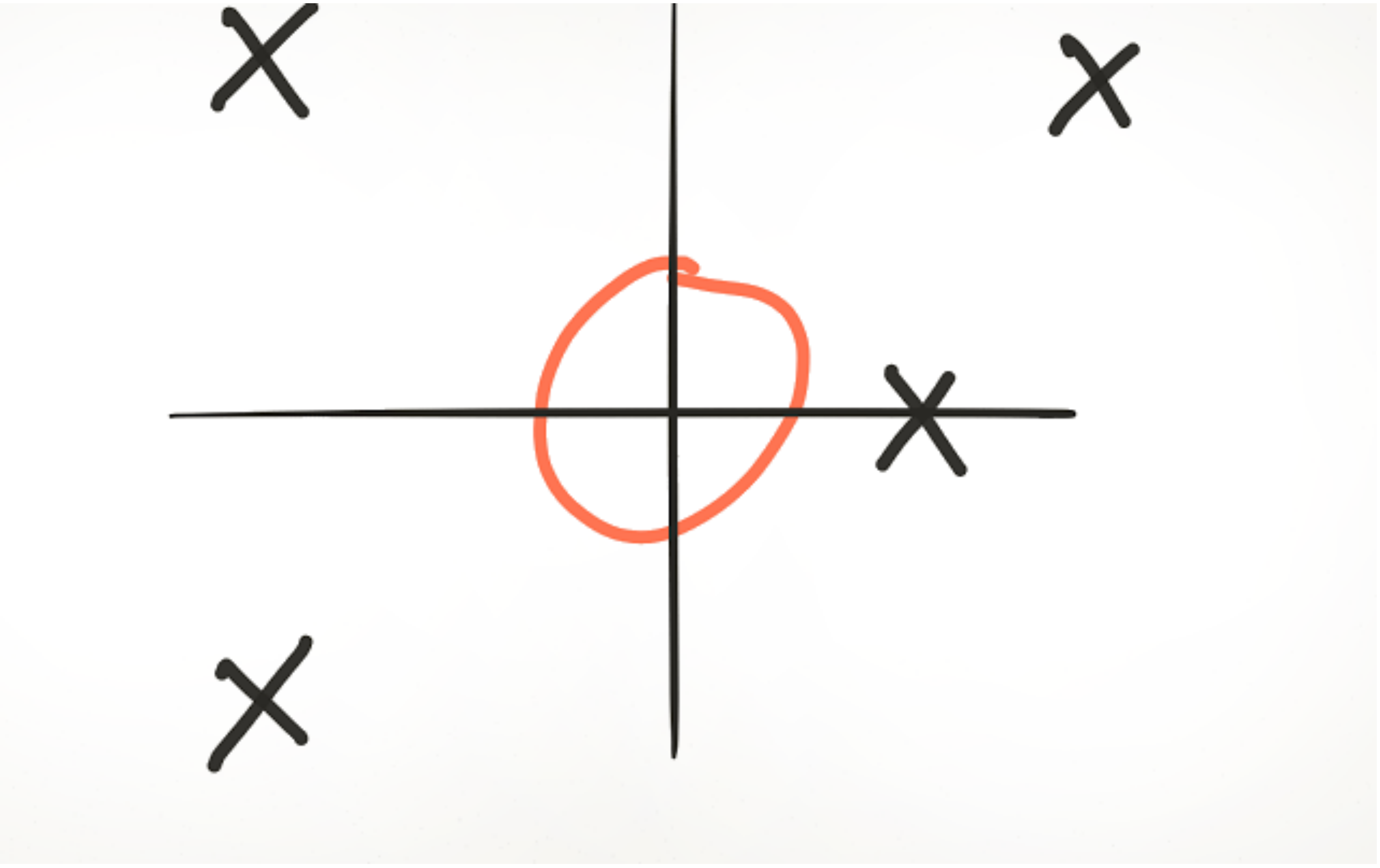
Univariate Analytic Combinatorics

If $S(z) = \sum_{k\geq 0} s_k z^k$ is analytic then

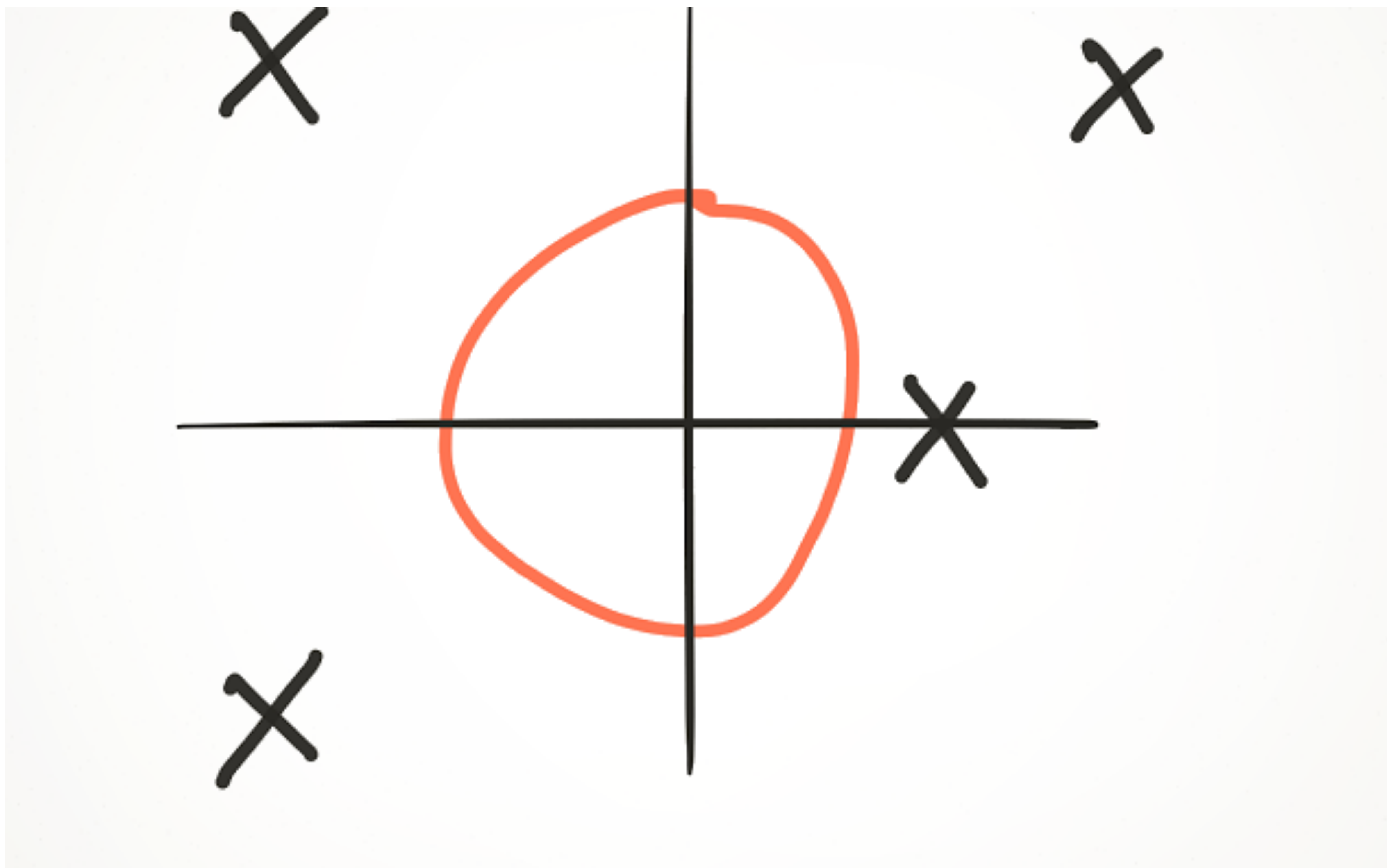
$$\begin{aligned} \frac{1}{2\pi i}\int_C S(z)\frac{dz}{z^{n+1}} &= \frac{1}{2\pi i}\int_C \sum_{k\geq 0} s_k z^{k-n-1}\,dz \\ &= \sum_{k\geq 0}\frac{1}{2\pi i}\int_C s_k z^{k-n-1}\,dz \\ &= s_n \end{aligned}$$

for a suitable closed curve C around the origin.

"Transfer Theorem" for Asymptotics



“Transfer Theorem” for Asymptotics



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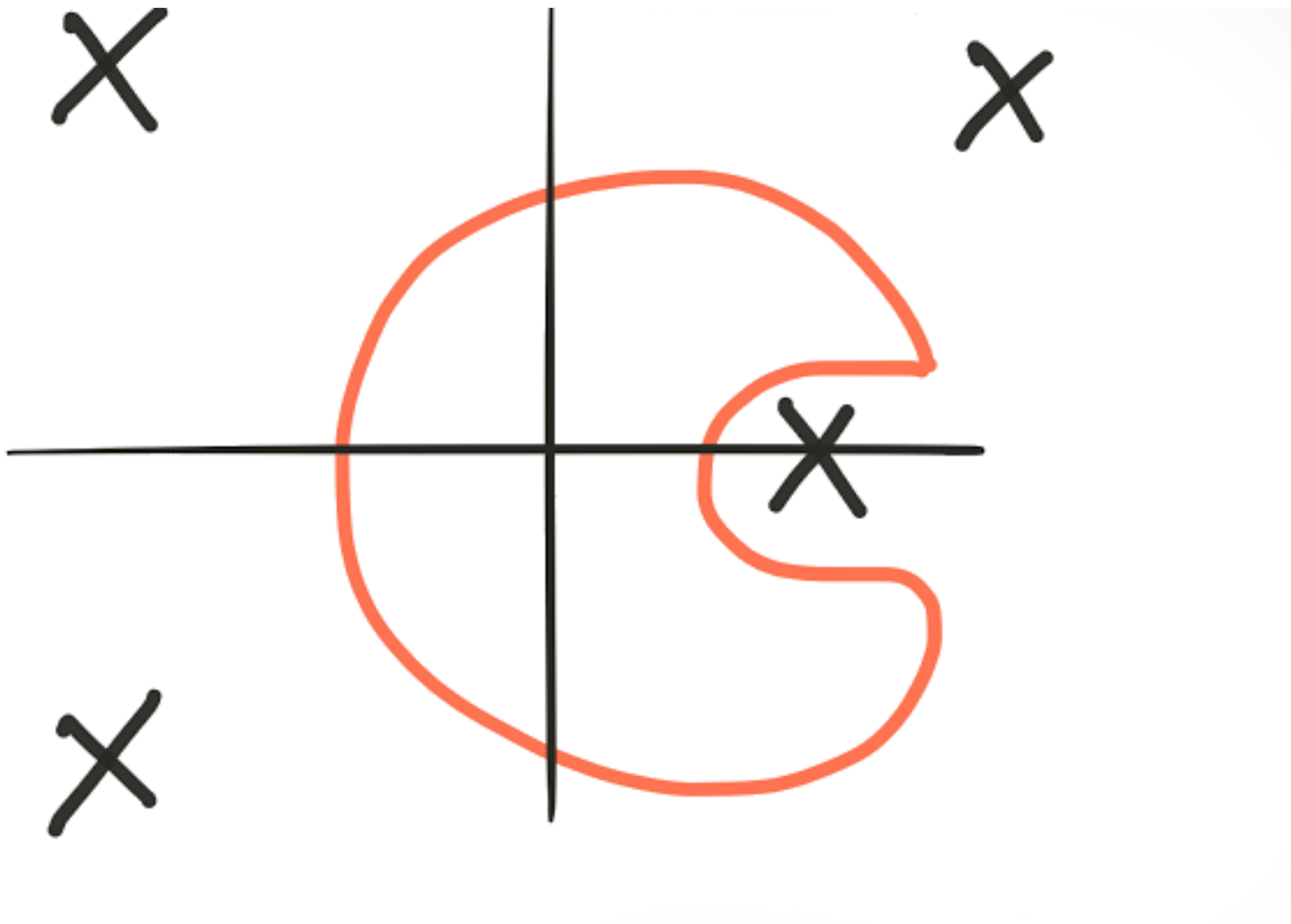

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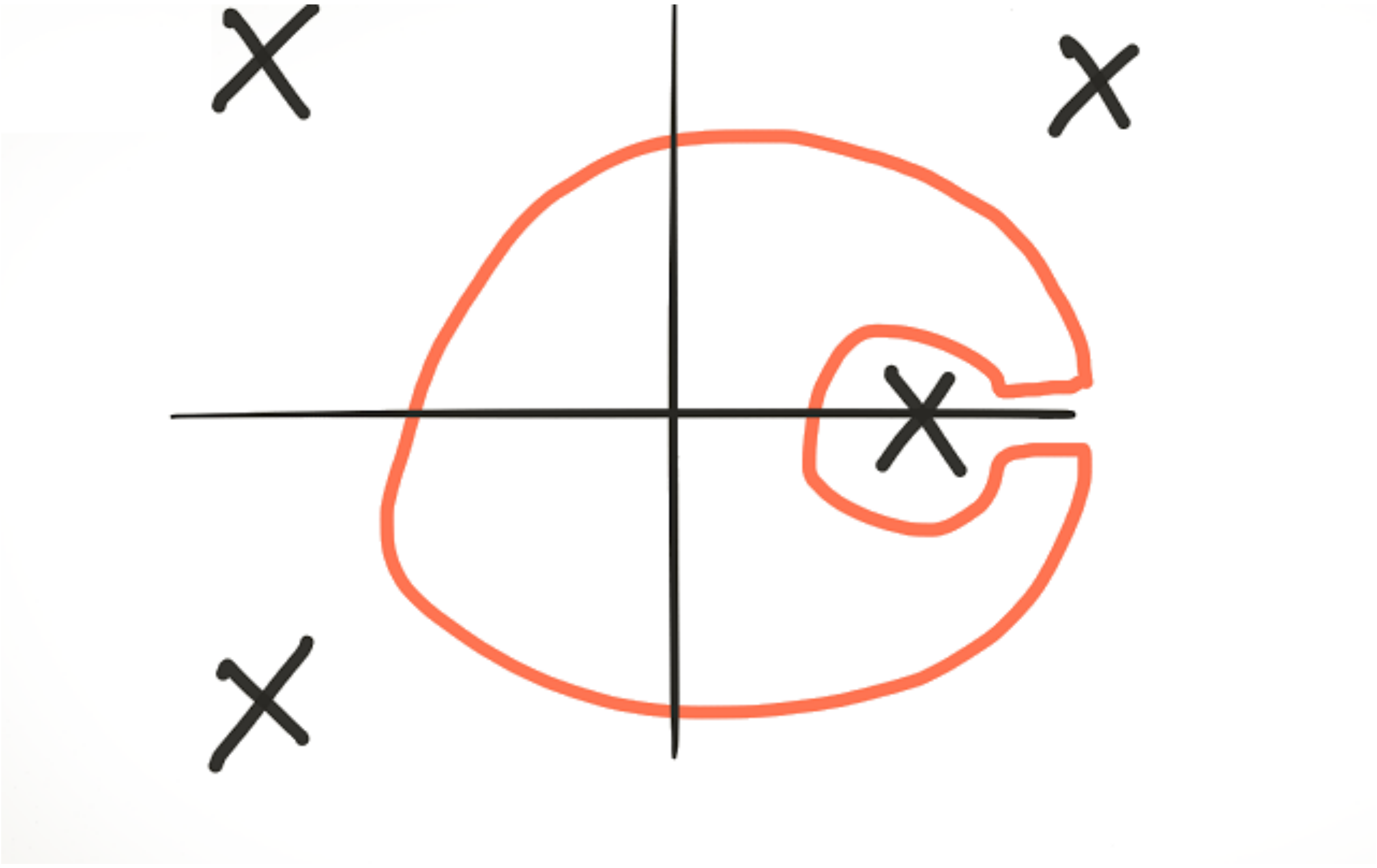

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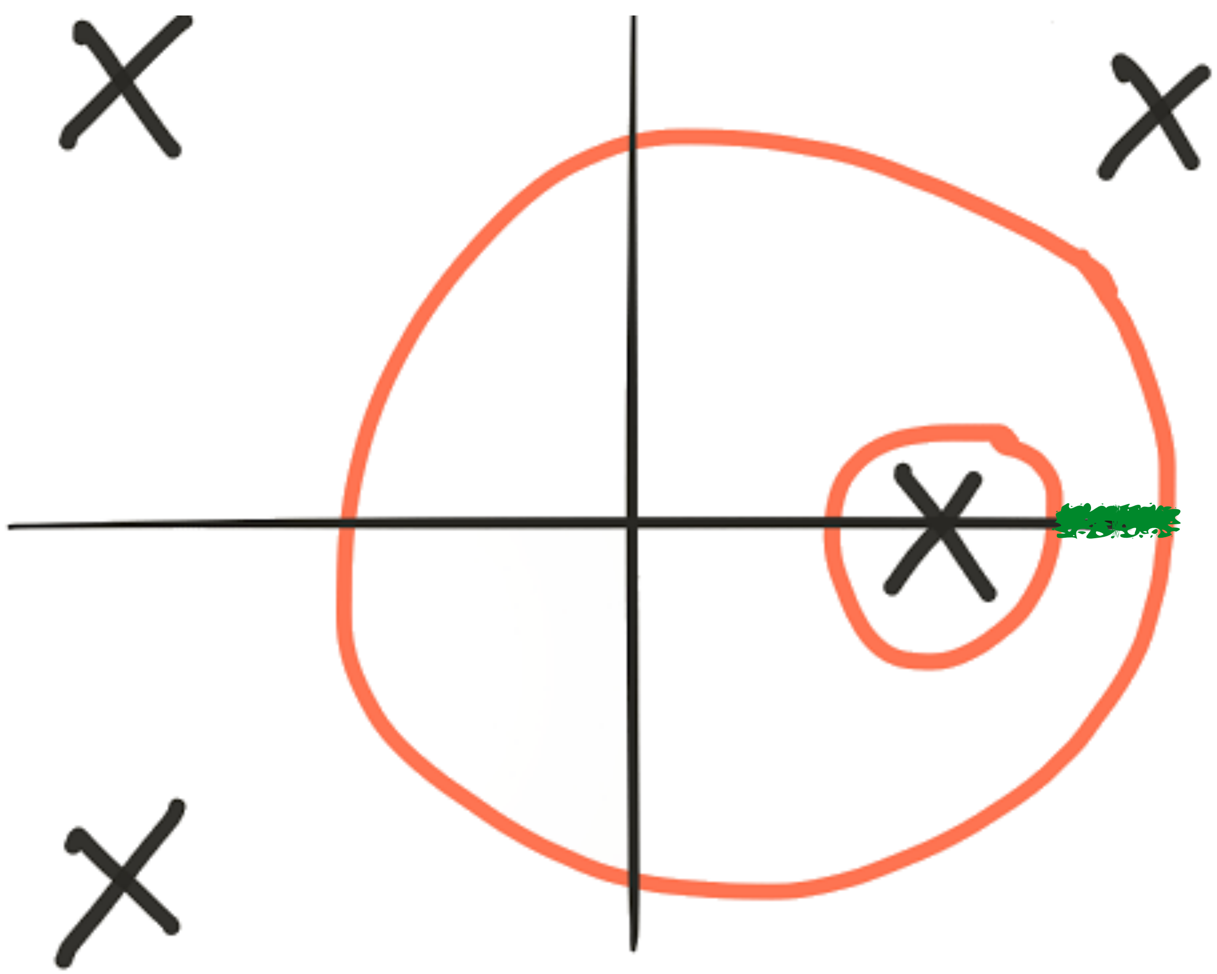

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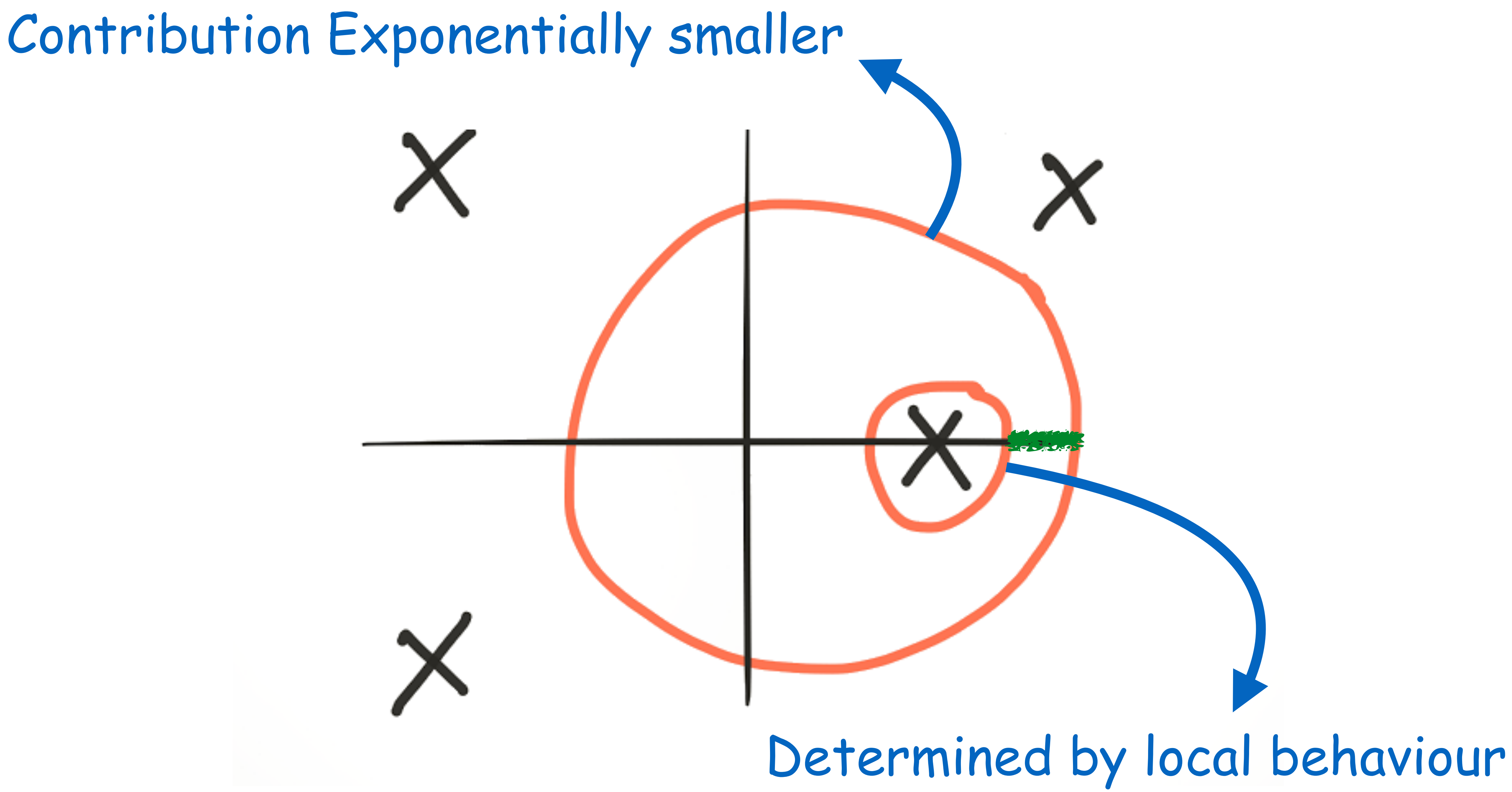


Analytic Combinatorics in Several Variables

Theorem (CIF In Several Variables)

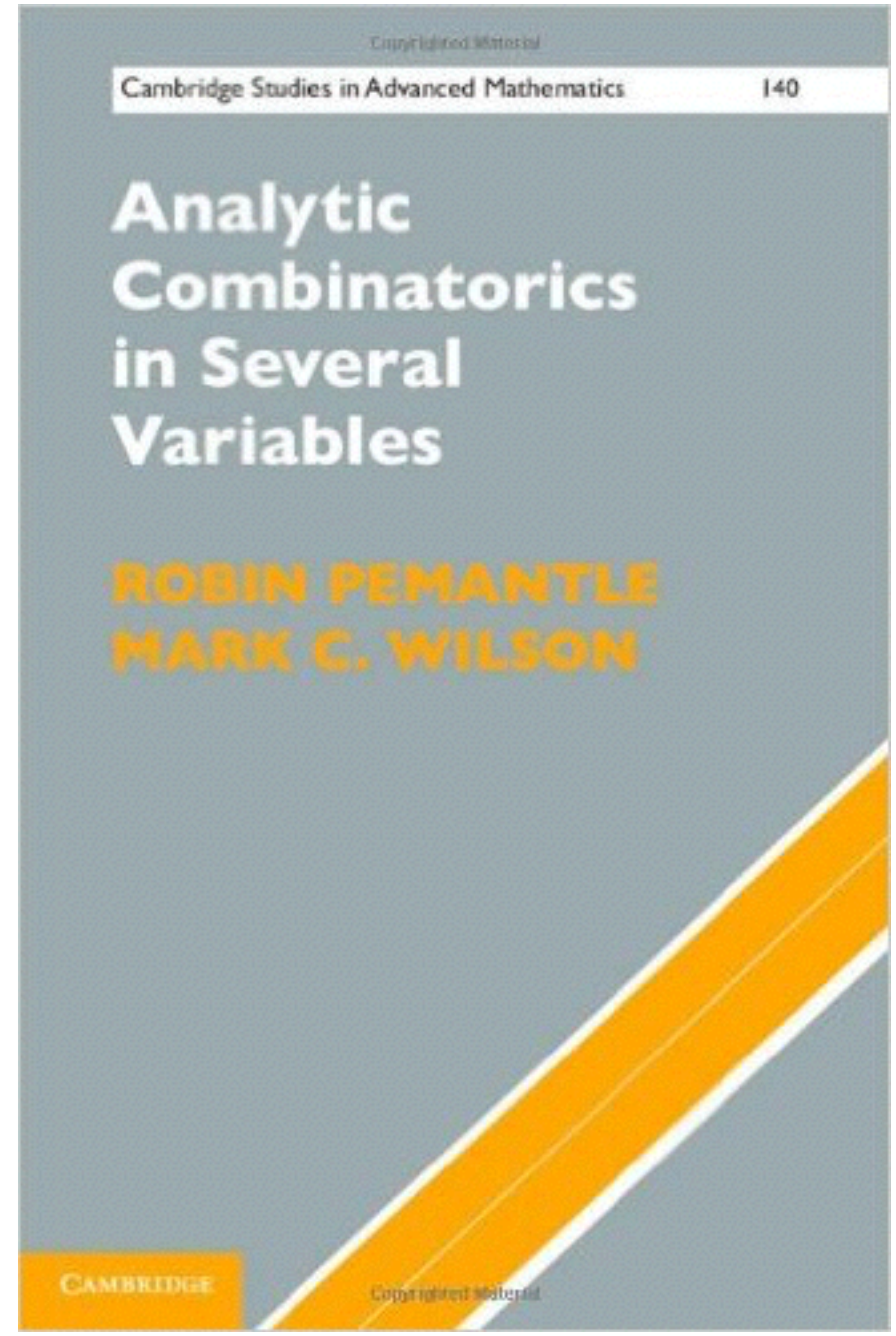
Let $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z}) \in \mathbb{Q}(\mathbf{z})$ be holomorphic at the origin. Then there is a unique series $F(\mathbf{z}) = \sum_{\mathbf{i}\in\mathbb{N}^n} a_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}$ converging absolutely in a neighbourhood of the origin, with

$$a_{\mathbf{i}} = \left(\frac{1}{2\pi i}\right)^n \int_T \frac{f(\mathbf{z})}{\mathbf{z}^{\mathbf{i}}} \cdot \frac{d\mathbf{z}}{z_1\cdots z_n}$$

The goal is to determine the points where local behaviour determines asymptotics, and deform T to be close to such points. ***In general this is very very hard!***

Analytic Combinatorics in Several Variables

There has been much work in the last few decades translating results from Complex Analysis in Several Variables to this context.



Essentially, the important points are the zeroes of $H(\mathbf{z})$ which are **on the boundary of the domain of convergence** and **minimize the product** $|\mathbf{z_1} \cdots \mathbf{z_n}|$, which encodes exponential growth.

$$c_{k,\ldots,k} \sim |z_1 \cdots z_n|^k \cdot \theta(k)$$

Analytic Combinatorics in Several Variables

$F(\mathbf{z})$ is ***combinatorial*** if every coefficient $c_{\mathbf{i}} \geq 0$.

Assuming F is combinatorial + generic conditions, asymptotics are determined by the points in $\mathcal{V} = \{\mathbf{z} : H(\mathbf{z}) = 0\}$ such that

$$\begin{cases} (1) \quad z_1 H_{z_1}(\mathbf{z}) = \cdots = z_n H_{z_n}(\mathbf{z}) \\ \\ (2) \quad \mathbf{z} \text{ on boundary of domain of convergence} \end{cases}$$

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$\mathbf{z}$ is coordinate-wise minimal in $\mathcal{V}$

(1) is an **algebraic** condition.

(2) is a **semi-algebraic** condition (can be expensive).

Kronecker's Approach to Solving (RUR)

Generically there are a finite number of solutions of (1).

"Kronecker's Approach" to Solving (1880s):

Results in linear form

$$u = \lambda_1 z_1 + \lambda_2 z_2 + \cdots + \lambda_n z_n, \qquad \lambda_i \in \mathbb{C}$$

and parametrization

$$\begin{aligned} P(u) &= 0 \\ P'(u) z_1 - Q_1(u) &= 0 \\ &\vdots \\ P'(u) z_n - Q_n(u) &= 0 \end{aligned}$$

square-free

History and Background: see Castro, Pardo, Hägele, and Morais (2001)

The Kronecker Representation

Compute a Kronecker Representation of the system

$$H(\mathbf{z}) = 0, \qquad z_1 H_{z_1}(\mathbf{z}) = \cdots = z_n H_{z_n}(\mathbf{z})$$

Suppose

$$\deg(H) = d, \qquad \max \operatorname{coeff}(H) \leq 2^h \qquad D := d^n$$

Then in $\tilde{O}(hD^3)$ bit ops there is a prob. algorithm to find:

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$$\left.\begin{aligned} P(u) &= 0 \\ P'(u)z_1 - Q_1(u) &= 0 \\ &\vdots \\ P'(u)z_n - Q_n(u) &= 0 \end{aligned}\right\} \begin{array}{ll} \text{Degree} & \leq D \\ \text{Height} & \leq \tilde{O}(hD) \end{array}$$

Giusti, Lecerf, Salvy (2001)
Schost (2001)

Example (Lattice Path Model)

The number of walks from the origin taking steps $\{NW,NE,SE,SW\}$ and staying in the first quadrant has

$$F(x,y,t) = \frac{(1+x)(1+y)}{1-t(x^2+y^2+x^2y^2+1)}$$

One can calculate the Kronecker representation

$$\begin{aligned}
P(u) &= 4u^4 + 52u^3 - 4339u^2 + 9338u + 403920 \\
Q_x(u) &= 336u^2 + 344u - 105898 \\
Q_y(u) &= -160u^2 + 2824u - 48982 \\
Q_t(u) &= 4u^3 + 39u^2 - (4339/2)u + 4669/2
\end{aligned}$$

so that the solutions of (1) are described by:

$$P(u) = 0, \quad x = \frac{Q_x(u)}{P'(u)}, \quad y = \frac{Q_y(u)}{P'(u)}, \quad t = \frac{Q_t(u)}{P'(u)}$$

Example (Lattice Path Model)

The number of walks from the origin taking steps {NW,NE,SE,SW} and staying in the first quadrant has

$$F(x,y,t) = \frac{(1+x)(1+y)}{1-t(x^2+y^2+x^2y^2+1)}$$

One can

Which of these points are on the domain of convergence?

so that t

$$P(u) = 0, \quad x = \frac{Q_x(u)}{P'(u)}, \quad y = \frac{Q_y(u)}{P'(u)}, \quad t = \frac{Q_t(u)}{P'(u)}$$

Numerical Kronecker Representation

These bounds allow for efficient computation via efficient algorithms for polynomial solving and root bounds.

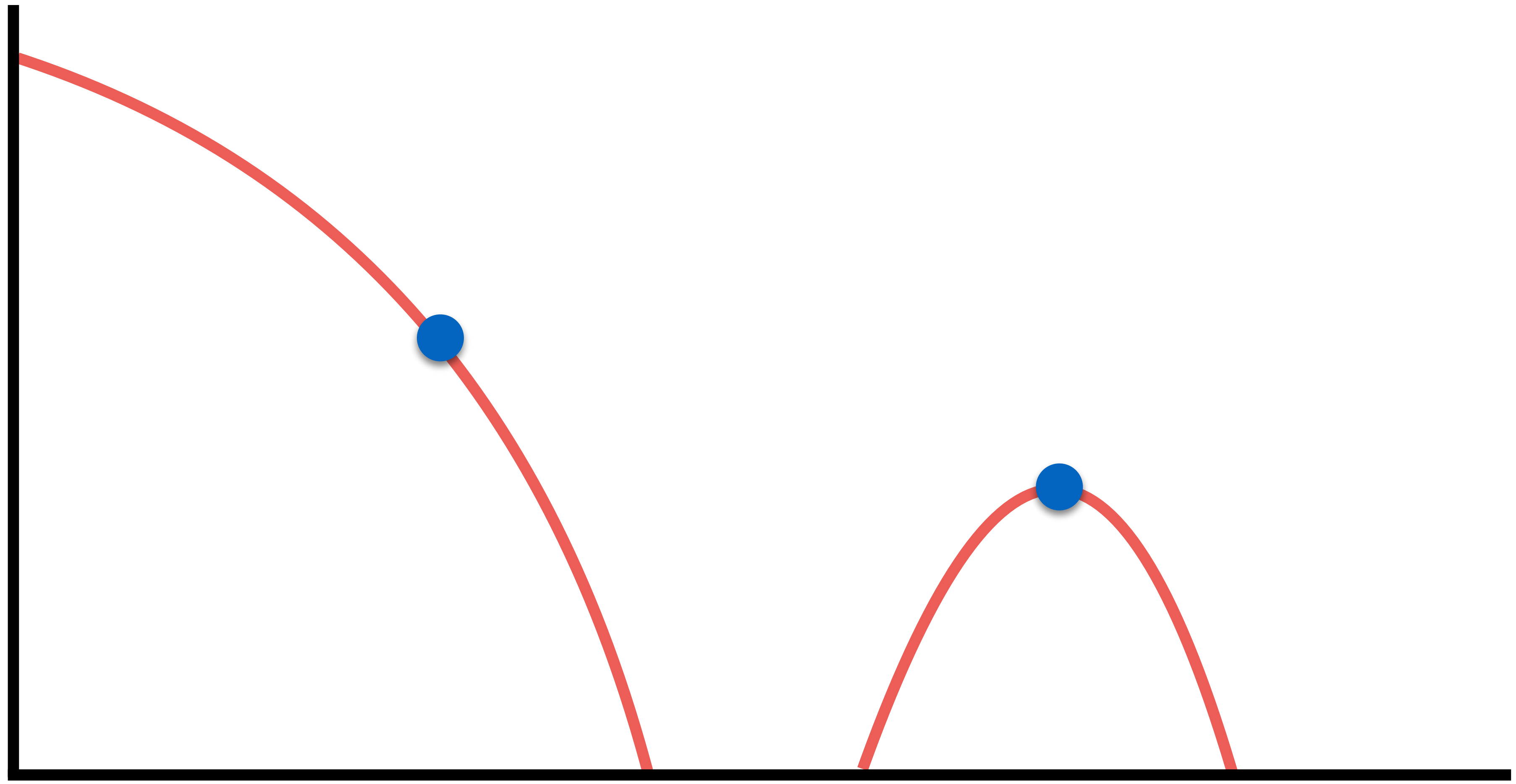
Operation on Coordinates	BIT COMPLEXITY
Determine 0	$\tilde{O}(hD^3)$
Determine sign	$\tilde{O}(hD^3)$
Find all equal coordinates	$\tilde{O}(hD^4)$
Find $\lvert z_j - a\rvert \leq 2^{-\kappa}$	$\tilde{O}(D(hD^2 + \kappa))$

Fast univariate solving: Sagraloff and K. Mehlhorn (2016)

Back to ACSV

In the combinatorial case*, it is enough to examine only the positive real points on the variety to determine ***minimality***.

Thus, one adds the equation $H(tz_1, \ldots, tz_n) = 0$ for a new variable t and finds the positive real point(s) $\mathbf{z}$ with no $t \in (0, 1)$.



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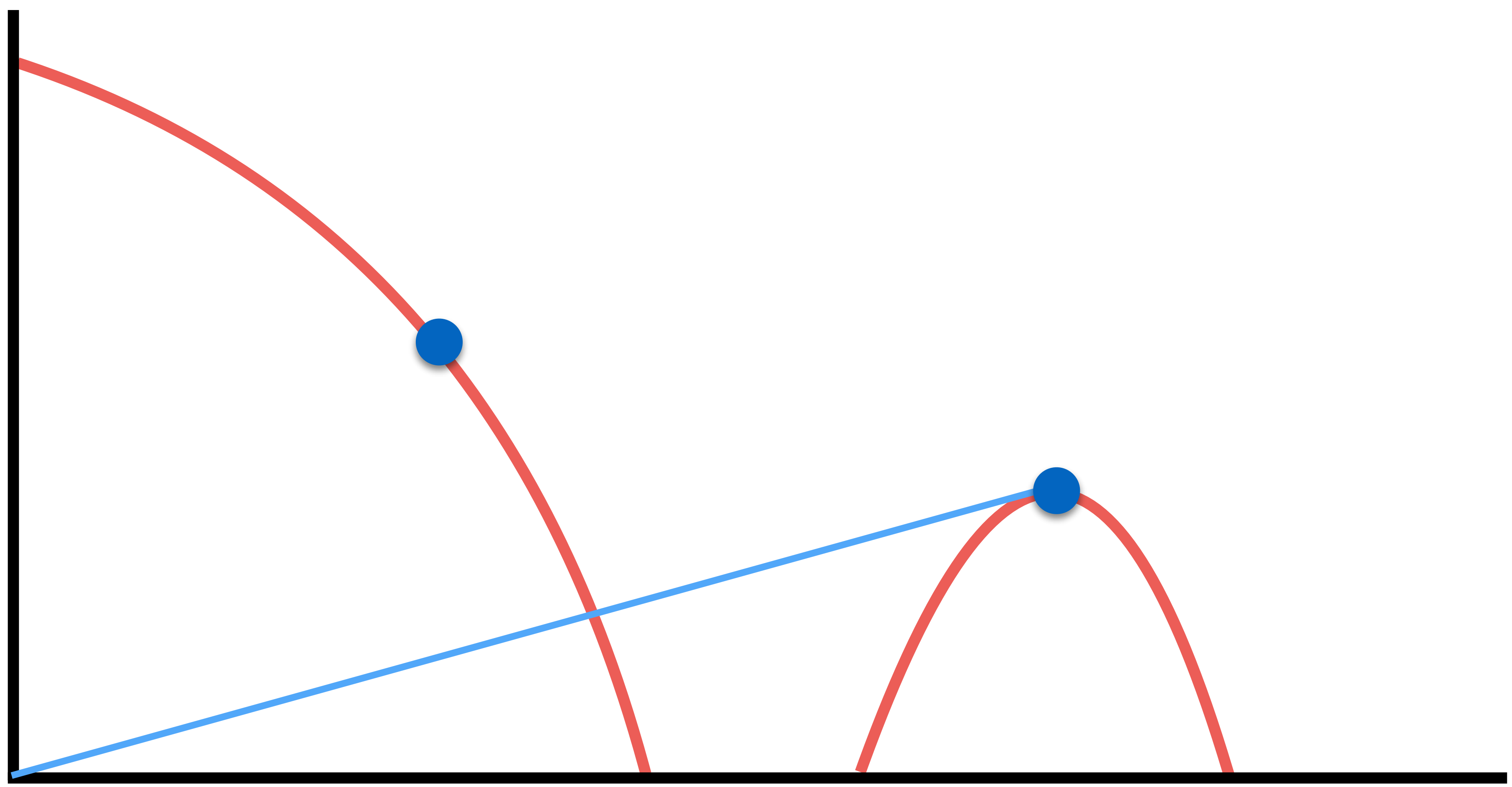

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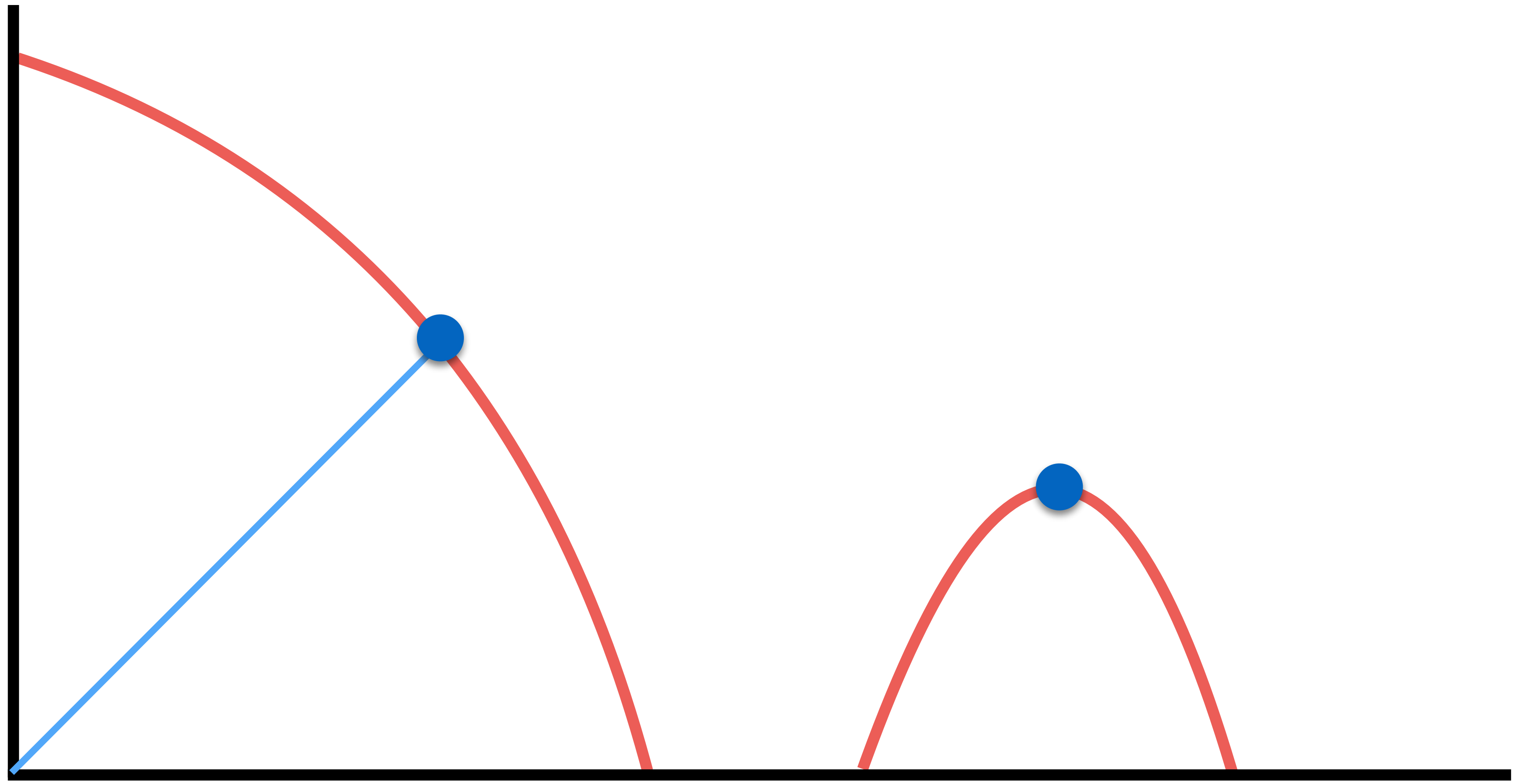

First Complexity Results for ACSV

Theorem (M. and Salvy, 2016)

Under generic conditions, and assuming $F(\mathbf{z})$ is combinatorial, the points contributing to dominant diagonal asymptotics can be determined in $\tilde{O}(hD^5)$ bit operations. Each contribution has the form

$$A_k = \left(T^{-k} k^{(1-n)/2} \cdot (2\pi)^{(1-n)/2}\right) (C + O(1/k))$$

and can be found to $2^{-\kappa}$ precision in $\tilde{O}(h(dD)^3 + D\kappa)$ bit ops.

The genericity assumptions heavily restrict the form of the asymptotic growth. Removing more of these assumptions is ongoing work.

Example 1

Example (Apéry)

$$F(w,x,y,z) = \frac{1}{1 - z(1+w)(1+x)(1+y)(wxy + xy + x + y + 1)}$$

For simplicity, we add the variable $T = xywz$ and use the linear form $u - T = 0$. The Kronecker representation is

$$P(u) = u^2 - 34u + 1$$

$$a = \frac{2u - 82}{2u - 34}, \quad b = c = \frac{24}{2u - 34}, \quad z = \frac{-164u + 4}{2u - 34}, \quad T = u$$

There are two real critical points, and one is positive. After testing minimality, one has proven asymptotics

$$A_k = \frac{(17 + 12\sqrt{2})^k}{k^{3/2}} \cdot \frac{\sqrt{34 + 24\sqrt{2}}}{8\pi^{3/2}}(1 + O(1/k))$$

Example 1

Example (Apéry)

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```
DiagonalAsymptotics(numer(F),denom(F),[a,b,c,z],u,k, useFGb);
```

$$\frac{1}{4} \frac{\left(\frac{2u + 182}{34u - 2234}\right)^k \sqrt{2} \sqrt{\frac{2u + 182}{-96u + 6360}}}{k^{3/2} \pi^{3/2}}, \left[RootOf\left(\_Z^2 + 182\,\_Z - 16361, 65.9777\right.\right.$$

(Non-)Example 2

Example (Restricted Words in Factors)

$$F(x,y) = \frac{1 - x^3y^6 + x^3y^4 + x^2y^4 + x^2y^3}{1 - x - y + x^2y^3 - x^3y^3 - x^4y^4 - x^3y^6 + x^4y^6}$$

One can get a more direct representation for the variables x, y , as in the system

$$P(u) = 32u^{21} - 240u^{20} + \cdots - 98400u^2 + 4176u$$
$$x = \tilde{Q}_1(u), \qquad y = \tilde{Q}_2(u)$$

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But, the (lowest terms) coefficients of $\tilde{Q}_1, \tilde{Q}_2$ have numerators of size 10^{20} !

Example 3

Example (Restricted Words in Factors)

$$F(x,y) = \frac{1 - x^3y^6 + x^3y^4 + x^2y^4 + x^2y^3}{1 - x - y + x^2y^3 - x^3y^3 - x^4y^4 - x^3y^6 + x^4y^6}$$

Using the Kronecker representation gives

$$P(u) = 32u^{21} - 240u^{20} + \cdots - 98400u^2 + 4176u$$
$$P'(u)x - Q_1(u) = P'(u)y - Q_2(u) = 0$$

where Q_1, Q_2 are polynomials with largest coeff around 2500.

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where Q_1, Q_2 are polynomials with largest coeff around 2500. There are two points with positive coordinates, adding $H(xt, yt) = 0$ allows one to show which is minimal, and that

$$A_k = \frac{(.6029\ldots)(3.9101\ldots)^k}{k^{1/2}}(1 + O(1/k))$$

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Example (Restricted Words in Factors)

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```
> ASM, U := DiagonalAsymptotics(numer(F),denom(F),indets(F),u,k,true,u-T,T):
  ASM;
```

$$\frac{1}{2}\Bigg(\Bigg(\frac{84u^{20} + 240u^{19} - 285u^{18} - 1548u^{17} - 2125u^{16} - 1408u^{15} + 255u^{14} + 756u^{13} + 2509u^{12} + 2856u^{11} + 605u^{10} + 2020u^{9} + 1233u^{8} - 1760u^{7} + 9}{-12u^{20} + 30u^{19} + 258u^{18} + 500u^{17} + 440u^{16} - 102u^{15} - 378u^{14} - 1544u^{13} - 2142u^{12} - 550u^{11} - 2222u^{10} - 1644u^{9} + 2860u^{8} - 1848u^{7} + 1230}$$

$$\sqrt{\frac{84u^{20} + 240u^{19} - 285u^{18} - 1548u^{17} - 2125u^{16} - 1408u^{15} + 255u^{14} + 756u^{13} + 2509u^{12} + 2856u^{11} + 605u^{10} + 2020u^{9} + 1233u^{8} - 1760u^{7} + 92}{-162u^{18} - 612u^{17} - 902u^{16} - 616u^{15} + 254u^{14} + 548u^{13} + 2054u^{12} + 2156u^{11} + 898u^{10} + 2268u^{9} + 2462u^{8} - 2088u^{7} + 1312u^{6} - 5}}$$

$$- 255u^{16} - 190u^{15} - 19u^{14} + 46u^{13} + 461u^{12} + 628u^{11} + 133u^{10} + 374u^{9} + 161u^{8} - 384u^{7} + 146u^{6} - 138u^{5} - 285u^{4} - 40u^{3} + 91u^{2} - 30u + 32)$$

$$+ 756u^{13} + 2509u^{12} + 2856u^{11} + 605u^{10} + 2020u^{9} + 1233u^{8} - 1760u^{7} + 924u^{6} - 492u^{5} - 675u^{4} + 632u^{3} - 249u^{2} + 24u + 16)\Bigg)$$

```
> U;
```

$$[\mathit{RootOf}(4\_Z^{21} + 12\_Z^{20} - 15\_Z^{19} - 86\_Z^{18} - 125\_Z^{17} - 88\_Z^{16} + 17\_Z^{15} + 54\_Z^{14} + 193\_Z^{13} + 238\_Z^{12} + 55\_$$

0.2557418387670495142539197612059706034569501897187817325149891776278724062164321975380157608214281272

65078956751963482051943316805725807875808917044164397051701428417717043336467054424811549764821932906

05597982521159814229442365606027410080708124216701893396901412120640024489362393793733177653017308562

99734374355850189724258427758618089253738395794473630503792093715946038342747035565025302472725126106

Conclusion

We

- Give the first complexity bounds for methods in analytic combinatorics in several variables

- Combine strong symbolic results on the Kronecker representation with fast algorithms on univariate polynomials in a novel way to create a symbolic-numeric data structure

Lots of room for extensions on the analytic combinatorics side and the computer algebra side!

FIN

(MERCI)

Example (Lattice Path Model)

$$F(x,y,t) = \frac{(1+x)(1+y)}{1-t(x^2+y^2+x^2y^2+1)}$$

The asymptotic contribution of a point given by a root of

$$P(u) = 4u^4 + 52u^3 - 4339u^2 + 9338u + 403920$$

is

$$\Phi(u) := \left(\frac{16\,u^3 + 156\,u^2 - 8678\,u + 9338}{23\,u^2 - \frac{3061\,u}{2} + \frac{12931}{2}} \right)^k \frac{16\,u^3 + 424\,u^2 - 11632\,u - 119680}{2k\pi\,\left(16\,u^3 + 156\,u^2 - 8678\,u + 9338\right)}$$

All 4 roots contribute to the asymptotics up to exponential decay, however only the first determines the dominant polynomial growth, which is

$$a_k = \frac{2}{\pi}\frac{4^k}{k}\left(1 + O\left(\frac{1}{k}\right)\right)$$